Asset Liability Management for Icelandic Pension Funds

The Stochastic Programming Approach

by

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in Partial Fulfillment of the Requirements for the Degree of

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Abstract

In this thesis a modeling framework to aid Icelandic pension funds in their asset allocation decisions is introduced. The framework is based on stochastic programming and asset liability management methodologies, where emphasis is placed on the uncertainty in the liabilities of the pension funds as well as the current Icelandic regulatory and supervisory environment.

The modeling framework is based on three interacting components: modeling the underlying stochastic variables; a scenario generator; and a stochastic programming model which is formulated as a multistage recourse problem in discrete state space and is solved with a scenario-based technique. Computational experiments presented in this thesis are carried out for a hypothetical pension fund, where market valued liabilities and premiums are used and the current capital controls are not accounted for.

The credibility of the model is analyzed in terms of in-sample stability and analysis on spurious changes in the optimized asset allocation. This vital attribute has received little attention in similar studies. The model performance is measured by comparison with partly dynamic fixed-mix investment strategies in terms of stronger actuarial position at the horizon. In addition, analysis of the most influential constraints is carried out with sensitivity analysis.

The weakest component of the framework proved to be the scenario generator which was unable to generate a sufficient number of scenarios for satisfactory in-sample stability. In terms of performance and credibility, the model outperformed the partly dynamic fixed-mix benchmarks, in terms of stronger actuarial position at the horizon with no drastic changes in the optimal asset allocation. The most influential constraint is maximum allowed purchases in each year. Finally, comparison based on actuarial position, where liabilities and premiums are valued with fixed 3.5 percent interest rate, suggests that the interest rate risk is underestimated in the actuarial valuation methods currently used by the Icelandic Financial Supervisory Authority.

Preface

This dissertation is in fulfillment of the requirements for the M.Sc. degree in Financial Economics at the University of Iceland. The dissertation is to the value of 30 ECTS credits. The M.Sc. degree requires 90 ECTS credits. It is accomplished under the supervision of Prof. Dr. Helgi Tómasson and Asst. Prof. Dr. Steinn Guðmundsson. I would like to express my utmost gratitude for their support, guidance and constructive criticism during the writing of this dissertation.

I also would like to thank Eiríkur Dór Jónsson and Ólafur Örn Jónsson at Arion Bank for their professional opinions regarding real pension fund modeling as well as, providing me with necessary data.

Finally I would like to thank my family and friends for their support and encouragement during the last weeks of writing.

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List of Notation

Stages (time periods) are denoted by an index t with an initial stage denoted by $t = 0$ and and a final stage (the horizon) denoted by T. All stages considered are periods of one year, which is the time span from the time $t-1$ to time t.

Individual assets are denoted with an index j. The number of individual assets is denoted with J. To ease the notation a little, is j also used to denote a stochastic variable. The distinction is however clear in context.

Asset classes are denoted by an index q. This is done to clearly separate asset classes from assets. The reason is that one asset class, foreign stocks, is composed of four stock indices.

In addition we have the frequently used notation

Sets

$\mathcal{T}$: Set of times from the present to the horizon, $\mathcal{T} = \{1, 2, \ldots, t, \ldots, T\}$.

$\mathcal{J}$: Set of stochastic variables, $\mathcal{J} = \{1, 2, \ldots, j, \ldots, J\}$.

$\mathcal{N}$: Set of simulated paths, $\mathcal{N} = \{1, 2, \ldots, n, \ldots, N\}$.

$\mathcal{S}$: Set of scenarios, $\mathcal{S} = \{1, 2, \ldots, s, \ldots, S\}$.

$\mathcal{Q}$: Set of asset classes, $\mathcal{Q} = \{1, 2, \ldots, q, \ldots, Q\}$.

Stochastic variables

ξ^j: Stochastic variable j.

$\boldsymbol{\xi}_t^j$: Stochastic process coherent with stochastic variable j,

$\tilde{\boldsymbol{\xi}}^j$: Simulated paths of stochastic variable j, $\hat{\boldsymbol{\xi}}_n^j = \{\tilde{\xi}_1^j, \ldots, \tilde{\xi}_N^j\}$

$\tilde{\xi}_n^j(t)$: Value of the nth simulated path of stochastic variable j at stage t.

The scenario tree

$\mathbb{B}$: Branching structure, $\mathbb{B} = \{b_1, \ldots, b_t, \ldots, b_T\}$.

$C^{(t,s)}$: Cluster at stage t in scenario s.

$C_o^{(t,s)}$: The center of a cluster $C^{(t,s)}$(its position)

p^s: Unconditional probability that scenario s occurs.

$\pi^{(t,s)}$: Inflation in cluster $C^{(t,s)}$.

$r_{usd}^{(t,s)}$: Exchange rate returns in cluster $C^{(t,s)}$.
$\pi^{(t,s)}$: Inflation in cluster $C^{(t,s)}$.
$\text{spot}^{(t,s)}$: One year real yields in cluster $C^{(t,s)}$.
g_{m_2,m_1}: A m_2-to-m_1 real yield spread. $(m_2 > m_1)$.
q_{m_2,m_1}: A m_2-to-m_1 nominal yield spread. $(m_2 > m_1)$.
$y_r^{(t,s)}(m)$: Real yield for maturity m in $C^{(t,s)}$.
$y_n^{(t,s)}(m)$: Nominal yield for maturity m in $C^{(t,s)}$.

The Optimization Model

Decision variables

$h_{t,j}$: Holdings in asset j at stage t, where $h_{t,j} \geq 0$.
$p_{t,j}$: Purchase in asset j at a stage t, where $p_{t,j} \geq 0$.
$s_{t,j}$: Sales in asset j at a stage t, where $s_{t,j} \geq 0$.
Z^{target}: Penalty variable for funding ratios which do not reach the pre-specified target funding ratio at the horizon.
$Z^{\min}$: Penalty variable for funding ratios which fall below the pre-specified minimum ratio.
W_t: Wealth of the pension fund at stage t.

Stochastic variables

$R_{t,j}^{(t,s)}$: Return on asset j at stage t in scenario s.
$\tilde{L}_i$: Present value of future liabilities of the pension fund at stage time t.
$\tilde{I}_i$: Present value of premiums of the pension fund at stage time t.

Parameters

h_j^0: Initial holdings in asset j.
c^j: Transaction cost for asset j.
l_j: Lower bound (as a percentage of wealth) for holdings in asset j.
u_j: Upper bound (as a percentage of wealth) for holdings in asset j
l^q: Lower bound (as a percentage of wealth) for holdings in asset class q.
u^q: Upper bound (as a percentage of wealth) for holdings in asset class q.
u^p: Upper limit (as a percentage of wealth) for purchase.
$\text{FR}^{\text{target}}$: Funding ratio of the pension fund at decision time t.
$\text{FR}^{\min}$: Funding ratio of the pension fund at decision time t.

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Chapter 1. Introduction

In a world where financial markets have become seemingly increasingly complicated and more volatile, the need for professional financial management is becoming vital for a standard of living. As a result of a rise in life expectancy, pension funds are essential to ensure decent retirement years. From that perspective, individuals rely on pension fund managers to responsibly invest their contributions and they expect fair pension payments in return when they retire. Faced with the fact that financial crises have occurred fairly frequently the necessity for responsible allocation decisions made by pension fund managers are greater than ever. These decisions are made with respect to portfolio management theory as well as market experience. Rapid developments have taken place in the field of portfolio management and a vast number of sophisticated tools have emerged to aid managers in their decisions. Moreover, growing awareness of the importance of transparency, trust and fiduciary responsibility in portfolio management have led to a change in the attitude of pension funds managers toward their responsibilities.

Maginn *et al.* (2007) define portfolio management as a *process* which applies to all types of portfolio investments. They suggest that the process is based on an integrated set of steps undertaken in a consistent manner to create and maintain appropriate combinations of investment assets. The steps are

1. The planing step
2. Identifying and specifying the investors objectives and constraints
3. Creating investment policy statements
4. Forming Capital Market Expectations
5. Creating the Strategic Asset Allocation

Briefly stated, the planing step is governed by the specifications of investor, market and economic related input factors which are incorporated into the investor's objectives and constraints. The investment policy statement serves as a document for all investment decision making, which includes for example performance measures and guidelines for rebalancing. In the Capital market expectations step, long term forecasts of risk and return measures are established. These forecasts serve as benchmarks for the strategic asset allocation decision (SAA) which is the final step. SAA lists the percentage of wealth invested in each asset class (portfolio policy) and is integrated with long term capital market expectations.

By itself, SAA can be defined as a process[1] with certain well-defined steps, the objective of which is to define the portfolio policy. Another type of asset allocations is tactical asset allocation (TAA), which deals with short term adjustments of the portfolio policy. For example, a pension fund's SAA might state that it is coherent with the fund's objective and risk aversion to hold 60% of its wealth in domestic government bonds. If at any time, the actual proportions held in this asset class drifts away, say to 55%, it is referred to as a tactical decision. The portfolio policy is often stated both in terms of SAA and TAA. In such a format, SAA are expressed as the target holdings in each asset class whereas TAA define the tolerance of deviation from the SAA. This gives the pension fund manager the flexibility of adjusting the portfolio according to his beliefs while staying within the limits of the portfolio policy.

Defining portfolio management as a process emphasizes the fact that it requires constant consideration and revision through feedback. The process is centered around the the investment policy statement (IPS), and the necessity for feedback and maintenance are vital features of the process. The IPS serves as the governing document for all investment decision making. Icelandic pension funds are bound by law[2] to create and accompany the IPS which is kept at the forefront in the event of a dispute between the fund and its members.

1.1 The Asset liability management framework

In recent years, Icelandic pension fund managers have shifted the focus from return objectives, such as reaching a pre-specified average return over a few years, to integrated asset and liability objectives, such as minimizing the risk of reduction in pension payments (Ó. Ö. Jónsson and E.D. Jónsson, personal interview, June 24, 2011). Such objectives generally concentrate on the asset-liability ratio, also known as the funding ratio. This shift in focus results in different risk measures. Before, the main risk factor was the uncertainty or the volatility of asset returns, as apposed to liability risk measures. The latter being a result of the integrated asset liability objectives. This growing awareness of the interplay between assets and liabilities is

[1]The definition of SAA varies in the litereture. One can either think of SAA as an individual process inherent in portfolio management or as the result of the portfolio management process itself.

[2]Article 20 of Act 129/1997. `www.althingi.is`

more consistent with the fundamental purpose of the pension system, which is to provide pension and disability payments to retirees as well compensation to spouses and children.

Consider the following example which further illustrates the necessity for an integrated view of assets and liabilities, in terms of interest rate risk. Suppose that a pension fund portfolio of assets has a duration of eight years and the fund is faced with future liabilities with duration of twenty five years. If yields decrease by one percent, then the asset portfolio increases by roughly eight percent and the liabilities increase by twenty five percent[3]. A Focus on return objectives, where only the asset side is considered, results in overconfidence since the asset-liability ratio has decreased.

The pension fund investment policy must take into account the interplay between the asset and the liability sides and this has given rise to the asset liability framework. In this framework, more emphasis is paid to balancing the asset and liability sides which results in increased awareness of other risk factors. Asset liability management studies are not only relevant for pension fund applications, but also for a wide range of applications where one is faced with future liabilities, e.g. insurance companies.

1.1.1 Modeling techniques

In general, all pension funds reinvest their collected premiums and pay old age pensions as well as compensation. However each, pension fund is unique. Apart from the different types of pension funds each fund is a reflection of its members. The age composition of the pension fund members' has a great impact on the fund. Young members typically prefer more aggressive investment strategies and have a higher risk tolerance, whereas members closer to retirement are more concerned with risk of reduction in pension payments and generally prefer more conservative investment strategy. Also each pension fund is subject to preferences, beliefs and investment guidelines imposed by the fund's manager and its board of directors. Moreover the ability to predict future liabilities is subject to the age compositions of the fund's member. Such predictions are essential in asset liability management. This variability between pension funds requires a custom-designed model for each fund, where objectives and risk preferences can be modeled in sufficient detail.

A number of modeling methods and techniques have been suggested in the asset liability management literature. Frequently used methods in pension fund applications are simulation and optimization under uncertainty. A fundamental difference

[3]Based on the first term taylor expansion, $(\Delta\text{Price})/\text{Price} = -\Delta(\text{yield}) \cdot \text{duration}$.

is that in simulation, the portfolio policy is an input, which corresponds to static decision making. In optimization under uncertainty, the portfolio policy is the result, and the method allows for dynamic decision making. In addition Klein Haneveld, W.K., Streutker, M.H., van der Vlerk, M.H (2010) argue that optimization under uncertainty provides suitable and flexible framework in terms of incorporating numerous restrictions.

A study by Fabozzi, F.J. and Focardi, S.M. and Jonas, C.L. (2005) on 28 pension funds in four countries reveals that two thirds of the participating funds used optimization but many funds still prefer simulation over optimization, the main reason being that optimization models are time consuming and hard to implement.

1.1.2 Optimization under uncertainty

Optimization addresses the problem of finding a decision, often is subject to some constraints, that minimizes (or maximizes) a function. Using matrix notation, this is represented mathematically as

$$\begin{array}{lll} \min\limits_{x} & f(x) & \\ \text{subject to} & & \\ & g_i(x) = 0 & , i = 1, \dots, n \\ & h_j(x) \leq 0 & , j = n+1, \dots, m \end{array} \tag{1.1.1}$$

where x, $f(x)$ are referred to as decision variable(s) and the objective function and $g_i(x), h_j(x)$ are constraints. In general, optimization models are classified according to the nature of the objective function and constraints. For example, when the objective function and all constraints are linear, problem (1.1.1) is referred to as a linear optimization problem.

The problem of optimal decision-making in face of uncertainty is referred to as optimization under uncertainty. Optimization problems that include uncertainty have been applied in many fields, such as production planning, transportation, scheduling and finance to name a few. Uncertainty governs for instance the availability of raw materials in production planning, the demand for goods or the return of assets. In such problems, the objective function and/or constraints are represented as functions of stochastic variable(s), either discrete or continuous. Over the years a number of modeling philosophies and approaches have been suggested to cope with the complexity of such optimization problems. Three main methods are schematically laid out in figure (1.1).

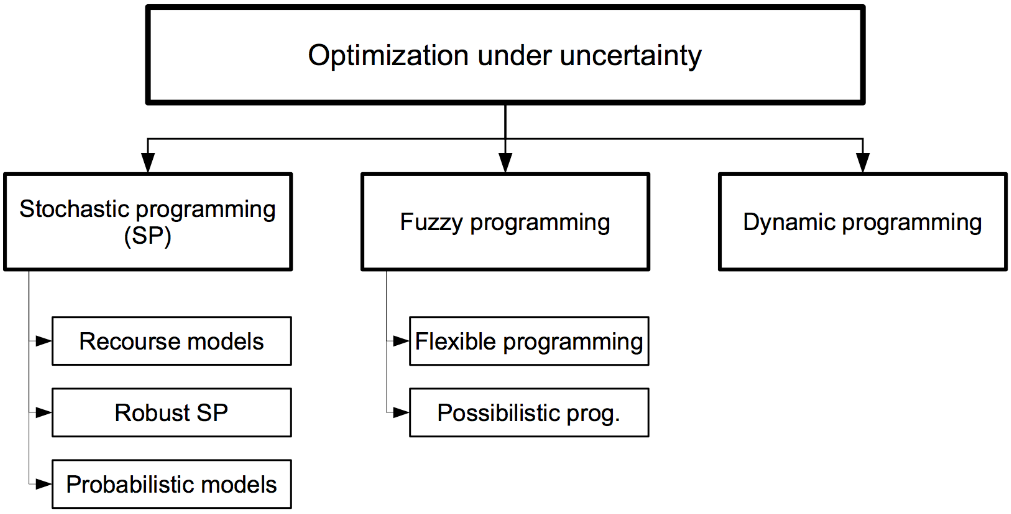

Figure 1.1. Taxonomy of methods for optimization under uncertainty.

The main methods differ on how the uncertainty is modeled. A sensible discussion of the different aspects of each method requires detailed formulations as well as numerous definitions and concepts, especially in the case of fuzzy programming. Instead the interested reader is referred to a comprehensive discussion and extensive survey presented by Sahinidis (2004). It is worth mentioning that publications in the literature governed by fuzzy programming for financial applications are scarce. The choice is therefore mostly between dynamic and stochastic programming.

In this thesis, the stochastic programming method is used for the following two main reasons. Asset liability management applications for pension funds presented in the literature are highly concentrated on stochastic programming[4], e.g. see surveys in Sodhi (2005); Ziemba & Mulvey (1998) or Zenios & Ziemba (2007). In addition a class of stochastic programming models can be reformulated into a large scale linear optimization problem. This is highly desirable since linear optimization problems are well studied and powerful solution methods exists.

[4]A number of applications presented in the literature are titled dynamic stochastic programming. This is to emphasize that stochastic programming in general is dynamic, rather than dynamic programming is being used, which might be misleading.

1.2 Motivation

The collapse of the financial markets in late 2008 left many Icelandic pension funds severely underfunded in terms of their actuarial position[5]. Many funds are running a high deficit and only two funds out of 24 showed a positive actuarial position at the end of year 2010 (Financial Supervisory Authority, 2011). This has forced Icelandic pension funds to reduce accrued pension rights. In late 2011 the fund Stapi was forced to make a staggering six percent reduction in accrued pension rights from contributions prior to 2010,[6] and in the same year, a two point five percent reduction was made by the fund Almenni lífeyrissjóðurinn[7].

A study by Frank J. Fabozzi & Jonas (2005) revealed that 90% of the private-sector defined-benefit pension plans in the UK and US were underfunded. The reason mostly bad modeling or the absence of modeling, together with difficult conditions in the financial markets. Although one cannot assume that this was the case for all the Icelandic pension funds and the current crisis, this issue is worth investigating further.

Tower Perrin-Tillinghast constructed an asset liability management system using a stochastic programming model to aid its pension plan clients, described in Mulvey *et al.* (2000). In the same paper, savings of USD 450 to USD 1,000 million in opportunity costs in US WEST's pension plan was reported. Another example of successful application in asset liability management using stochastic programming is the Russell-Yasuda Kasai model. It was designed for Japanese insurance company and savings of USD 79 million was reported during the models first two years in use (Carino *et al.* , 1994). Although such large scale successes are rarely reported in academic papers, this increases the confidence that such modeling systems could prove very useful.

1.3 Problem description

Pension fund managers are concerned with the issue of forming an investment strategy which benefits the funds. More precisely, how should a pension fund allocate its wealth among the different asset classes such that the risk of reduction in pension payments is minimized?

[5] Discussed in more detail in section [2.2]

[6] The funds news report: `http://www.stapi.is/is/news/rettindi-laekka-um-6-/`

[7] Specification was conditional on actuarial position at the end of year 2010 and are laid out in (Almenni lífeyrissjóðurinn, 2010)

The focus of this thesis is to introduce an alternative modeling framework based on stochastic programming. Its purpose is to aid a pension fund manager their asset allocation decisions. The model is based on asset liability management methodology, which is ideal for pension funds and on stochastic programming which is adjusted to the environment in which Icelandic pension funds operate. In addition, emphasis is placed on taking the current regulatory and supervisory constraints into account to make the model as realistic as possible.

Several models, based on asset liability management studies from a stochastic programming point of view, are available in the literature. However each model has to be custom designed according to the pension funds at hand. Specifications in terms of objectives and risk aversion of the fund, as well as the regulatory framework, must be incorporated into each model.

Little attention has been paid to the quality of asset allocation decisions suggested by the models in the literature. This aspect will be investigated in this thesis by analyzing the stability of the solution, as well as the effects of individual model parameters on the solution. Furthermore, an important question which has not received much attention in the literature is whether the resulting asset allocation decisions are realistic. In this thesis credibility of the asset allocations will be analyzed in terms of spurious asset allocation switching between time periods. This is essential if the model is to be used in practice. The performance of the model introduced in this thesis is measured by comparison with partly dynamic fixed-mix investment strategies in terms of stronger actuarial position at the horizon. The term fixed-mix applies to constant portfolio rebalancing to maintain fixed holdings in each asset class. The strategies are partly dynamic in the sense that although holdings in each asset class are fixed, no additional restrictions are placed on holdings in individual assets.

1.4 Thesis structure

The model introduced in this thesis is based on the following three interacting components:

1. Modeling the underlying stochastic variables
2. Scenario generation
3. A multistage optimization model

Each part is covered in detail in separate chapters, along with a brief description of the Icelandic pension system. The interplay between the three parts is shown in figure 1.2, where references to relevant chapters are shown in brackets.

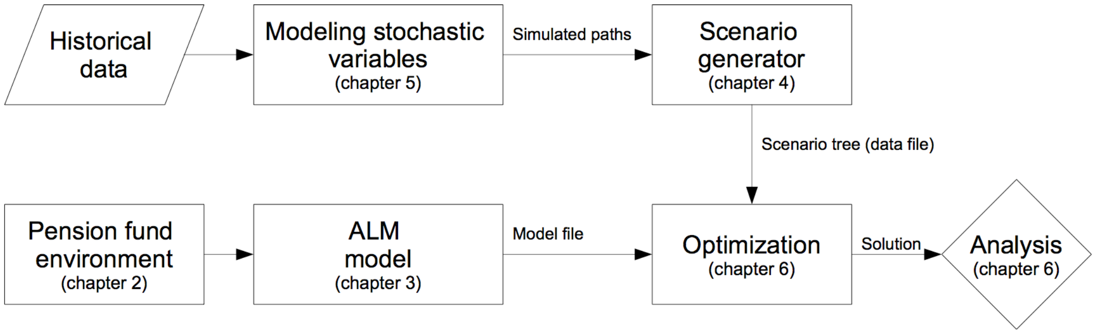

Figure 1.2. Interplay between the components of the multistage stochastic programming framework.

Chapter 2. The Icelandic pension system

The present day pension system in Iceland dates back to the general wage settlement of the spring of 1969, a result of negotiations between the State, labour unions and the Federation of Icelandic Employers. In the beginning of 1970 the labor unions traded a wage increase for the setting up of a fully funded mandatory occupational pension fund which is the dominant feature of today's Icelandic pension system. The occupational pension funds became general in 1970, then mandatory by law for wage earners in 1974 and were extended to the self employed in 1980. A Law was adopted in 1991 on the annual accounts and auditing of pension funds (V. Árnason, 2010). This gave the Bank Inspectorate of the Central Bank of Iceland some supervisory role over the pension funds. A comprehensive reform took place in 1997 and 1998 which resulted in the current Pension act, Mandatory Guarantee of Pension Rights and the Operations of Pension Funds. This adoption proved difficult, in fact the work on legislation framework on pension funds had started in 1976 (Guðmundsson, 2001). The reform affected the mandatory occupational pension funds as well as introduction of tax incentives for voluntary individual pension savings. Since then, minor changes in the pension fund legislation have been imposed.

The current Icelandic pension system is based on three pillars. The First pillar is the Icelandic Social Security, generally referred to as the public pension system. It is a tax financed pension scheme which aims to secure each individual minimum pension payments which are reduced in relation to other income. It provides an old age pension along with survivor's and disability pension. The public pension is divided into a basic pension and supplementary pension, both are paid from the age of 67. The Second pillar is a mandatory occupational pension scheme. It is mandatory by law[1] to pay at least 12% of all wages and salaries from the age of 16, which is split between a 4% contribution from the employee and a 8% contribution

[1] Article 2 of Act 129/1997

from the employer. According to Icelandic Law[2] pension funds must pay a minimum coverage of 56% of monthly wages for 40 years of work. Occupational pension schemes are managed by private, fully funded, pension funds governed jointly by unions and employers and provide lifelong retirement and disability pension as well as pension to children and spouses. The majority of pension funds in the Second pillar are based on a defined contribution scheme without guarantee by the Treasury and Municipal Authorities. However, a few funds are based on defined benefit schemes with an employer guarantee. The Second pillar is the dominant feature of the Icelandic pension system. The funds are monitored by the Icelandic Financial Supervisory Authority[3]. The Third pillar forms a voluntary individual pension saving scheme with tax incentives. The maximum deduction by the employee is 4% and in wage settlements employers have agreed to contribute 2% to these voluntary pensions if the employee matches the amount with at least the same percentage, making the maximum total contribution of 6%. The voluntary pension savings are in most cases a defined contribution pension scheme with individual accounts, where pension savings have to be paid in equal payments over of at least seven years from the age of 60 (Jónsdóttir, 2007). At the end of year 2010 the voluntary pension saving scheme accounted for 15.5% of the whole pension system (Financial Supervisory Authority, 2010).

2.1 Recent developments

The Icelandic pension system has grown dramatically in the past 20 to 30 years. At end of year 2010 net assets amounted to roughly ISK 2,021 billion compared to ISK 522 billion at the end of 2000. The pension funds' net assets as a share of gross domestic product (GDP) were 123.91% compared to 83.97% in 2001 and 11.6% in 1986. The growth is largely due to the advantageous age composition of the Icelandic nation and relative young age of the system, which results in positive cash flow in the system. Furthermore, indexation gave pension funds unusually good returns in the early years of the system. The pension system is dominated by a few large pension funds. At the end of year 2010, the three largest pension funds accounted for 47.2% of the pension system assets and the assets of the ten largest pension funds accounted for 80.1% of the whole pension system, (Financial Supervisory Authority, 2011). The Pension funds' real return, accounted for inflation, was 2.65% at the end of year 2010 compared with 0.34% at the end of year 2009. Figure 2.1 shows the yearly real return

[2] Article 4 of Act 129/1997
[3] `www.fme.is`

from 2000.

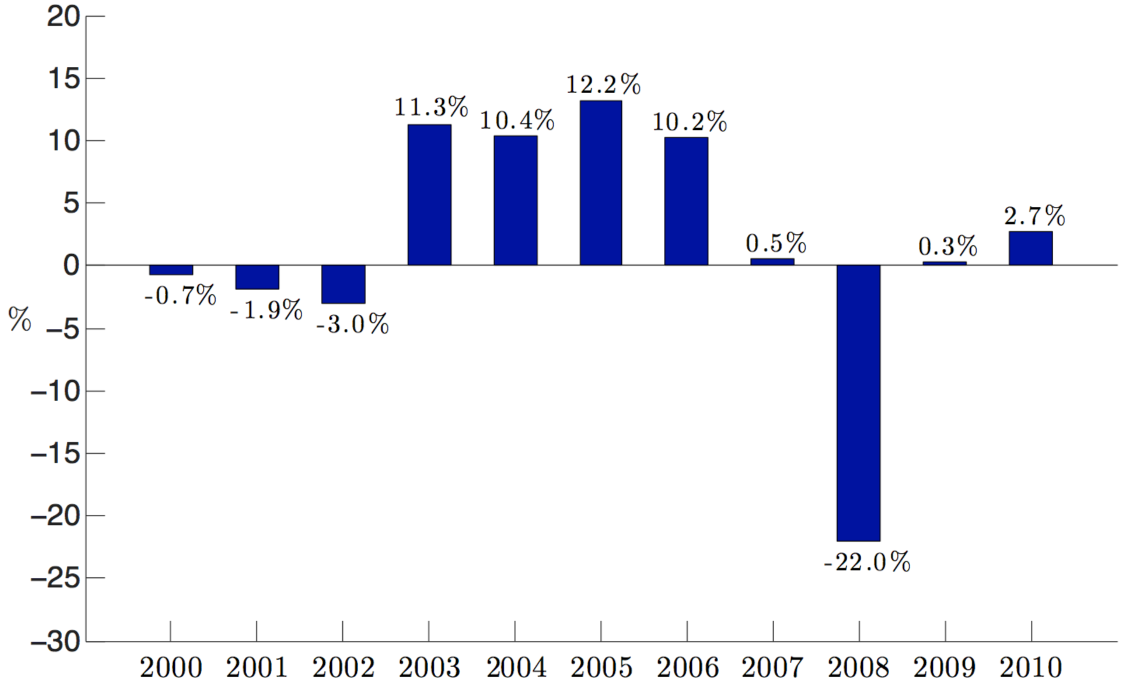

Figure 2.1. Yearly real return from end of year 2000 to end of year 2010. **Source:** The Icelandic Financial Supervisory Autority, `www.fme.is`

In the spring of 2008, pension funds managers realized that difficulties in the Icelandic economy were severe, particularly when the Currency Market collapsed in the spring. The fear became a reality when the Icelandic commercial banks fell in October of the same year. The pension funds, which had invested heavily in the Icelandic Financial sector, lost their shares in the three banks and had to depreciate much of the bank's bonds. The collapse was not only tied to the financial sector, the entire domestic stock market plunged. The value of domestic stocks in pension fund portfolios were estimated at nearly ISK 24 billion at end of year 2008, compared to ISK 141 billion in September of the same year. Similarly the Icelandic bond market took a big downswing, particularly corporate bonds. Corporate bonds accounted for ISK 144 billion in pension funds portfolios in late December 2008, compared to ISK 189 billion in September. In 2008 the Icelandic krona fell by 80.24% and the average inflation was 16.34% (Óskarsdóttir, 2011). This led to a negative real increase of 22% of the pension funds' net assets in 2008 as shown in figure 2.1.

Since the collapse, foreign exchange transactions have been subjected to capital controls, requested by the International Monetary Fund (IMF). Therefore new foreign investments are not allowed, and pension funds can only reinvest their wealth which was present before the controls. Together with the collapse of the domestic stock market, investment opportunities are limited. Therefore pension funds increased their shares in government and housing bonds as well as deposits relative to previous years as shown in figure 2.2.

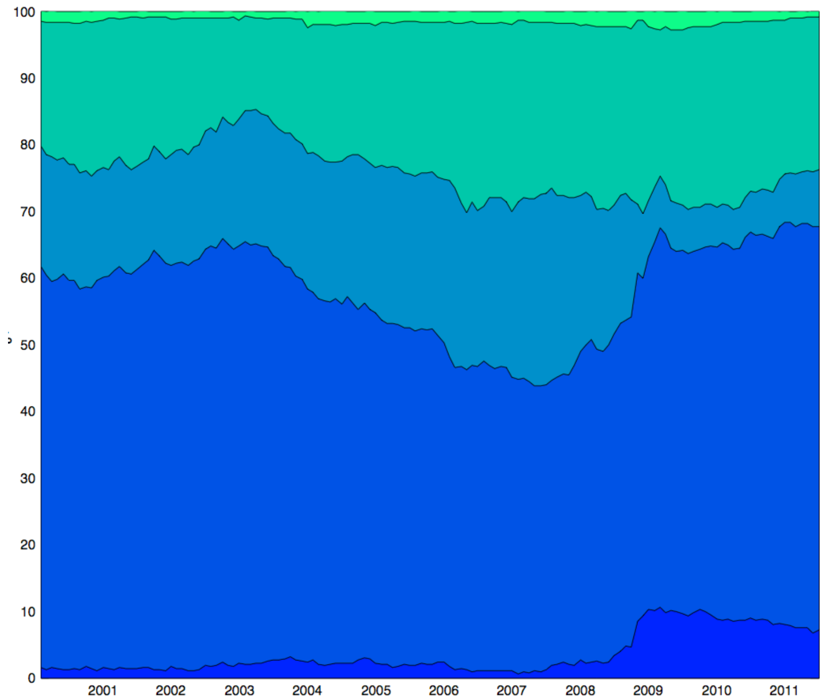

Figure 2.2. Historical asset allocation of Icelandic pension fund with the following legend. (■) Deposits, (■) domestic fixed income securities, (■) domestic variable income securities, (■) foreign variable income securities and (■) other asset classes. **Source:** The Icelandic Financial Supervisory Authority, `www.fme.is`

According to a study conducted by the Organisation for Economic Co-operation and Development (OECD) on pension funds' assets as a share of GDP, Iceland's pension system is large in comparison to the OECD nations at the end of year 2010 as shown in figure 2.3.

It is evident that many challenges await the Icelandic pension system in the near future. The liberation of the capital controls are in motion (Central Bank of Iceland, 2011), and the Icelandic stock market is just a fraction of its previous size. Until it regain some of its former size, investment opportunities are limited. On the liability side, changes in the age composition of the Icelandic nation and a possible increase in disability pension payments will affect the funds' financial position. This requires that the funds invest their wealth responsibly to be able to cover future liability payments.

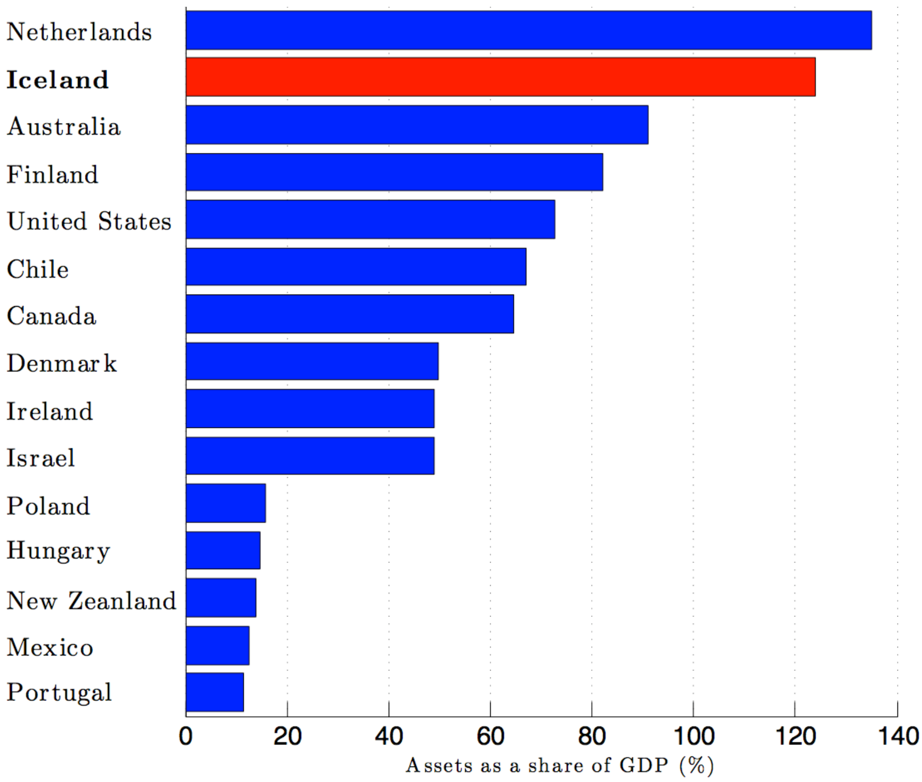

Figure 2.3. Assets as a share of GDP in % (fifteen highest ratios) at the end of year 2010. **Source:** OECD Global Pension Statistics, `www.oecd.org`

2.2 Actuarial position

In July each year Icelandic pension funds conduct an actuarial survey within the Mutual Insurance Division of the pension funds, as requested by the Icelandic Financial Supervisory Authority. The actuarial survey is pursuant to the principal feature (article 24) of Act 129/1997 on the Mandatory Guarantee of Pension Rights and the Operations of Pension Funds, which took effect on 1 July 1998.

Pension funds liabilities are pension payments to retirees, disability pensions as well as compensations to spouses and children in case of the passing of a fund member. Premiums are paid contributions by fund members as mandated by law. Pension fund liabilities consist of accrued liabilities and future liabilities. Accrued liabilities are based promised on pension payments on already paid premiums. Future liabilities are estimated present values of liabilities which correspond to premium payments in the future. The estimates are based according to the articles of association of each pension fund and are based on members who contributed to the fund in the previous year. Future premiums are estimated present value of future contributions from active pension fund members. Figure 2.4 shows the structure of the total actuarial position[4] which is used as an indicator to measure the funds' ability to meet its liabilities.

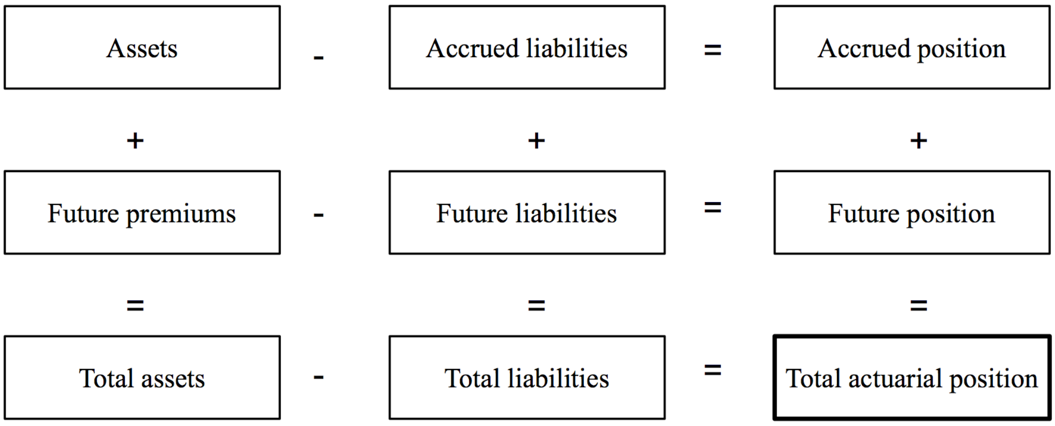

Figure 2.4. The structure of the total actuarial position. **Source:** (Baldvinsson, 2004, p. 53)

Accrued position is thus calculated as the difference between assets and liabilities, and future position is calculated as estimated present value of future premiums less estimated present value of future liabilities. The total actuarial position is based on the sum of assets and future premiums less accrued and future liabilities (Financial Supervisory Authority, 2010). Table 2.1 shows the accrued position, future position and total actuarial position for ten largest pension funds as well as for the Icelandic pension system at the end of year 2010. In all cases, the accrued and total actuarial position is negative.

[4]The provisions of the 39th Article of Act 129/1997 about actuarial position only take into account total actuarial position

Table 2.1. Accrued position, future position and total actuarial position for the ten largest pension funds and the Icelandic pension system at the end of year 2010. All figures in ISK thousand. **Source:** (Financial Supervisory Authority, 2011)

Pension fund	Accrued position	Future postion	Total actuarial position (in ISK)
Lífeyrissjóður starfsmanna ríkisins	-324,654	-73,159	-397,813
Lífeyrissjóður verslunarmanna	-7,924	-12,009	-19,933
Gildi lífeyrissjóður	-25,929	-10,366	-36,295
Stapi lífeyrissjóður	-17,917	-6,242	-24,159
Sameinaði lífeyrissjóðurinn	-11,097	787	-10,310
Almenni lífeyrissjóðurinn	-6,868	1,406	-5,462
Frjálsi lífeyrissjóðurinn	-4,275	472	-3,803
Stafir lífeyrissjóður	-16,733	4,558	-12,175
Söfnunarsjóður lífeyrisréttinda	-1,709	-1,571	-3,280
Festa lífeyrissjóður	-6,507	-3,184	-9,697
Icelandic pension system	-524,171	-127,465	-651,637

The total actuarial position (in %) is the ratio between total assets less total liabilities divided by total liabilities. Total assets are the sum of assets and future premiums and total liabilities are the sum of accrued and future liabilities. If the ratio is positive then the net assets at each time are sufficient to cover already promised liability payments which indicate strong actuarial position. According to article 39 of Act 129/1997, all pension funds without a guarantee, showing deficit of 10% or higher in a single year or a deficit higher than 5% for five consecutive years, must amend their Articles of Association in order to achieve a satisfactory actuarial position. In late December 2008, a transitional provision was added which uthorized pension funds to run a deficit up to 15% based on actual valuation for the year 2008, without making changes to the Articles of Association of the fund (Financial Supervisory Authority, 2009). Historical actuarial position for all pension funds without an employer guarantee is shown in table 2.2 from the year 2005 to 2009.

Table 2.2. Historical actuarial position, for all pension funds without guarantee, from end of year 2005 to end of year 2010. **Source:** Icelandic Financial Supervisory Authority (`fme.is`).

	2005	**2006**	**2007**	**2008**	**2009**	**2010**
Positive	22	25	22	3	3	2
Deficit between 0,1% - 5%	12	8	4	7	3	8
Deficit between 5,1% -10%	3	2	3	7	6	10
Deficit between 10,1% - 15%	1	0	0	8	13	4
Deficit in excess of 15%	-	-	-	4	3	0
Total:	38	35	29	29	28	24

The effects of the financial crisis on the actuarial position is evident. Prior to the crisis the majority of funds had a positive actuarial position, but after the crisis many funds showed a high deficit. Almost all funds with a guarantee from the Treasury and Municipal Authorities show a high deficit at the end of 2010. The highest deficit amounts to a staggering 99.3% which is however guaranteed (Financial Supervisory Authority, 2011).

Chapter 3. The Stochastic programming approach

The Committee on Stochastic Programming (COSP) (2011) describes, stochastic programming as a framework for modeling optimization problems that involve uncertainty. Stochastic programming models allow for progressive revelation of information in time and multiple decision stages, where each decision is adapted to the available information (Kouwenberg & Vorst, 1998). It is a general framework for modeling optimization problems that involve uncertainty. Several different methods for formulation exists for stochastic problems.

First explanations on the important concepts of stochastic programming is given. Stages correspond to the time periods when decisions are made and the horizon refers to the number of stages. It is assumed that at any stage, finitely many states of the system exist, and the states are described by (multidimensional) state variables. In stochastic programming the state variables are affected by uncertainty. Given the initial state of the system, the overall objective is to maximize (or minimize) some objective function of an immediate return for all stages and states, (Kall & Wallace, 1994).

Stochastic programming systems can be categorized, based on how information is reveled through time, (Yu *et al.* , 2003). In an *anticipative* formulation, also referred to as a static model, the decision does not depend in any way on future observations of the environment. In an *Adaptive* formulation, information related to the uncertainty becomes partially available before the decision making, so optimization takes place in a learning environment. The *recourse* formulation combines the former two models in a common mathematical framework, which seeks a policy that not only anticipates future observations but also takes into account temporarily available information to make recourse decisions.

For financial application, the recourse formulation is the most relevant. For example a portfolio manager considers both future movements of stock prices (anticipation) as well as the rebalancing of the portfolio as prices change (adaptation).

3.1 General multistage recourse formulation

In this thesis we consider a stochastic programming problem in the form of a multistage recourse problem in discrete stage and state space, following (Dupačová, 1999; Kall & Wallace, 1994; Shapiro *et al.* , 2009).

Suppose there are $T \geq 2$ discrete stages and that the uncertainty is expressed by the (multidimensional) random variable $\xi_1, \dots, \xi_T$, which is revealed gradually over time during the T stages. The decision process $x_1, \dots, x_T$ is adapted to the revelations of the random variables and has the form

$$\begin{array}{ccccccc} \text{decision} & \rightsquigarrow & \text{realization} & \rightsquigarrow \cdots \rightsquigarrow & \text{decision} & \rightsquigarrow & \text{realization} \\ (x_1) & & (\xi_2) & & (x_T) & & (\xi_T) \end{array}$$

The sequence ξ_t, for $t = 1, \dots, T$ is viewed as a stochastic process[1]. Let $\xi^{\lfloor t \rfloor} := (\xi_1, \dots, \xi_t)$ denote the history of the process up to time t. At the first stage, the state of the system is known and the outcome of the decision completely depends on the future realizations of the underlying stochastic process. Thereafter, for each realization of the history $\xi^{\lfloor t \rfloor}$ up to time t, a recourse decision is made which is only allowed to be a function of the observed realizations $(x^{\lfloor t-1 \rfloor}, \xi^{\lfloor t \rfloor})$. In other words, the recourse decisions depend on the current state of the system as determined by previous decisions $(x^{\lfloor t-1 \rfloor})$ and by the information $(\xi^{\lfloor t \rfloor})$ available up to time t but not future observations. This adaption to the available information is the basic requirement for nonanticipativity, which will be discussed further later on.

The generic form of a T stage stochastic programming model can be written in nested formulation as

$$\min_{x_1 \in \mathcal{X}_1} f_1(x_1) + \mathbb{E}\left[\inf_{x_2 \in \mathcal{X}_2(x_1, \xi_2)} f_2(x_2, \xi_2) + \mathbb{E}\left[\cdots + \mathbb{E}\left[\inf_{x_T \in \mathcal{X}_T(x_{T-1}, \xi_T)} f_T(x_T, \xi_T) \right]\right]\right] \tag{3.1.1}$$

where $\mathbb{E}$ is the expectation operator, the function $f_1 : \mathbb{R}^{n_1} \to \mathbb{R}$ and the set $\mathcal{X}_1 \subset \mathbb{R}^{n_t}$ are deterministic, $x_t \in \mathbb{R}^{n_t}$, $t = 1, \dots, T$ are decision variables and $f_t : \mathbb{R}^{n_1} \times \mathbb{R}^{m_1} \to \mathbb{R}$ are continuous functions for $t = 2, \dots, T$. For a linear multistage stochastic program we have

$$\begin{aligned} f_t(x_t, \xi_t) &:= c_t^T x_t && t = 2, \dots, T \\ \mathcal{X}_1 &:= \{x_1 : A_1 x_1 = b_1,\ x_1 \geq 0\} && \\ \mathcal{X}_t(x_{t-1}, \xi_t) &:= \{x_t : B_1 x_{t-1} + A_t x_t = b_t,\ x_t \geq 0\} && t = 2, \dots, T \end{aligned}$$

[1]A stochastic variable that evolves in time is known as a stochastic process

where $\xi_1 := (c_1, A_1, b_1)$ is known and $\xi_t := (c_t, B_t, A_t, b_t) \in \mathbb{R}^{m_t}$, $t = 2, \ldots, T$ are data vectors or matrixes, whose elements are random.

The problem can be reformulated to (3.1.2), if the decision variables at time t are a function of the (data) process up to time t, $x_t = \boldsymbol{x}_t(\xi^{\lfloor t \rfloor})\,, t = 1, \ldots, T$

$$\begin{array}{ll} \min\limits_{x_1, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_T} & \mathbb{E}\left[f_1(x_1) + f_2(\boldsymbol{x}_t(\xi^{\lfloor 2 \rfloor}), \xi_2) + \cdots + f_T(\boldsymbol{x}_T(\xi^{\lfloor T \rfloor}), \xi_T)\right] \\ \text{Subject to} & x_1 \in \mathcal{X}_1, \quad \boldsymbol{x}_t(\xi^{\lfloor t \rfloor}) \in \mathcal{X}_t(\boldsymbol{x}_{t-1}(\xi^{\lfloor t-1 \rfloor}), \xi_t), \quad t = 2, \ldots, T. \end{array} \tag{3.1.2}$$

This formulation is often used in the stochastic programming literature. However unless the data process, $\xi_1, \ldots, \xi_T$, has a finite number of realizations, formulation (3.1.2) leads to an infinite dimensional optimization problem (Shapiro *et al.* , 2009).

In the case of discrete stages and a discrete states, one assumes that the probability distribution of ξ is a discrete and is concentrated at a finite number of points. In such a case, the stochastic process is conveniently represented as a scenario tree.

3.2 Scenario based recourse formulation

For a discrete time state a scenario based formulation is a standard solution technique. An alternative approach is to assume a continuous time stage, which leads to distribution-based solution techniques. In scenario based stochastic programming one assumes that the finite number of realizations of the stochastic process, $\xi_1, \ldots, \xi_T$, is concentrated on a finite number of points, denoted $\xi^1, \ldots, \xi,^S$.

This allows for a derivation of a deterministic equivalent formulation on the form

$$\begin{array}{llllll} \min\limits_{x_1, \ldots, x_T} & \sum\limits_{s=1}^{S} p^s \left[f_1(x_1^s) + f_2(x_2^s) + f_3(x_3^s) + \cdots + f_T(x_T^s)\right] & & & & \\ \text{s.t} & A_1 x_1 & & & & = b_1 \\ & B_2^s x_1 \;+\; A_2^s x_2^s & & & & = b_2 \\ & & B_3^s x_2 \;+\; A_3^s x_3^s & & & = b_3 \\ & & \ddots & \ddots & & \vdots \\ & & & B_T^s x_{T-1} \;+\; A_T^s x_T^s & & = b_T \\ & & l \le x_t^s \le u \quad t = 1, \ldots, T & & s = 1, \ldots, S & \end{array} \tag{3.2.1}$$

where p^s denotes the probability of scenario s. In problem (3.2.1) all parts of the decision vector are allowed to depend on all parts of the random data. However, as previously discussed, decision x_t made in stage t, should only be allowed to depend on the data known up to stage t, i.e. $\xi^{\lfloor t \rfloor}$. To correct for this, nonanticipativity constraints on the form

$$x_t^{\lfloor i \rfloor} = x_t^{\lfloor l \rfloor}, \quad \text{for all } i \text{ and } l \text{ where } \xi^{\lfloor i \rfloor, s} = \xi^{\lfloor l \rfloor, s}, \qquad t = 1, \ldots, T \tag{3.2.2}$$

are included. Problem (3.2.1) together with the nonanticipativity constraints (3.2.2) are equivalent to the original formulation (3.1.1).

The term scenario tree might seem somewhat mysterious at this point, but this will be described in detail in the next chapter.

3.3 The asset liability model

The asset liability management model is formulated as an linear multistage stochastic problem. We assume that at the beginning ($t = 0$) the asset allocation for the next years is already determined. That is the asset portfolio is not optimized at the beginning of the study. Most of the constraints are standard in asset liability management studies and the notation follows (Hilli *et al.* , 2007) for most parts.

Before the model is presented three sets must be defined. Let $\mathcal{J}$ denote a set of assets where and $\mathcal{T}$ denote the set of discrete stages when decisions are made. However sometimes it is more relevant to express the constraints in terms of asset classes rather than individual assets. To that end let, $\mathcal{Q}$ denote the set of asset classes considered. The model will be introduced in terms of constraints and a objective function.

3.3.1 Constraints

Inventory constraints

Inventory constraints describe holdings, purchases and sales of each asset over time. The inventory constraint at stage $t > 0$ are

$$h_{t,j} = h_{t-1,j}(1 + R_{t,j}) + p_{t,j} - s_{t,j}, \qquad j \in \mathcal{J}, \quad t > 0$$

where $R_{t,j}$ denotes (random) return on asset j in stage t and $h_{t,j}$, $p_{t,j}$, $s_{t,j}$ are decision variables, which describe the holdings, purchases and sales in asset j at stage t respectively. The initial holdings in each asset are denoted by h_j^0 and are assumed to be known.

Budget constraints

Budget constraints describe the cash inflow (revenues) and cash outflow (expenses) of the pension fund and ensure that the cash inflow never exceed the cash outflow. At stage $t = 0$ (precent) the budget constraints are

$$\sum_{j \in \mathcal{J}} (1 + c^q) p_{0,j} = \sum_{j \in \mathcal{J}} (1 - c^q) s_{0,j}$$

and at stage $t > 0$

$$\sum_{j \in \mathcal{J}} (1 + c^j) p_{t,j} + L_t = \sum_{j \in \mathcal{J}} (1 - c^j) s_{t,j} + I_t$$

where c^j denotes the transaction cost for purchase and sales of asset class j, L_t denotes the liability payments to pensioners and I_t denotes the premiums paid by the pension fund's members. The transaction costs are assumed to be equal for purchase and sales.

Portfolio constraints

Upper and lower limits of holdings in various asset classes for Icelandic pension funds are bound both by law and by the funds' investment policy. Those limits are expressed as a proportion of the total wealth of the pension fund

$$l^q \sum_{j \in \mathcal{J}} h_{t,j} \leq h_{t,q} \leq u^q \sum_{j \in \mathcal{J}} h_{t,j}, \qquad q \in \mathcal{Q}, \quad t \in \mathcal{T}$$

where the parameters l^q and u^q denote the upper and lower bounds for different asset classes, and are assumed to be constant. Similar constraints are placed on holdings in foreign investments. To prevent spurious asset switching, the maximum amount allowed for purchase, as a percentage of the wealth, is introduced in stage t

$$p_{t,j} \leq u^p \sum_{j=1}^{J} h_{t,j}, \qquad j \in \mathcal{J}, \quad t \in \mathcal{T}$$

where the constant u^p denotes the maximum holdings (percentage). The wealth at time t is given by

$$W_t = W_{t-1} + \sum_{j \in \mathcal{J}} (1 + R_{j,t}) h_{t,j-1}, \quad j \in \mathcal{J}, \quad t \in \mathcal{T}$$

where W_0 is the funds initial wealth.

Nonanticipativity constraints

Nonanticipativity constraints ensure that groups of scenarios[2] with identical values of the uncertain parameters in stage t must yield the same decision up to that period. Let $x_t^{\lfloor s \rfloor}$ denote a decision variable in scenario s from the precent ($t = 0$) up to stage t. The nonanticipativity constraints can be expressed mathematically as

$$h_{t,j}^{\lfloor s \rfloor} = h_{t,j}^{\lfloor s' \rfloor}, \qquad p_{t,j}^{\lfloor s \rfloor} = p_{t,j}^{\lfloor s' \rfloor}, \qquad s_{t,j}^{\lfloor s \rfloor} = s_{t,j}^{\lfloor s' \rfloor}$$

for scenarios s and s' inheriting an identical past up to stage t.

3.3.2 Objective function

The objective of the pension fund is to obtain a pre-specified target funding ratio at the end of the horizon. The funding ratio[3] is denoted by FR_t and given by

$$\mathrm{FR}_t = \frac{W_t - \tilde{I}_t}{\tilde{L}_t} - 1 \tag{3.3.1}$$

where $\tilde{L}_t$ and $\tilde{I}_t$ denote present value of future liability and premiums respectively. To model the pension fund risk aversion, two penalty functions are introduced. First the objective is penalized for a funding ratio which falls below a pre-specified minimum, denoted $\mathrm{FR}^{\mathrm{min}}$. Secondly the objective is penalized for a funding which does note reach the target, $\mathrm{FR}^{\mathrm{target}}$ at the horizon. Both penalty functions consists of a constant multiplied with the first downward moment of the funding ratio. The two penalty constraints are obtained by rewriting equation (3.3.1) and introducing two positive decision variables, Z^{target} and Z^{min}, i.e.

$$\begin{aligned} (W_T + \tilde{I}_T) &\geq (1 + \mathrm{FR}^{\mathrm{target}})\tilde{L}_T - Z_T^{\mathrm{target}}, \qquad & Z_T^{\mathrm{target}} &\geq 0 \\ (W_T + \tilde{I}_T) &\geq (1 + \mathrm{FR}^{\mathrm{min}})\tilde{L}_T - Z_T^{\mathrm{min}}, \qquad & Z_T^{\mathrm{min}} &\geq 0. \end{aligned}$$

If the funding ratio falls below the minimum or does not reach the target at the horizon, Z^{target} and Z^{min} become positive. Since the deterministic equivalent formulation is based on a number of scenarios the expected funding ratio is maximized and penalized of the expected downside moments (times a constant) of the funding ratio.

[2]Scenarios are defined in chapter [4].

[3]The term funding ratio is similar to the total actuarial position (in %). However, the term funding ratio is used in this thesis to emphasize that they differ in terms of valuation of future liabilities and premiums. This will be discussed in detail in chapter 5.

The the objective function becomes

$$\max \quad \mathbb{E}\left[\left(\frac{W_T+\tilde{I}_T}{\tilde{L}_T}-1\right)-\lambda_1\left(\frac{Z_T^{\text{target}}}{\tilde{L}_T}\right)-\lambda_2\sum_{t=1}^{T}\left(\frac{Z_t^{\min}}{\tilde{L}_t}\right)\right]$$

where λ_1 and λ_2 are the respective penalty constants. The two penalty components are strictly positive and are therefore subtracted from the expected funding ratio at the horizon. Similar penalty functions could be presented in terms of target wealth and or deviation from a specified target wealth.

An alternative way to model the risk aversion is to use chance constraints. Such constraints state that the funding ratio should at all times be higher than a pre-specified minimum value with *high probability*,

$$\mathrm{P}\left\{(W_t+\tilde{I}_t)\geq(1+\mathrm{FR}^{\min})\tilde{L}_t\right\}\geq\alpha.$$

Chance constraints have a more intuitive appeal than penalty constraints. Moreover the reliability constant, α, is easier to determine than the penalty constants λ_1 and λ_2. However change constraints are computationally much more inefficient and complex. When included in the model, one has two options: Firstly one can make distributional assumptions to reformulate the chance constraints. However, few distributions will lead to convex and well behaved constraints. In the case of the normal distribution, the variance-covariance matrix for the different assets has to be estimated, which a non-trivial task. The second option is to reformulate the chance constraints into a mixed-integer or integrated change constraints, which need special algorithms and are generally harder to solve. This approach is described in detail in (Dert, 1995; Drijver, 2005; Klein Haneveld *et al.* , 2010).

Chapter 4. Scenario tree generation

In this chapter important aspects of scenario tree generation are discussed and the scenario generation method used in this thesis introduced. The resulting scenario tree is used as input to the scenario based multistage recourse formulation of the stochastic programming model.

Before reviewing the scenario tree generation methods in the literature let's address two important concepts, a scenario *fan* and a scenario *tree* which are shown in figure (4.1).

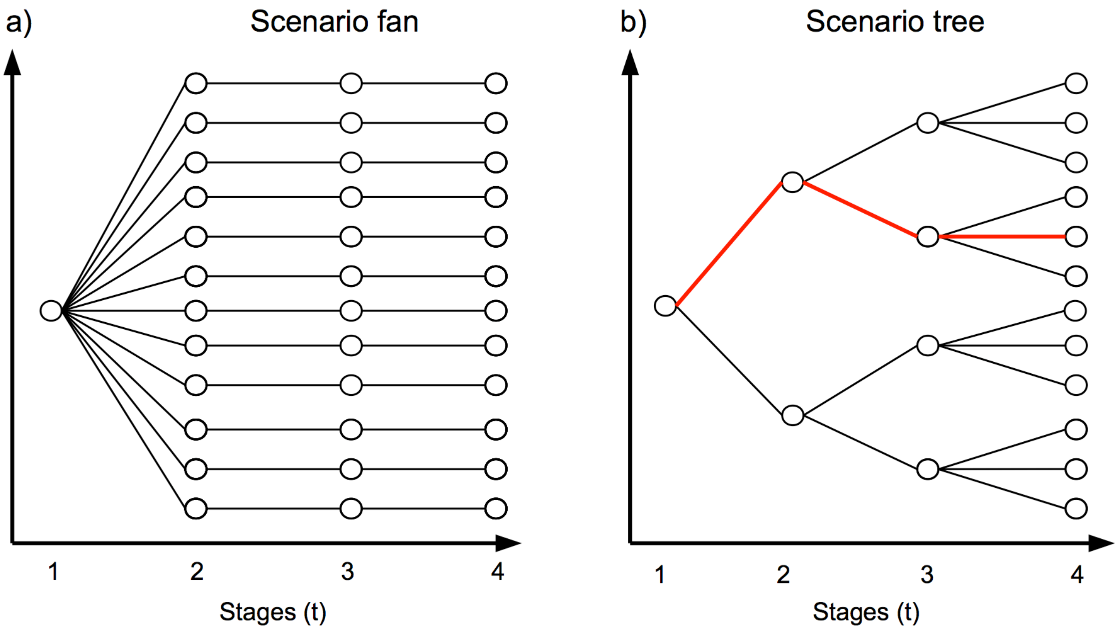

Figure 4.1. Schematic representation of a scenario tree and a scenario fan.

A scenario tree consists of clusters[1] (marked with circles) and branches linking the clusters together (marked as black/red lines). The rightmost cluster represents the present and is referred to as the root cluster, whereas the leftmost clusters represent the horizon and are referred to as the leaf clusters. A scenario is defined as a route from the root cluster to one of the leaf clusters. Each cluster has a one-to one

[1] Also known as nodes

relation to certain stage t and certain scenario s, denoted by (t, s). In a scenario fan, all clusters at stage t are connected to one cluster at stage $t-1$ and one cluster at stage $t+1$. In a scenario tree, b_{t-1} clusters at time t are connected to one cluster at time $t-1$ and branch of to b_{t+1} clusters at stage $t+1$. For example in figure 4.1, a pair of clusters at stage two are connected to one cluster at stage one and each cluster at stage two branches off to three clusters at stage three.

It is commonly understood that a scenario fan is only suitable in two-stage scenario based stochastic problems. A Two-stage problem might seem misleading since the scenario fan in figure 4.1 is represented by four stages. However the dependancies between clusters at stage two, three and four are the same. Therefore the scenario fan is argued to only adequately represent two stages, one and four in the figure (Sutiene & Pranevicius, 2007). In the case of a multistage scenario based on stochastic programming, a scenario tree is needed to represent the relation between clusters for all stages. The necessary notation for scenario trees is introduced in section 4.3.1.

The rest of this chapter is organized as follows. In section 4.1 the vast number of scenario generation methods in the literature is briefly introduced and discussed. In section 4.3 the scenario generator used in this thesis is explained in detail.

4.1 Literature survey

Scenario tree generation is not an easy task and a numerous methods have been suggested in the literature. The methods reviewed in this thesis can be classified into roughly four classes: sampling methods; statistical methods; methods based on probability or approximation theory and; other methods. In addition, a closely related subject of scenario reduction methods is briefly introduced.

4.1.1 Sampling methods

Various sampling methods can be used to construct the scenario tree. The simplest method is to sample randomly from historical data, known in the financial literature as bootstrapping. Although easy to implement, it is seldom used in practical application due to its limitations. The main limitation of bootstrapping is that it is unable to generate samples different from those that already have occurred in the past. Therefore using bootstrapping to generate scenarios assumes that historical data is representative of the future and only identical observations can be resampled.

This means, for example, that more extreme values than observed from historical data are not available.

To overcome the drawbacks of bootstrapping, (Kouwenberg, 2001) suggests random sampling and adjusted random sampling methods. The resulting scenario trees are used to solve an asset liability management model for a Dutch pension fund, where a vector autoregressive time series model (VAR) was used to estimate the stochastic parameters in the model. A random sampling method is based on sampling from the error distribution of the estimated VAR model. An adjusted random sampling method is based on antithetic sampling in order to fit every odd moment of the underlying distribution of the stochastic parameters by assuming an even number of clusters in the scenario tree. This method is based on one of the scenario generating methods considered in (Carino *et al.* , 1994) where the mean and the variance of the underlying distribution are fitted with an adjusted random sampling scheme. Backtesting, with rolling horizon simulation, revealed that random sampling can lead to extensive asset mix and spurious profits while the adjusted random sampling method led to less asset mixing and more stable solutions.

Dupačová *et al.* (2000) proposed a sequential sampling algorithm. This sampling method is an iterative algorithm which is part of a framework that uses a scenario tree nodal partition matrix, conditional sampling, stochastic programming generator and a solver. The scenario tree is modified through an update of the nodal partition matrix in each iteration and the stochastic programmin model solved. Another variation of a sampling algorithm discussed is based on the expected value of perfect information (EVPI). For further discussion see Dempster (1998); Dempster & Thompson (1999) and Heitsch & Römisch (2009) as well as their references.

Various Monte Carlo (MC) and Quasi Monte Carlo (QMC) sampling schemes have been considered in the literature. QMC methods have very good and well documented convergence properties and are therefore well suited for scenario generation. QMC schemes that epi-converge to the original problem are presented in (Pennanen, 2005, 2009). MC based methods are presented in Chiralaksanakul (2003); Chiralaksanakul & Morton (2004).

4.1.2 Statistical methods

Moment matching

Arguably the most commonly used scenario tree generation method was initially introduced by Kouwenberg & Vorst (1998). The idea is to estimate the first two moments of the underlying distribution in the scenario tree. This is obtained by solving small systems of the scenario tree recursively. The method was generalized by Høyland, K. and Wallace, S.W. (2001) and is known as the moment matching method. The moments and co-movements of the unknown distributions are matched with historical data or values pre-specified by the user. The matching is obtained by solving a non-linear optimization problem which penalizes deviation from the desired moments. The authors approximate the first four moments of the underlying distributions and the correlation between the assets. The resulting scenario tree is obtained by solving sequential non-linear optimization problems or by solving one large non-linear optimization problem. The computing time grows hand in hand with the size of the scenario tree and quickly becomes large when dealing with realistic problems. To overcome this problem, a somewhat successful attempt to decrease the solution times, by introducing a heuristic algorithm, was introduced in Høyland, K. and Kaut, M. and Wallace, S.W. (2003). Another suggestion, using parallel implementation, was proposed by Beraldi *et al.* (2010).

The main drawback to this method is that convergence to the optimal solution is not guaranteed. This means that the method may diverge, resulting in a spurious scenario tree. In addition, increasing the number of scenarios will not lead to a better match of the observed and desired moments. Another drawback, pointed out by Hochreiter & Pflug (2007), is that matching the first four moments of a distribution can result in many different distributions which have only the first four moments in common. For example, uniform and normal distribution may have the same four moments but have very different characteristics. Underlying distributions for financial data indicate unimodal features. Ji *et al.* (2005) point out that if only the first four moments are used to estimate the scenario tree, the resulting distribution may have a multimodal feature which is not representative of financial data. They argue therefore that descriptive features such as unimodality must be included in the moment matching procedure. This is achieved with partition of the distribution space of the stochastic parameters and imposing additional constraints in the moment matching optimization.

In financial applications, scenario trees should be free of arbitrage opportunities[2]. Klaassen (2002) demonstrates that the moment matching method does not strictly prevent arbitrage opportunities. To prevent arbitrage opportunities, two additional non-linear optimization problems must be solved at each node of the scenario tree to detect and prevent arbitrage. Those two non-linear optimization problems can also be added as constraints to the main moment matching model. Ji *et al.* (2005) modified

[2]Arbitrage free scenario tree is discussed further in subsection 4.3.4

the moment matching method to prevent arbitrage opportunities by solving a linear optimization during every iteration of the sequential version of the moment matching method. The additional constraints mentioned above will increase the already high computing times.

In addition to the random sampling and adjusted random sampling methods, discussed previously, Kouwenberg (2001) also construct a scenario tree with the moment matching method. The performance, of those three methods, on the solution of the asset liability management model for the Dutch pension fund are analyzed. Kouwenberg reports that the moment matching method slightly outperformed the adjusted random sampling technique. A fixed-mix strategy was used as a benchmark for comparison. On the positive side the solution of the asset liability management model that used the moment matching scenario generation method was the only one whish strictly outperformed the fixed-mix benchmark.

Clustering

Another method proposed by Dupačová *et al.* (2000) is a multi-level clustering method. In general clustering methods consist of two phases, the simulation phase and the clustering phase. In the simulation phase, many different paths are simulated from the underlying time series model, where each simulated path represents one possible evolution of the future. All the simulated paths are generally referred to as a scenario fan. In the second phase the scenario fan is molded into a scenario tree by bundling together similar scenario paths into clusters at different points in time. Usually the clustering method assumes that some time series models have been estimated beforehand. Simulated paths which are nearest to a cluster by some distance measure are assigned to each cluster. A similar approach is proposed by Gulpinar *et al.* (2004) where a randomized clustering algorithm is used in the clustering phase. Two methods are proposed for the simulation phase, parallel simulation and sequential simulation. In the parallel simulation, all paths are simulated before clustering begins whereas in the sequential simulation, the simulation phase and clustering phase are performed alternately until the whole scenario tree is completed. In more detail, a large number of paths are simulated and then bundled into so called first stage clusters. Then the simulation is initialized at each first stage cluster and new scenario fans are simulated which are then bundled together into second stage clusters. This is repeated until the scenario tree is completed. The difference between the two methods is that parallel simulation requires storage of large number of simulated paths whereas in sequential simulation only scenarios from the relevant clusters are stored at each time. Furthermore the parallel method will produce a wider clustering structure as opposed to a more homogenous one when sequential simulation is used. A wider clustering scheme corresponds to more extreme values of each stochastic variable and is desirable in risk management such as in asset liability management applications. Gulpinar *et al.* (2004) also uses a moment matching scenario tree generation method employing both sequential and overall optimization where the first four conditional moments and correlation are matched with historical data or pre-specified values. The overall optimization is argued to be superior to the sequential method because there is a risk that the sequential method might produce a sub-optimal scenario trees. The final method considered in their paper is a hybrid approach which combines the simulation and optimization approaches. The random variables are obtained with simulation and clustering methods. They are then substituted for decision variables in a optimization problem. Overall or sequential optimization can either be used in the hybrid approach. All six variations of the three methods were used to solve a mean-variance optimization model. The computing times required to generate the same scenario tree were compared. The overall optimization in the hybrid and moment matching methods required much more computing time compared to the simulation and clustering method. However backtesting indicated that no perceptible gains are obtained using the overall optimization variant of the moment matching method, over the much faster heuristics of sequential optimization, simulation and hybrid simulation/optimization methods.

Sutienė, K. Makackas, D and Pranevivcius, H. (2010) introduce a similar clustering method based on K-means clustering algorithm is a which well documented algorithm to perform clustering. The K-means clustering algorithm is modified to cope with multi dimensional scenario paths. The method is based on earlier research by (Pranevicius & Šutiene, 2007). In-sample stability tests for a small asset liability management problem, proposed by (Kaut & Wallace, 2003)[3], are performed with promising results.

4.1.3 Probabilistic methods

Probabilistic methods approach scenario generation from an approximation point of view. The difference between the true optimal value of the underlying problem and the solution obtained by solving the stochastic optimization, using a scenario tree should be as small as possible. This difference is known as the the approximation er-

[3]Stability is discussed in subsection 4.2.1

ror. Pflug (2001) suggests that the scenario tree should be constructed by minimizing the approximation error. The approximation error is impossible to calculate directly for realistic problems because it requires the true optimal value. However an upper bound for the approximation error is obtained by solving non-linear optimization problem. Hochreiter & Pflug (2007) apply the method to a simple stochastic asset allocation optimization problem. As a robust check convergence of the objective value is shown. However, as in the moment matching method no formal proof or convergence guarantee is available. This is a major drawback since the method may diverge.

Pennanen & Koivu (2002) introduce a discretization method that weakly converges to the original stochastic process as the number of branches in the scenario tree increases. The method is based on techniques of numerical integration and uses a low-discrepancy sequence in sample tree construction to reduce the approximation error. This method is used in asset liability management for a Finnish pension fund in Hilli *et al.* (2007).

4.1.4 Other methods

Recently, Mitra & Ji (2010) proposed a new scenario generating technique based on Gaussian Mixture Hidden Markov Model. In each stage of the hidden Markov model, observations are represented by a Gaussian mixture probability density function which is composed of a weighted sum of univariate Gaussians. For financial data it is natural to classify the state of the economy into three phases, growth, recession and a transitional state. These phases are used in the hidden markov model and transaction probabilities are assigned to each phase. This method captures important reversion properties like autoregression, conditional heteroscedasticity, cyclic behaviour and jumps. Numerical experiments are presented and although that this method seems promising no proper research on stochastic programming problems has been conducted. The impact on the solution of the stochastic programming model and stability has not been carried out.

In recent years copulas have been popular in the financial literature. Scenario generation using coupulas are presented in Sutiene & Pranevicius (2007) and Kaut & Wallace (2011). However those methods will not be discussed in this thesis.

4.1.5 Scenario reduction

The above mentioned scenario generation methods often lead to extremely large scenario trees. Scenario reduction techniques aim at reducing the size of the existing scenario tree while maintaining the important properties of the tree. The methods use either forward or backward reduction. In forward, reduction one starts by reducing scenarios, from the root cluster of the tree and works down to the leaf clusters. In backwards reduction, one starts from the leaf clusters and works back to the root cluster. Dupačová *et al.* (2003) and Heitsch & Römisch (2003) approache the subject form the theoretical point of view, using probability measures. A method described in Growe-Kuska *et al.* (2003) is based on reducing the scenario fan by bundling together similar scenarios into a scenario tree. Algorithms for forward and backward scenario tree reduction are presented and the method used to construct a scenario tree for a power management application.

4.2 Minimal requirements

4.2.1 Stability and bias

Surprisingly little attention had been paid to the practical evaluation of the scenario generation methods until 2003. What are the minimal requirements for reliable scenario tree? Kaut & Wallace (2003) addressed this issue from a practical point of view. They argue that there are at least two minimal requirements that have to be considered before the scenario tree is used in practical applications, namely bias and stability. They redirected the discussion to whether the scenario tree will lead to a good decision. Two stability concepts are presented, in-sample stability and out-of-sample stability. In-sample stability implies that the solution of the stochastic model should not depend too much on the scenario tree itself. If several different scenario trees are generated with the same method and the stochastic optimization is solved, then the value of the objective function should not change dramatically. In the absence of in-sample stability, the scenario generation method needs improvements. Out-of-sample stability is much harder to examine, because one has to be able to evaluate the true objective function. One method to evaluate out-of-sample stability is to perform backtesting with simulation methods such as Monte Carlo. However, the authors argue that in most practical applications one will have either both stabilities or none. In-sample stability tests should therefore be sufficient to detect instability, and out-of-sample tests should be considered whenever possible. The second requirement, bias, is the difference between the true solution of the optimization problem and the one obtained by solving the stochastic problem with a scenario tree. Kaut & Wallace (2003) suggest that one should construct a huge reference scenario tree and then solve the stochastic optimization model. Then one can compare the solution obtained when the reference tree is used to the solution given by the original scenario tree. If both solutions are similar one can be more confident that the scenario tree constructed by the scenario generation method is free of bias. In their paper, stability and bias tests are performed for a simple one-period asset allocation optimization problem. The scenario tree method used was the moment matching method. The results show that the method is both stable and unbiased.

As opposed to practical evaluation methods, several papers address the issue of stability from theoretical grounds. The reader is referred to the following papers and the references within, Heitsch *et al.* (2007) and Heitsch & Römisch (2009).

4.2.2 Arbitrage free scenario trees

For financial applications such as asset liability management, it is essential that the scenario tree is free of arbitrage. Arbitrage is defined as a financial strategy that has zero payout and strictly positive probability of profit. In the absence of arbitrage free scenario tree the optimization routine will most certainly take advantage of the arbitrage opportunities. This will lead to spurious profits and overconfident solutions. Klaassen (2002) addresses the issue of arbitrage-free scenario trees and proposes that arbitrage tests should be incorporated into the scenario generation process. Klaassen introduces two non-linear optimization problems which can be solved to detect if a scenario tree is free of arbitrage. The optimization problems are solved at each discrete time after clusters have been formed.

4.3 The scenario generator

The method proposed in this thesis is a extended version of the modified multi stage K-means clustering method presented by (Sutienė, K. Makackas, D and Pranevivcius, H., 2010, hereafter: SMP). The reasons this method is preferred over the wide range of methods available are as follows.

1. In-sample stability results for a small asset liability management model presented in their paper are promising.
2. The method is able to cope with multidimensional variables and is based on the K-means algorithms, which is well studied.
3. The computation time needed to construct a realistic scenario tree seems fairly moderate and much smaller for than e.g. moment matching scenario generation method.

However this method, has some drawbacks, e.g. no arbitrage test where presented in SMP. In an attempt to avoid confusion the original K-means algorithm will be referred to as the K-means, also the modified version introduced by SMP will be referred to as the modified K-means method. In addition the method used in this thesis will be referred to as a extended K-means method since it features improvements on the modified K-means method.

4.3.1 Notations for scenario trees

The (time series) models, introduced in chapter 5, are used to simulate a finite number of paths, which form a scenario fan. The extended K-means clustering method is used to convert the scenario fan into a scenario tree[4]. At this point lets assume that a time series model readily available to construct a scenario fan.

To distinguish between the simulated paths and the resulting scenario tree further notation has to be introduced. Let N denote the number of paths simulated for each stochastic variable. All the stochastic variables considered will have the same number of simulated paths and a further distinction between the variables is unnecessary. Let $\mathcal{S}$ denote the set of scenarios in the scenario tree. Recall that one scenario is a route from the root cluster of the scenario tree to one of the leaf clusters. Here $\mathcal{J}$ is used to denote the set of stochastic variables considered, such as the price of a stock or interest rates. Simulated paths related to stochastic variable j are denoted by $\tilde{\boldsymbol{\xi}}^j = \{\tilde{\xi}_1^j, \ldots, \tilde{\xi}_n^j, \ldots \tilde{\xi}_N^j\}$ and the value of the n-th path at stage t of stochastic variable j is denoted by $\tilde{\xi}_n^j(t)$.

In addition to the stages used, a scenario tree is defined by a branching structure, $\mathbb{B} = \{b_1, \ldots, b_t, \ldots, b_T\}$. The branching structure defines how many branches lead from each cluster in each stage. Let $\hat{C}^{(t,s)}$ denote a cluster present in stage t and scenario s in the scenario tree. Each cluster corresponds to a J-dimensional vector of

[4]Recall that scenario fans are only suitable two two stage scenario based stochastic programming. The multistage scenario based recourse stochastic programming model thus requires a scenario tree.

the values of all stochastic variables. The number of clusters at stage t is determined by the branching structure.

Figure 4.2 gives a schematic representation of the process of generating a scenario tree from a scenario fan. The scenario fan consists of nine simulated paths, $\tilde{\boldsymbol{\xi}}^j = \{\tilde{\xi}_1^j, \ldots, \tilde{\xi}_9^j\}$, for stochastic variable j, shown in panel (a). The resulting scenario tree, shown in panel (b), has three stages $\mathcal{T} = \{1, 2, 3\}$ and branching structure, $\mathbb{B} = \{1, 3, 2\}$. This branching structure results in six scenarios, $\mathcal{S} = \{1, \ldots, 6\}$.

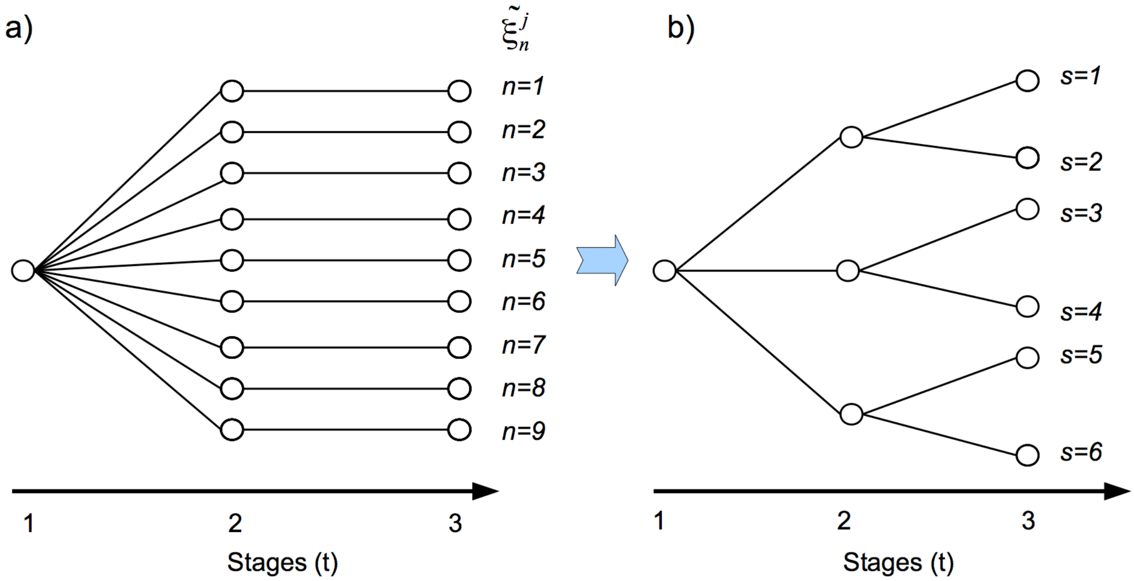

Figure 4.2. Schematic representation of a three stage scenario fan, composed of nine simulated paths, and a resulting scenario tree with branching structure $\mathbb{B} = \{1, 3, 2\}$.

In the next three sections the foundations for the scenario generation method used in this thesis is presented.

4.3.2 K-means

The K-means clustering algorithm is a method for partitioning observations into a pre-specified number of clusters, in which each observation belongs to the cluster with the nearest mean. Here the observations are the values of simulated stochastic variables at certain stages.

Let $C_o^{(t,s)}$ denote the center of cluster $C^{(t,s)}$ (its position) and let $\mathbf{C}_o$ denote a vector of all the clusters centers, $\mathbf{C}_o = (C_o^{(t,s)}, C_o^{(t,s)}, \ldots, C_o^{(t,s)})$. The K-means algorithm attempts to solve the following global optimization problem

$$\arg\min_{\mathbf{C}_0} \sum_{k=1}^{\bar{k}_t} \sum_{\tilde{\boldsymbol{\xi}}^j(t)\in C^{(t,s)}} ||\hat{\boldsymbol{\xi}}^j(t) - \boldsymbol{C}_o||_2^2, \tag{4.3.1}$$

where $||\cdot||_2^2$ denotes the squared Euclidean distance, defined as

$$||\tilde{\boldsymbol{\xi}}^j(t) - \boldsymbol{C}_o||_2^2 = \sum_{n=1}^{N}(\tilde{\xi}_n^j(t) - C_o^{(t,s)})^2.$$

The K-means algorithms is a two step procedure, an assignment step and an update step. In the beginning, clusters centers are randomly chosen and then the two steps are performed alternately in an iterative manner. In the assignment step, each observation is assigned to its nearest cluster center,

$$C_o^{(t,s)} = \left\{\hat{\xi}^j(t) : ||\hat{\xi}^j(t) - C_o^{(t,s)}||^2 \leq ||\hat{\xi}^j(t) - C_o^{(t,s')}||^2 \quad \text{For all } (t,s') = 1,\ldots,s'\right\} \tag{4.3.2}$$

Then each cluster center is evaluated as the mean of all observations assigned to it,

$$C_o^{(t,s)} = \frac{1}{\left|C_o^{(t,s)}\right|} \sum_{\hat{\xi}^j(t)\in\mathbf{C}_o^{(t,s)}} \hat{\xi}^j(t), \quad \text{for all clusters.} \tag{4.3.3}$$

A global minimum is reached when no observations are assigned to new clusters in an iteration. As a safety stopping criteria only finite number of iterations are allowed. Pseudo code (1) summarizes the K-means algorithm.

Pseudo Code 1 K-means

Set $itr_{\max}$
Assign initial cluster centers randomly
while tol>$tol_{\min}$ and $itr < itr_{\max}$ **do**
  Assignment step: Assign a observation to its nearest cluster according to (4.3.2)
  Update step: Update each cluster center according to (4.3.3)
  $itr = itr + 1$
end while
Store the clusters centers $\mathbf{C}_0$ (the solution)

Here $tol_{\min}$ denotes the allowed number of observations to be assigned to new clusters at each iteration and $itr_{\max}$ denotes the maximum numbers of iterations allowed. Convergence to a global minimum is not guarantied. Therefore the algorithm is sometimes started a few times and the best solution is chosen. The solution consists of the vector $\mathbf{C}_o$ along with a vector of cluster indices.

4.3.3 Modified K-mean clustering

Multiple scenario paths are simulated and form a scenario fan. The scenario fan is then converted into a scenario tree where similar simulated paths are bundled into clusters at each decision time. As a result multiple clustering tasks have to be performed to generate the tree. Secondly, probabilities have to be assigned to each cluster to indicate the conditional probability of arriving at cluster a cluster at stage t from a cluster at stage $t-1$ in the same sceneario.

During the first stage all simulated paths for a stochastic variables are assigned to a cluster according to the branching structure. This is called first stage clustering. After all paths have been assigned to a cluster, the second stage clustering can begin. All the paths which were assigned to the same cluster at the first stage are collected and bundled, independently of the other paths who were assigned to other clusters in the first stage, into clusters at the second stage. This forms the first task of the second stage clustering. In the second clustering task in the second stage, paths who were assigned to the same cluster in the first stage are collected and bundled into clusters. This continues until all clusters in the second stage have been formed. Thus, in the second stage the number of clustering tasks is equal to the number of clusters at the first stage. Illustrative example on the evolution of the modified K-means method can be found in Appendix I.

This requires the knowledge of which paths where assigned to which cluster at each stage before the clustering can begin. As a result, a numbering scheme that keeps track of which paths where assigned to which cluster in each stage is required. SMP introduce one such numbering system. However, the numbering system is not very intuitive and therefore a new numbering system will be introduced in this thesis.

Probability Assignment

Another modificaiton that has to be made to the K-means method is the assignment of probabilities to each cluster. At stage t a finite number of paths have been assigned to each cluster. The conditional probability of arriving at a certain cluster at stage t from a cluster at stage $t-1$ of the same scenario is simply the number of paths assigned to the cluster at stage t divided by all the paths present in the cluster at stage $t-1$. In other words, the conditional probabilities depend on the number of paths assigned to a cluster as a fraction of all paths which belongs to the same cluster at stage t. This ensures that the conditional probabilities sum to one in each clustering task. The probability of a scenario s is the product of all conditional probabilities related to that scenario in all stages.

Other distance measures

Instead of the squared Euclidean distance used in equation (4.3.1) alternative distant measures might be used. For example the one-norm, two-norm or the maximum norm. However the Euclidean distance is well suited since it affects the center positions of each clusters are strongly affected by extreme values. This will result in a wider scenario tree (with same branching structure $\mathbb{B}$) than if other distance measures are used.

4.3.4 Extended K-means clustering

As discussed earlier the modified K-means scenario generation method has a few drawbacks. This extended version attempts to overcome the most important ones.

The cluster matrix

As previously stated, numbering system has to be introduced to keep track of which simulated paths are assigned to which clusters at at all decision times. The numbering system proposed in this thesis is similar to the one presented in (Dupačová *et al.* , 2000). To keep track of all cluster in each stage and each scenario s, a cluster matrix M is used, where each row corresponds to one path and each column represents each stage. The cluster matrix is a $N \times (T+1)$ matrix and each cluster is assigned a index, which is used to label every that it inherits. When each sub clustering takes place one needs to extract the values of each path that has the same number (same parent cluster) in the previous column of the cluster matrix before sub clustering can begin.

Detecting arbitrage opportunities in the scenario tree

It is vital that the resulting scenario trees are free of arbitrage, but the modified K-means method does not directly prevent arbitrage opportunities. As a result arbitrage tests were implemented as a part of the extended K-means method. Before the arbitrage tests are present some notation presented in chapter 3 will be reviewed. Let W_t denote the pension funds wealth at stage t. Let $h_{t,j}$ denote the proportion of the wealth allocated to asset j at stage t. Let $R^s_{t,j}$ denote the return on asset j which is reveled between stages $t-1$ and t in scenario s. Here the index s is added tho emphasize the different returns are present in different scenarios.

The definition and formulation of arbitrage follow (Klaassen, 2002), which distinguish between two types of arbitrage. Arbitrage of the first type is defined as follows. If there exists a zero investment portfolio that has non-negative payoffs in all scenarios and strictly positive payoffs in at least one state then arbitrage exists. Arbitrage of the first type in stage t can be expressed as

$$\begin{aligned}
&\text{a)} \quad \sum_{j\in\mathcal{J}} h_{t,j} && = 0 \\
&\text{b)} \quad \sum_{j\in\mathcal{J}} h_{t,j}R^s_{t,j} && \geq 0 \quad \text{for all } s \in \mathcal{S} \\
&\text{c)} \quad \sum_{j\in\mathcal{J}} h_{t,j}R^s_{t,j} && > 0 \quad \text{for at least one } s \in \mathcal{S}
\end{aligned}$$

Where a) represent the zero investment, b) represents non-negative payoff at all states of the world and c) represents strictly positive payoff in at least on state of the world.

Arbitrage of the second type occurs if there exists an asset allocation $h_{t,1}, \ldots, h_{t,J}$ that has non-negative payoffs in all scenarios at time t while providing an immediate positive cash flow to the investor. Arbitrage of the second type can be expressed as

$$\begin{aligned}
&\text{d)} \quad \sum_{j\in\mathcal{J}} h_{t,j} && < 0 \\
&\text{e)} \quad \sum_{j\in\mathcal{J}} h_{t,j}(1 + R^s_{t,j}) && \geq 0 \quad \text{for all } s \in \mathcal{S}
\end{aligned}$$

Where d) and e) represent the immediate positive cash flow and non-negative payoff in all states of the world respectively.

The two types of arbitrage have to be tested separately. We will follow (Klaassen, 2002), and construct two linear optimization models that have to be solved at every stage t, after all clusters have been formed. To detect arbitrage of the first type, the following linear optimization is solved

$$\begin{aligned}
\max_{h_{t,1},\ldots,h_{t,J}} \quad & \sum_{s\in\mathcal{S}} \sum_{j\in\mathcal{J}} h_{t,j}R^s_{t,j} \\
\text{subject to:} \quad & \sum_{j\in\mathcal{J}} h_{t,j} = 0 \\
& \sum_{j\in\mathcal{J}} h_{t,j}R^s_{t,j} \geq 0 \quad \text{for all } s \in \mathcal{S}.
\end{aligned} \tag{4.3.4}$$

If the solution yields a positive objective function then there exist arbitrage of the first type. To detect arbitrage of the second type the following linear optimization model is solved,

$$
\begin{aligned}
&\min_{h_{t,1},\ldots,h_{t,J}} && \sum_{j \in \mathcal{J}} h_{t,j} \\
&\text{subject to:} \quad \text{e)} && \sum_{j \in \mathcal{J}} h_{t,j}(1 + R_{t,j}^{s}) \geq 0 \quad \text{for all}\, s \in \mathcal{S}
\end{aligned}
\tag{4.3.5}
$$

If the objective value at the optimum is negative, then there exists arbitrage of the second type. If either arbitrage of the first or the second type is detected in the scenario tree then the extended K-mean method is restarted. Since both (4.3.4) and (4.3.5) are linear optimization problems they can be solved efficiently. Here the `CVX` package for Matlab introduced by Grant & Boyd (2008, 2011) was used.

4.4 Summary

This chapter is devoted on scenario generation. Numerous methods in the literature were reviewed and discussed. After introducing the necessary notation for scenario generation, the statistical K-means algorithm was discussed along with the modified K-means method method introduced by SMP. Lastly, the clustering method used in this thesis was introduced as an extended version of the modified K-means clustering, which includes arbitrage tests and a cluster matrix to keep track of the numbering system. The following pseudo code summarizes the extended K-means clustering method.

Pseudo Code 2 Extended K-means clustering

Set $\mathbb{B}, \mathcal{T}$ and $itr_{\max}$
Initialize the cluster matrix, M
for every stage $t \in \mathcal{T}$ **do**
    for every cluster **do**
        Get each observation from its parent cluster (ready for clustering)
        Clustering: Call K-means (pseudo code 1)
        Update cluster matrix, M
        Calculate probabilities $\pi(C^k)$ and p^s
    end for
    Store cluster centers and probability estimates
    Test for arbitrage of the first type by solving optimization (4.3.4)
    Test for arbitrage of the second type by solving optimization (4.3.5)
end for

Chapter 5. Modeling stochastic variables

In this chapter models for the underlying stochastic variables are introduced. The chapter is broken down into three sections according to the nature of the variables. In section 5.1 models for stochastic variables related to the state of the economy in scenario s at stage t are introduced. The state of the economy is assumed to be determined by the nominal and real yield curves, inflation and exchange returns.

In section 5.2 valuation of future liabilities and future premiums is discussed and in section 5.3, methods and models to simulate asset returns are introduced.

5.1 Economic variables

5.1.1 Inflation

Due to the small size of the Icelandic economy, inflation is assumed to be determined by the outside world, that is inflation is assumed to be exogen.

The model proposed in this thesis is a random walk with stochastic trend used to model the logarithm of the Consumer Price Index (CPI). The model is presented on state space form and is used to simulate CPI paths, which are clustered according to the extended K-means method. After clustering, yearly inflation is calculated in each clusters according the dependencies between the cluster, which is defined by the branching structure, $\mathbb{B}$.

Linear Gaussian state space models

State-space models are a popular and widely used methodology in time series analysis. Linear Gaussian state space models can be expressed in their general form as

$$
\begin{array}{lll}
(1.1) & y_t = Z_t\alpha_t + \varepsilon_t, & \varepsilon_t \sim N(0, H_t) \\
(1.2) & \alpha_{t+1} = c_t + T_t\alpha_t + R_t\eta_t, & \eta_t \sim N(0, Q_t) \\
 & \alpha_1 \sim N(a_1, P_1), & t = 1, \ldots, n
\end{array}
$$

where y_t is the observed time series, ε_t is the measurement error, α_t is the unobserved state sequence, η_t is the state disturbance, and the matrixes Z_t, T_t, R_t define the model. Equations (1.1) and (1.2) are referred to as observation equation and state equation respectively. The state equation governs the time evolution of the state variable while the observation equation provides the link between the data y_t and the state α_t, (Tsay, 2005). The state, α_t, is unobservable and the aim is to infer properties of the state given a certain model and a time series y_t, $t = 1 \ldots T$.

In its state-space form the logarithm of the consumer price index at stage t is assumed to be given by

$$
\begin{aligned}
&\log(\mathrm{CPI}_t) = \mu_t \\
&\mu_t = \mu_{t-1} + \beta_{t-1} + \upsilon, \quad \upsilon \sim N(0, \sigma_\upsilon^2) \\
&\beta_t = \beta_{t-1} + \xi, \qquad\qquad \xi \sim N(0, \sigma_\xi^2)
\end{aligned}
\tag{5.1.1}
$$

Here μ_t and β_t denote the state level and state trend respectively, where the trend in consumer price index (β_t) followes pure random walk.

Estimation

Gaussian state-space models are estimated with maximum likelihood procedure where Kalman filter provides an efficient way to evaluate the likelihood function. The Kalman filter is an algorithm used to update the knowledge of the state variable recursively when new data points become available. That is, the Kalman filter recovers the state variable (here μ_t and β_t) given the information available at time t, $y^{\lfloor t \rfloor}$. Derivation of the Kalman filter is outside the scope of this thesis and to keep the discussion focused, the interested reader is refered to any advanced textbooks on time series analysis or state space analysis, e.g. (Tsay, 2005).

The data used to estimate (5.1.1) consists of monthly values of the Icelandic Consumer Price Index from May 1988 to November 2011 (283 data points)[1]. The estimated parameters are displayed in table 5.1.

[1]Statistic Iceland, `http://www.statice.is/`

Table 5.1. Estimated parameters for (5.1.1). Note that here the present state variables are indexed with a zero rather than T which might be misleading since they are estimated using the entire data series. This is however more convenient notation since the zero represents the beginning of the asset liability management study.

$\hat{\mu}_0$	$\hat{\beta}_0,$	$\hat{\sigma}_\upsilon^2$	$\hat{\sigma}_\xi^2$
5.952	$1.522 \cdot 10^{-3}$	$3.940 \cdot 10^{-2}$	$1.355 \cdot 10^{-4}$

The model is used to simulate inflation scenarios. To incorporate the modelers beliefs about future inflation (optional) restrictions on maximum and minimum allowed inflation are introduced on the form

$$\pi^{\min} \leq \pi^{(t,s)} \leq \pi^{\max}$$

where $\pi^{(t,s)}$ denotes simulated inflation at stage t in scenario s. For example the modeler might think that inflation could not exceed $\pi^{\max}$. Similar restrictions can be imposed directly to the simulated consumer price index scenarios, although such restrictions are harder to estimate. The restrictions are assumed to be constant for all stages. However, stage dependent restrictions could easily be incorporated. For example restrictions on maximum allowed inflation in at stage one, $\pi^{\max,1}$ could be imposed while inflation at other stages is unrestricted.

5.1.2 Exchange rate returns

It is assumed that all foreign assets of the pension fund are assumed to be listed in the US dollar[2] to avoid modeling multiple currencies. Return of a denominated asset due to change in the value of the dollar, at stage t in scenario s, is

$$r_{\text{usd}}^{(t,s)} = \pi^{(t,s)} - \bar{\pi} + N\left(0, \sigma_{\text{usd}}^2\right) \tag{5.1.2}$$

where $\bar{\pi}$ denotes mean inflation and $\pi^{(t,s)}$ denotes the inflation at stage t in scenario s. This simple model assumes that when inflation is above average the exchange return is expected to be higher.

The average inflation is estimated using the same data as model (5.1.1) and is $\hat{\bar{\pi}} = 5.284\%$. The variance is harder to estimate and is set equal to $\hat{\sigma}_{\text{usd}} = 12\%$.

[2]All proxies are thus listed in the US dollars.

5.1.3 Yield curves

Yield curve describes the relation between the cost of borrowing and time to maturity, also known as term structure of interest rates. Market yield curves exhibit various shapes, upward or downward sloping and humps. Several hypotheses have been introduced in the economic and financial literature as to what determines the shapes of the curves. Three most common term structure theories are the pure expectations theory, the market segmentation theory and liquidity preference theory. In short (Hull, 2009) explains the tree theories as follows: The pure expectations theory argues that the forward interest rate corresponding to certain future period is equal to the expected future zero interest rate for that period. The market segmentation theory stats that there is no relationship between short-, medium-, and long-term interest rates. All rates are determined by supply and demand on its market, for example short-term interest rates are determined by the short-term bond market. According to this theory, major investors such as pension funds invest in bonds of a certain bond marked and do not readily switch from one maturity to another. The liquidity theory argues that forward rates should always be higher than expected future zero rates. This implies that investors should get a premium for holding long-term bonds.

In early asset liability management studies, a vector autoregressive model (VAR) was a popular approach for modeling yields and other economic variables, for example in (Kouwenberg & Vorst, 1998), (Gondzio & Kouwenberg, 2001) and (Koivu *et al.* , 2005). Using VAR model, bond yields are simulated by randomly drawing from the error distribution. Drijver (2005) pointed out that when scenarios are simulated using a VAR model, scenarios with unusually low or even negative yields might occur. Negative yields imply that investors are willing to purchase fixed income securities at a higher price than the redemption and coupon payments they will receive. To overcome this problem Drijver estimated the whole yield curve with a parametric model using nonlinear optimization. To model the dynamics of the curve and to simulate scenarios, parallel shifts in the yield curve were allowed and fixed income securities are then priced using the yield curve in each scenario. Other similar parametric approaches to simulate yield curve scenarios are suggested in the asset liability management literature. Rasmussen & Poulsen (2007) and Einarsson (2007) use the well known Nelson-Siegel model to estimate the yield curve. A first order VAR model was estimated for the level, slope and curvature parameters of the Nelson-Siegel model and then scenarios were simulated. In each cluster, different values for the parameters are present which define the whole yield curve.

Parametric approaches are, however, not suitable for modeling the Icelandic yield curves because few bonds are listed in the Icelandic bond market[3]. Therefore a alternative yield curve model is introduced in this thesis.

The Real yield curve

The method proposed in this thesis to model the real yield curve can be broken down into two main steps. In the first step, one year real yield, which determine the level of the real yield curve, is modeled and a large number of paths are simulated. The simulated paths are bundled into a scenario tree using the extended K-mean method, after which one year real yields are present in each cluster of the scenario tree. In the second step, the spot yields are used to simulate the whole real yield curve, one for each cluster.

Step one: The level

The one year real yield, hereafter referred to as the spot yield and denoted by spot_t, is modeled by the well known Cox–Ingersoll– Ross (CIR) short term interest rate model, introduced by (Cox *et al.* , 1985). The model is able to capture some fundamental features of short term yields such as mean reversion. The spot yield is assumed to be a solution of the following stochastic differential equation (SDE)

$$\mathrm{d}(\text{spot}_t) = \kappa(\gamma - \text{spot}_t)\mathrm{d}t + \varsigma\sqrt{\text{spot}_t}\,\mathrm{d}B_t \qquad (5.1.3)$$

where κ, γ and ς is the speed, level and volatility respectively and B_t denotes a univariate Wiener process[4] which captures the market risk dynamics. The drift part $\kappa(\gamma - \text{spot}_t)$ ensures mean reversion of the spot yield towards a long run average γ, with speed of adjustment governed by the parameters κ. The stochastic part $\sigma\sqrt{\text{spot}_t}$ avoids negative yields. That is, when the spot yields are low, the stochastic part becomes low and the drift part dominates, which drives the spot yield upwards.

The spot yield is not directly observable and has to be estimated by the observed market real yield curve. The level (γ) and volatility (ς) parameters are estimated with data provided by Arion Bank. The rate of reversion parameter (κ) is however estimated based on daily three month REIBOR rates. The parameter estimates are shown in table 5.2 below.

[3] See for example, `www.bonds.is`

[4] Also known as brownian motion. As before to keep the discussion focused the Wiener process is not further discussed. The unfamiliar reader is referred to any textbook on stochastic calculus, e.g. Cvitanic, J. and Zapatero, F. (2004)

Table 5.2. Estimates for CIR spot yield model (5.1.3), based on data provided by Arion Bank and three month REIBOR rates.

$\hat{\kappa}$	$\hat{\gamma}$	$\hat{\varsigma}$
0.9500	0.0161	0.0684

As before (optional) restrictions are introduced as

$$\text{spot}^{\min} \leq \text{spot}_t \leq \text{spot}^{\max}$$

on the simulated spot yields to incorporate the modelers beliefs on the future development in sport yields.

Step two: The curve

As discussed before, the CIR spot yield model is used to simulate spot rate realizations which are then bundled into a scenario tree using the extended K-means method. In the previous discussion, the spot rates were denoted as spot_t to emphasize the spot yields dynamics in time. However, after the generation of a scenario tree the spot rates are hereafter denoted as $\text{spot}^{(t,s)}$. This is done to distinguish between simulated spot yield paths and the resulting spot yields present in the scenario tree after clustering.

Given the level of the real yield curve in all clusters the whole real yield curve is simulated. Let $y_r^{(t,s)}(m)$ denote the real yield curve in cluster $C^{(t,s)}$ for maturity m. The derivation of the real yield curve includes two sub steps. Firstly, real yields for maturities $\{5, 10, 15, 20, 25\}$ are simulated and secondly the curve is estimated with spline interpolation.

Let's start by explaining the simulation of five year real yields. The five year yield is broken into two parts, the real spot yield and a five-to-one year real yield spread,

$$y_r^{(t,s)}(5) = \text{spot}_t^{(t,s)} + g_{5,1}$$

where $g_{5,1}$ denotes the real yield spread. It is assumed that the real yield curve is upwards sloping in each cluster. Therefore the spread is assumed to follow a gamma distribution with scale and shape parameters a and b respectively. This ensures that the resulting curve is always upwards sloping since the gamma distribution takes only positive values. The real yields in each cluster at maturity $\{10, 15, 20, 25\}$ are generated with the same method. To be more specific, the real yield at maturity m_2 is broken into the real yield at maturity m_1 plus a m_2-to-m_1 year spread,

$$y_r^{(t,s)}(m_2) = y_r^{(t,s)}(m_1) + g_{m_2,m_1}, \qquad m_2 > m_1$$

Note that the spread distributions are not indexed according to a cluster. The reason is that same same scale and shape parameters are used for each kind of spread distribution in all clusters. For clarification, a schematic illustration of the method is laid out in figure 5.1.

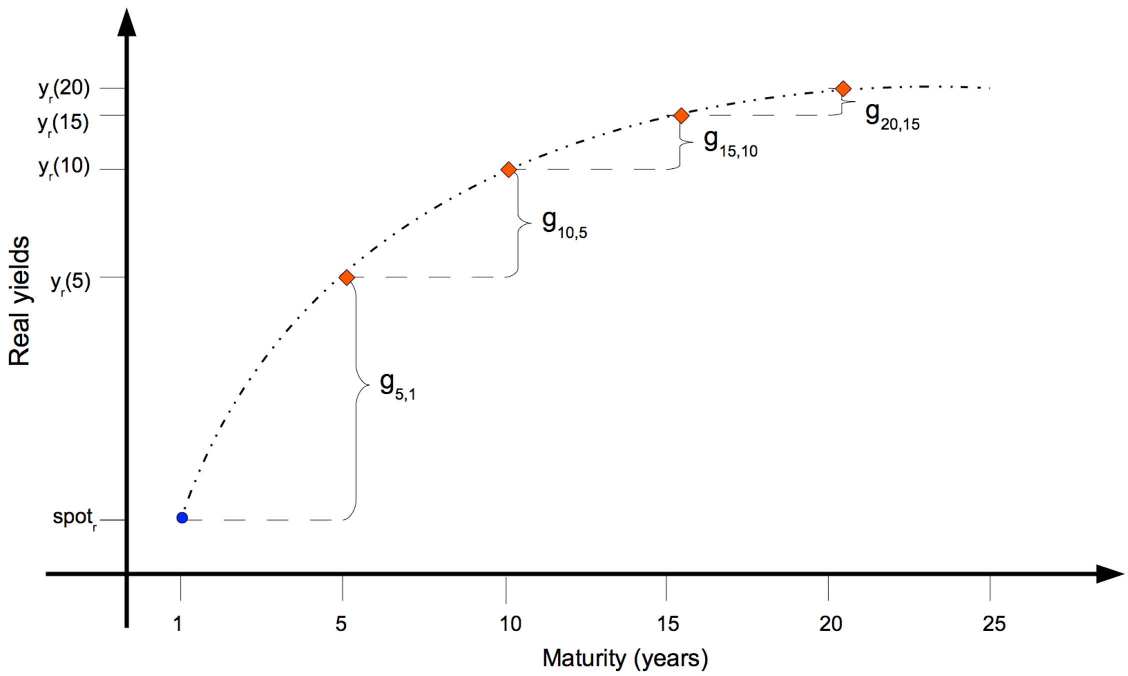

Figure 5.1. Structure of the real yield curve simulation method, where (●) denotes spot yield and (♦) denotes real yields $y^{(t,s)}(m)$, $m = \{5, 10, 15, 20\}$.

Note that although real yields for maturity of 25 years is not included in the figure, but are nevertheless used. After maturity of 25 years the real curve is assumed to be flat.

The parameters are determined, based on trial and error with Arion Banks' average real yield curve as a reference. The resulting parameters values as well as the mean (a/b) and variance (a/b^2) for each spread distribution are displayed in table 5.3.

Table 5.3. The shape and the scale parameters of the gamma spread distributions as well as the mean and the variance.

Distribution	$\hat{a}$	$\hat{b}$	Mean (%)	Variance (%)
$g_{5,1}$	25	333.3	0.750	$2.25 \cdot 10^{-4}$
$g_{10,5}$	35	500.0	0.700	$1.40 \cdot 10^{-4}$
$g_{15,10}$	20	500.0	0.400	$8.00 \cdot 10^{-5}$
$g_{20,15}$	30	10000.0	0.300	$3.00 \cdot 10^{-5}$
$g_{25,20}$	45	10000.0	0.045	$4.50 \cdot 10^{-7}$

The decrease in the mean and the variance of the spreads corresponds to flattening of the yield curve at longer maturities.

The whole yield curve is then derived (sub-step two) with spline interpolation where coefficients of a piecewise polynomial, which intersects all simulated real yields, are estimated. To keep the discussion focused, the spline interpolation method is not further discussed in this thesis but the reader is referred to any textbook on numerical analysis, e.g. Bradie (2006).

Eight simulated real yield curves are shown in figure 5.2 as well as the average real yield curve provided by Arion Bank.

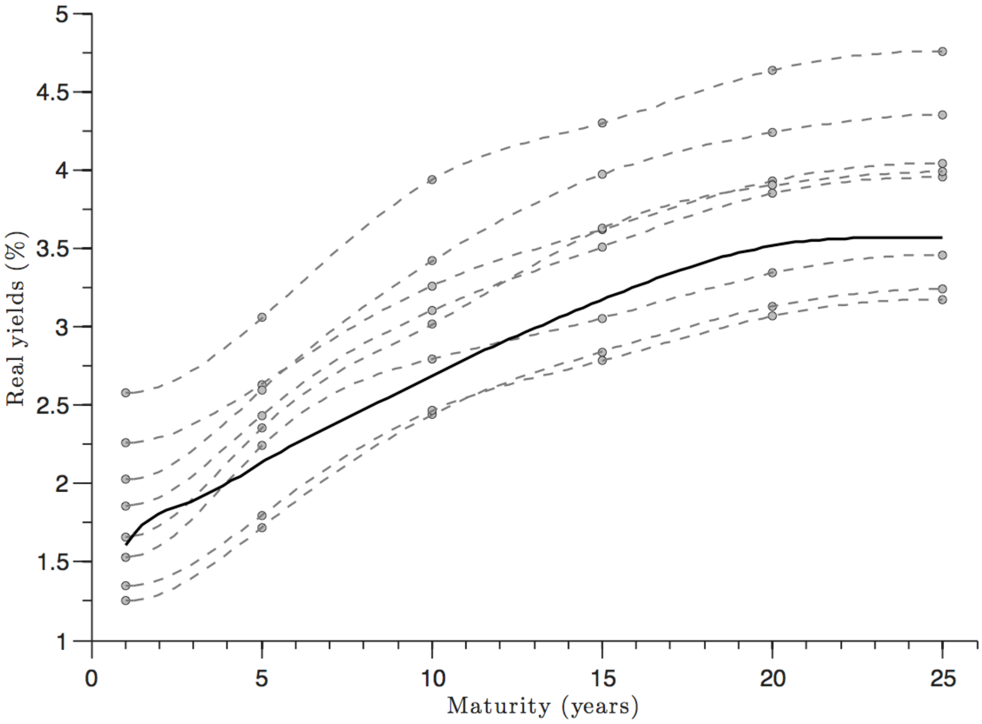

Figure 5.2. Eight simulated real yield curves (dashed lines) and Arion Banks average real yield curve (solid line). Simulated yields, $y_r^{(t,s)}(m)$ for $m = \{5, 10, \ldots, 25\}$ as well as spot yields are dented by (●).

In general, the parameters of the spread distributions can be functions of the stage, $a(t), b(t)$. This could prove useful to model an increasing in uncertainty as times goes by.

The Nominal yield curve

The real and nominal yield curve are connected by break-even inflation. The nominal yield curve is generated using a similar method as for the real yield curve. Let $y_n^{(t,s)}(m)$

denote the nominal yield for maturity m in cluster $C^{(t,s)}$. Unlike the real yield curve, we are only interested in part of the nominal yield curve. Namely the part between maturities of five and ten years. The reason for this will be discussed in section 5.3.3. Similar graphical demonstration is laid out in figure 5.3 for the generation of the nominal yield curve.

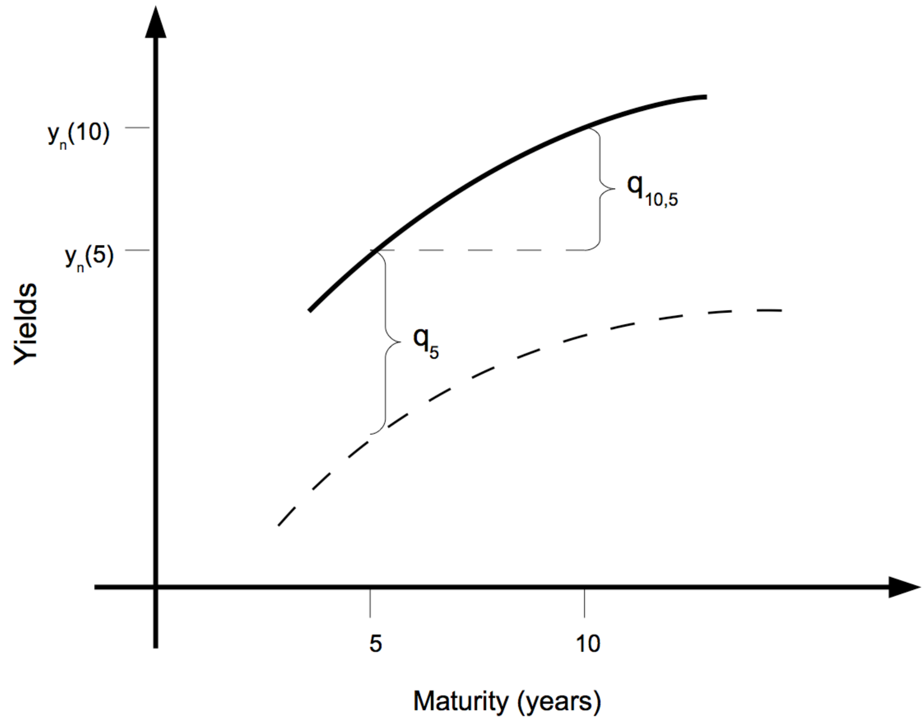

Figure 5.3. Structure of the nominal yield curve (solid line), the real yield curve (dashed line) and their relation.

Here q_5 denotes the five year real and nominal yield spread and $q_{10,5}$ denotes the ten-to-five year nominal yield spread. The five year real and nominal yield spread depends on inflation. To incorporate this relation the scale parameter of q_5 gamma distribution is a function of inflation. The scale and shape parameters for the nominal spread distributions are given in table 5.4.

Table 5.4. The scale, shape parameters as well as the mean and variance of the real-nominal spread distribution (q_5) and the ten-to-five year nominal yield spread distribution ($q_{10,5}$).

Distribution	$\hat{a}$	$\hat{b}$	Mean (%)	Variance (%)
q_5	$90 + 100\pi^{(t,s)}$	1400	–	–
$q_{10,5}$	80	5000	1.60	$3.20 \cdot 10^{-4}$

5.1.4 Summary

Scenario generation for economic variables is based on simulation of inflation and real spot yield paths, which are bundled into a scenario tree with the extended K-means method. In each cluster, the real and nominal yield curves along with exchange rate returns are modeled according to the dependencies represented in figure 5.4 and the models introduced in this chapter.

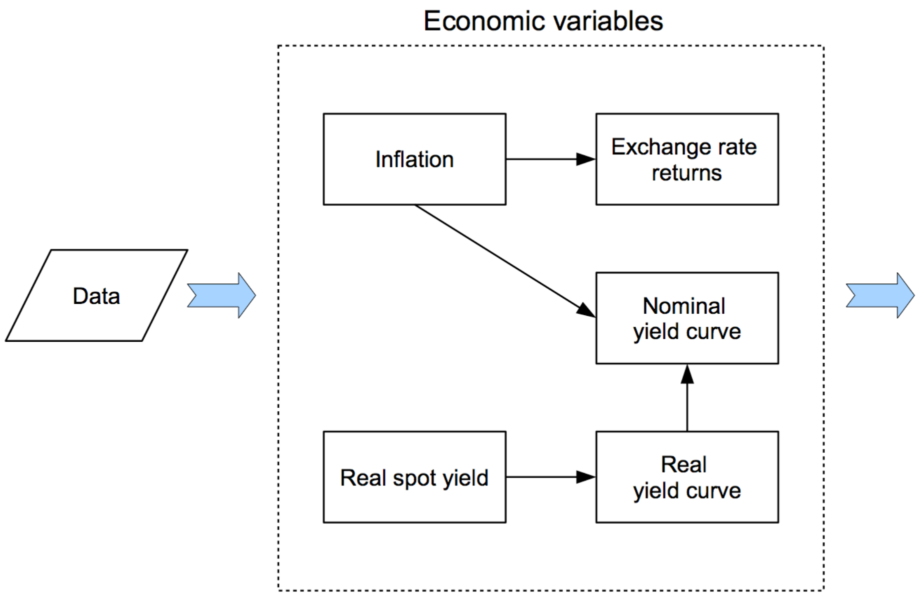

Figure 5.4. Dependencies between all models for the economic variables.

The dependencies ensure that in each cluster all economic variables are in line with economic theory. For example, the nominal and real yield curves are bounded by inflation which ensures that the curves do not intersect or drift unreasonably far apart.

5.2 Liabilities and premiums

A pension fund's total liabilities consist of accrued and future liabilites. Efficiently modeling future liabilities and premiums is an essential part of asset liability management. Both liabilities and premiums are affected by number of factors. The most influential risk factors for Icelandic pension funds are interest rate risk, inflation risk and mortality disability probabilities (Baldvinsson, 2004).

5.2.1 Risk factors

Interest rate risk

In the current regulatory framework[5] liabilities are valued with 3.5% fixed interest rate, whereas in many countries liabilities are valued using market interest rates (make-to-market) at each time. When fixed rates are used for valuation, it results in present value of future liabilities and premiums which is independent of market movements in interest rates.

Consider the following example based one (Kaupþing, 2006). Suppose a pension fund asset portfolio consists of single zero-yield coupon bond with maturity of ten years, a total of ISK 100. The pension fund faces a single liability payment in 25 years which amounts a total to of ISK 167.5. When a 3.5% interest rate is used for valuation, the present value of the fund's asset and liability are both ISK 70.9 and the funding ratio is 0%. This implies that the fund's asset is sufficient to meet the future liability payment. Now suppose that a 2.5% interest is used for valuation then the present values of the fund's asset and liability are ISK 78.1 and ISK 90.3 respectively. This results in a funding ratio of -13.5% which implies that the fund's asset is insufficient to meet the future liability payment. Figure 5.5 shows the funding ratio when interest rates ranging from 1% to 6% are used for valuation.

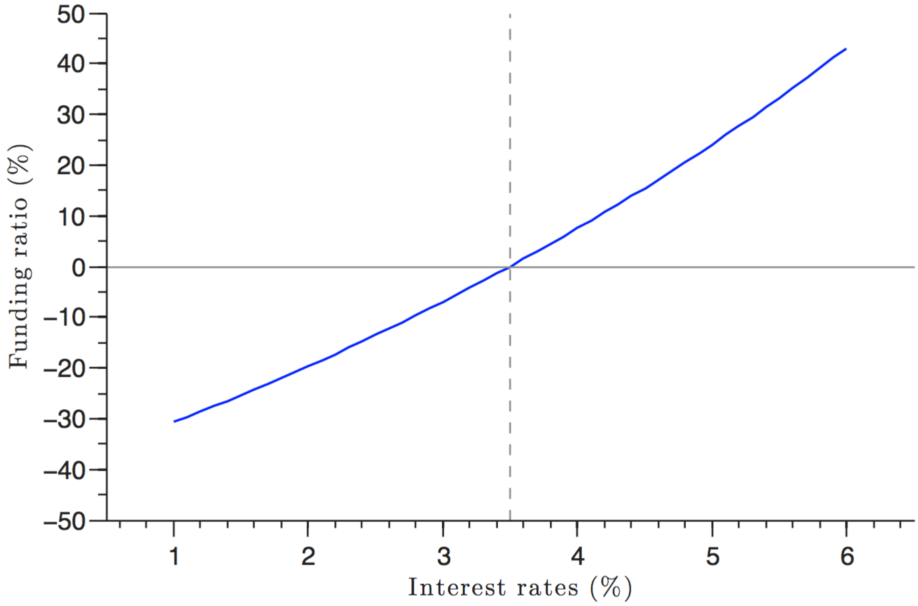

Figure 5.5. Pension fund actuarial position, measured with the funding ratio, valued for real interest rates ranging from 1% to 6%.

[5]More precisely principal feature (article 24) of Act 129/1997 on the Mandatory Guarantee of Pension Rights and the Operations of Pension Funds

This example clearly illustrates the interest rate risk facing the pension fund. Due to the pension systems relatively young age, future liability streams have high maturity making the liabilities more sensitive to movements in interest rates.

Suggestions on changing to market valuation of liabilities and premiums to accommodate the interest rate risk have grown in recent years. One such suggestion is put forward by (Valdimar & Möller, 2010).

Inflation risk

Inflation is another factor that has considerable impact on pension funds actuarial position. Liabilities and premiums are indexed-linked, due to valuation of 3.5% real interest rate. However only a proportion of the fund's assets is index-linked, namely real bonds. At the end of year 2010 roughly 60% of the wealth of Icelandic pension funds was allocated in bonds, which mainly consists of index-linked housing bonds. A pension fund with 85% of its bond portfolio allocated in domestic index-linked bonds makes for roughly half of total asset portfolio index-linked. This results in imbalance where high inflation results in a less favorable actuarial position.

Mortality and disability probabilities

The valuation of future liabilities and premiums is subject to assumptions on mortality and disability probabilities. Pension funds are obligated to provide payments until passing of the pensioners. Therefore an increase in longevity results in increased liabilities. The same holds for disability payments and their effects on future liabilities.

Life expectancy has risen steadily by about one percent since the 1960's in Europe and North America and on average, each additional year of life adds approximately 3-4% to the value of UK pension liabilities, (Loeys *et al.* , 2007). The importance of good estimates on future life expectancies and disability probabilities are vital for good estimates of future liabilities and premiums. A study by Thompson (2006) of FTSE100 companies revealed that assumptions about mortality rates were overly optimistic and that realistic longevity assumptions would more than double the aggregate deficit from GBP 46 billion to GBP 100 billion.

A number of studies on methods to forecast future mortality and disability rates are presened in the literature, e.g. Mitchell (2011) or Brockett *et al.* (2010) as well as references therein.

In this thesis future liabilities and future premiums are provided by Arion Bank, based on estimation methods used in the Bank's analysis. To account for the interest rate risk, market valuation is used in this thesis.

5.3 Modeling asset returns

It is assumed that the pension fund can invest its wealth in five assets classes: cash and deposits; domestic and foreign stocks: real and nominal bonds. These asset classes account for roughly 99% of the average portfolio of Icelandic pension funds at the end of year 2010, (Financial Supervisory Authority, 2011). The other one percent includes domestic corporate bonds, foreign bonds and alternative investments.

5.3.1 Return on cash and deposits

Icelandic pension funds invest small amounts of their wealth in cash and deposits[6], merely for liquidity reasons. However since the collapse of the Icelandic financial industry, the amounts held in cash has risen dramatically. The main reason is the lack of investment opportunities, due to the plunge in the Icelandic stock market and the current capital controls.

However cash is considered as one of the asset classes the pension fund can invest in. The returns for holdings in cash, in scenario s at stage t is approximated by the geometric average of one year real yields, or mathematically

$$R_{\text{cash}}^{(t,s)} = \sqrt{(1 + \text{spot}^{(t,s)})(1 + \text{spot}^{(t-1,s)})} - 1 \tag{5.3.1}$$

5.3.2 Stock returns

Stock returns exhibit some fundamental features, for example heavy tails, skewness and volatility clustering (Brooks, Chris, 2008) which have to be accounted for in modeling. Although such effects are clearly visible in high frequency (daily or intra-daily) data of stock returns, empirical research indicates that volatility clustering is also present in long data series with lower (monthly) frequency (Jacobsen & Dannenburg, 2003). The generalize autoregressive conditional heteroskedasticity, or the GARCH model was developed independently by (Bollerslev, 1986) and (Taylor, 1987) and is widely used in empirical studies. The model allows the conditional variance to be dependet upon previous own lags and is able to capture volatility clustering. The model is an empirical success and vast variations of the model have been introduced.

[6]Hereafter referred to as cash

In recent years, much effort has been devoted to extend the univariate GARCH model to multivariate one. The main issue regarding multivariate models is that the number of parameters that have to be estimated grows fast. Therefore many extensions are based on assumptions that limit the number of parameters that have to be estimated. The most widely recognized multivariate GARCH models are BEKK, Vector GARCH and the constant conditional correlation (CCC) GARCH and their extensions (Bauwens *et al.* , 2006; Francq & Zakoian, 2010).

The model suggested in this thesis is a dynamic conditional correlation multivariate GARCH, introduced by (Engle, 2002; Engle & Sheppard, 2001). The main reason for this are that the model is first of all in discrete time rather than continuous, which we prefer; the intuitive extension of the univariate GARCH model; the model's ability to capture volatility clustering and dynamic correlation between stock returns. It can be efficiently estimated by a two step procedure which is described in detail later. In addition numerous studies suggest that the DCC-GARCH model performs well (and sometimes better) compared to other multivariate GARCH models (Yilmaz, 2010).

DCC-GARCH

Let r_t denote a vector of filtered zero-mean return series of L asset and H_t denote the covariance matrix. In its basic form the DCC-GARCH model assumes that $r_t \sim N(0, H_t)$. Assuming normality is often considered too restrictive for stock returns. Various further extension on the model have been suggested in the literature to allow among other things asymmetric distributions, see e.g. (Cappiello *et al.* , 2006; Hafner & Franses, 2003). Orskaug (2009) reports that DCC-GARCH with *skewed-t* distributed error term performs better normal and *students-t* than in terms of various marginal goodness of fit mesures.

However the assumption on normally distributed error term is only considered in this thesis in which case the following factorization of the covariance matrix is assumed

$$
\begin{aligned}
H_t &\equiv D_t E_t D_t \\
&= \begin{bmatrix} \sigma_{1t} & 0 & \dots & 0 \\ 0 & \sigma_{2t} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \sigma_{Lt} \end{bmatrix}
\begin{bmatrix} 1 & \rho_{12t} & \dots & \rho_{1Lt} \\ \rho_{21t} & 1 & \dots & \rho_{2Lt} \\ \vdots & \vdots & \ddots & \vdots \\ \rho_{1Lt} & \rho_{2Lt} & \dots & 1 \end{bmatrix}
\begin{bmatrix} \sigma_{1t} & 0 & \dots & 0 \\ 0 & \sigma_{2t} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \sigma_{Lt} \end{bmatrix}
\end{aligned}
$$

where E_t denotes the time varying correlation matrix and D_t is $L \times L$ diagonal matrix of time varying standard deviation. The i^{th} standard deviation, σ_{it}, follows a univariate GARCH(1,1) model:

$$\sigma_{it}^2 = \omega_i + \alpha_i r_{it-1}^2 + \beta_i h_{it-1}$$

The correlation structure is (DCC(1,1) part)

$$E_t = Q^{\star-1} Q_t Q^{\star-1}$$

and the proposed dynamic correlation structure is

$$Q_t = (1 - \mathrm{dcc}_p - \mathrm{dcc}_q)\bar{Q} + \mathrm{dcc}_p(\varepsilon_t \varepsilon_t') + \mathrm{dcc}_q Q_{t-1}$$

where dcc_p and dcc_q are the dynamic correlation parametres.

Estimation

The DCC(1,1)-GARCH(1,1) model is estimated with two step quasi likelihood estimation following (Engle & Sheppard, 2001). The parameter vector is split in two, $\theta_{\text{dcc-garch}} = \{\theta_{\text{garch}}, \theta_{\text{dcc}}\}$ where elements of θ_{garch} correspond to the parameters of univariate GARCH and the elements in θ_{dcc} correspond to the dynamic conditional correlation parameters. In the first step univariate GARCH model is estimated for each asset using its filtered zero-mean return series, r_t[7]. Assuming normal distribution the quasi-loglikelihood function for the first stage is simply the sum of the log-likelihoods of univariate GARCH

$$\mathrm{QLL}_1(\theta_{\text{garch}}|r_t) = -\frac{1}{2}\sum_{l=1}^{L}\left(T\log(2\pi) + \sum_{t=1}^{T}\left(\log(\sigma_{lt}^2) + \frac{r_{lt}^2}{\sigma_{lt}^2}\right)\right)$$

where L and T are the number of assets and observations respectively. The resulting residuals, ε_t, from the first stage are standardized by $\varepsilon_t = D_t^{-1} r_t$ and used to estimate the conditional correlation parameters. The resulting second stage quasi-loglikelihood function is:

$$\mathrm{QLL}_2(\theta_{\text{dcc}}|\theta_{\text{garch}}, r_t) = -\frac{1}{2}\sum_{t=1}^{T}(k\log(2\pi) + 2\log|D_t| + \log|E_t| + \varepsilon_t' R_t^{-1}\varepsilon_t).$$

Note that the correlation matrix, E_t, has to be inverted in each iteration, in the second stage, and therefore has to be positive definite. For further information such as a proof of positive definiteness and consistency of the estimators, the reader is

[7] Note that other filtration methods can be used on the return series, such as ARMA filtration. However analysis indicate that such filtration is not needed.

referred to (Engle & Sheppard, 2001).

Data

It is assumed that the pension fund invests in diversified stock indices rather than individual stocks. The proxy for the available stocks are five Morgan Stanley (MSCI) indices, MSCI Nordic, World, Europe, North America and Far East. The data consists of monthly returns from 30 November 1981 to 30 November 2011, a total of 361 data points and is shown in figure 5.6. Since the collapse of the financial industry in late 2008, the Icelandic stock market is merely a fraction of its former size. Therefore the MSCI Nordic index is used as a proxy for the Icelandic stock market. The other four MSCI indices are used to approximate the selection of foreign stocks available for the pension fund.

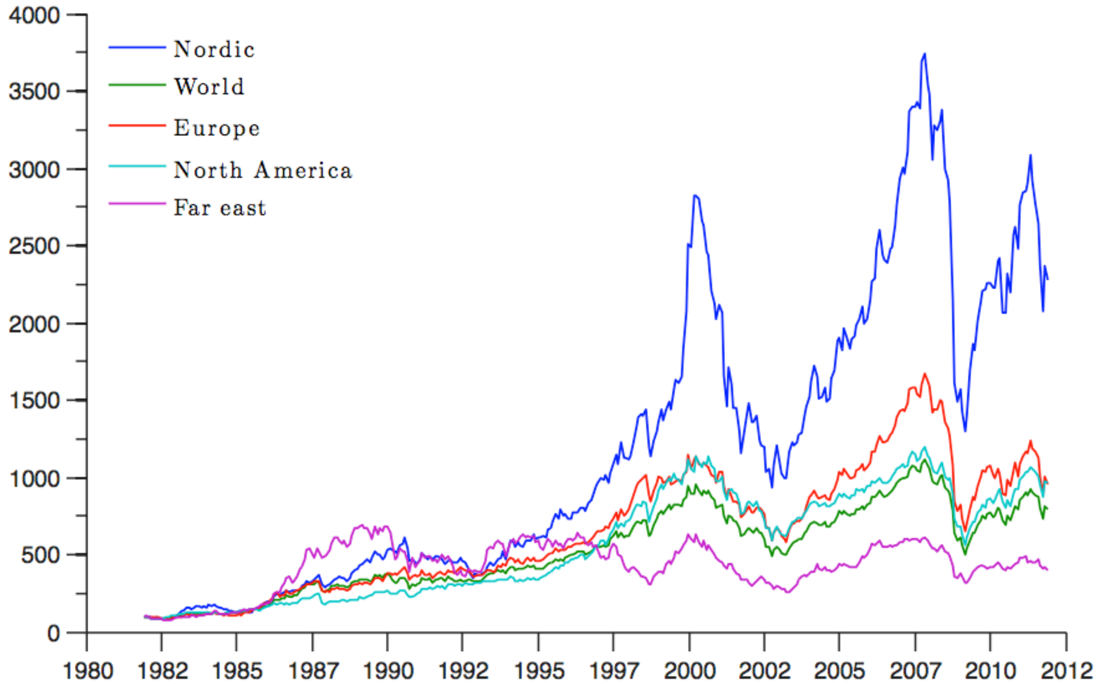

Figure 5.6. Historical prices of the five MSCI stock indices from 30 November 1981 to 30 November 2011. The starting values have been scaled to a value of 100 for comparison. Source: `www.mscibarra.com`

Historical return, volatility and correlation estimates are presented in tables 5.5 and 5.6. The parameter estimates are given in table 5.7

Table 5.5. Historical average returns and volatility of five MSCI indices based on monthly returns from 30 November 1981 to 30 November 2011. Yearly returns are estimated, by assuming normality, as $\mu_{\text{year}} = 12\mu_{\text{monthly}}$ and $\sigma_{\text{yearly}} = \sqrt{12}\sigma_{\text{monthly}}$. All values are reported as percentages (%).

MSCI indexes	Monthly return	Monthly volatility	Yearly return	Yearly volatility
Nordic	1.09	6.63	13.12	22.97
World	0.68	4.47	8.12	15.48
Europe	0.76	5.15	9.16	17.83
North America	0.73	4.52	8.82	15.65
Far East	0.57	6.24	6.89	21.60

Table 5.6. Historical estimated correlations, from 30 November 1981 to 30 November 2011

	Nordic	World	Europe	North America	Far East
Nordic	1.000	0.797	0.840	0.680	0.513
World	0.797	1.000	0.893	0.881	0.745
Europe	0.840	0.893	1.000	0.748	0.561
North America	0.680	0.881	0.748	1.000	0.435
Far East	0.513	0.745	0.561	0.435	1.000

Table 5.7. Estimated parameters and standard errors (in brackets) in the DCC(1,1)-GARCH(1,1)model for five MSCI world indices based on historical prices from 30 November 1981 to 30 November 2011 using *ccgarch* package for the statistical software R (`www.r-project.org/`) (Nakatani, 2009).

GARCH(1,1)	$\hat{\omega}_i$	$\hat{\alpha}_i$	$\hat{\beta}_i$
Nordic	$1.07 \cdot 10^{-3}$	0.144	0.620
	$(6.42 \cdot 10^{-4})$	$(7.99 \cdot 10^{-2})$	$(4.25 \cdot 10^{-2})$
World	$1.05 \cdot 10^{-4}$	0.101	0.853
	$(7.76 \cdot 10^{-2})$	$(1.12 \cdot 10^{-4})$	$(4.93 \cdot 10^{-2})$
Europe	$1.02 \cdot 10^{-4}$	0.090	0.878
	$(1.95 \cdot 10^{-1})$	$(5.84 \cdot 10^{-2})$	$(2.07 \cdot 10^{-4})$
North America	$5.32 \cdot 10^{-5}$	0.116	0.868
	$(7.47 \cdot 10^{-5})$	$(9.39 \cdot 10^{-2})$	$(6.50 \cdot 10^{-2})$
Far east	$4.07 \cdot 10^{-4}$	0.166	0.736
	$(5.50 \cdot 10^{-2})$	$(3.67 \cdot 10^{-5})$	$(9.60 \cdot 10^{-2})$
DCC(1,1)	$\hat{\text{p}}$	$\hat{\text{q}}$	
	0.085	0.913	
	(0.169)	(0.225)	

The DCC-GARCH model is used to simulate paths of the five MSCI stock indices, which are clustered by the extended K-means method. The MSCI Nordic index is used as a proxy for the Icelandic stock market and its return is only based on stock price movement. The total return of the other four indices, which correspond to foreign stocks, are based on returns from the change in stock prices and of from the prices and the exchange returns obtained by (5.1.2).

5.3.3 Bond returns

Icelandic pension funds have mostly invested in real (indexed linked) domestic housing and government bonds. However in recent years Icelandic pension funds have increased their shares in nominal bonds as well (Ó. Ö, Jónsson and E.D, Jónsson, personal interview, November 11, 2010).

It is assumed that the pension fund can invest in one nominal bond portfolio and one real bond portfolio. The portfolios are assumed to have constant maturity, thus rebalanced every year, with maturities of eight and ten years of nominal and real bonds respectively. The returns of the two portfolios are estimated using the log linear relation between holding-period returns and yield for coupon bonds, described in Campbell *et al.* (1997), and implemented in (Hoevenaars *et al.* , 2008), (Möller, 2006) and (Campbell *et al.* , 2001). The estimated gross logarithmic returns on nominal and real bond coupon portfolios, in stages t in scenario s is

$$\log\left(R_r^{(t,s)}\right) = \frac{1}{4}\log(y_r^{(t,s)}(8)) - D_r^{(8,t)}\left(\log(y_r^{(t,s)}(7)) - \log(y_r^{(t-1,s))}(8))\right)$$

$$\log\left(R_{nom}^{(t,s)}\right) = \frac{1}{4}\log(y_n^{(t,s)})(10) - D_{nom}^{(10,t)}\left(y_n^{(t,s)}(9) - \log(y_n^{(t-1,s)}(10))\right) + \log(\pi^{(t,s)})$$

where $D_r^{s,t}$ and $D_{nom}^{s,t}$ denotes duration of the real and nominal duration at stage t in scenario s respectively. The durations are which is approximated by

$$D_r^{s,t} = \frac{1-(1+y_r^{s,t}(8))^{-8}}{1-(1+y_r^{s,t}(8))^{-1}}, \qquad D_{nom}^{(t,s)} = \frac{1-(1+y_n^{s,t}(10))^{-10}}{1-(1+Y_n^{s,t}(10))^{-1}}$$

Here the nominal and real yields for each maturity are given by the nominal and real yield curves present in each cluster.

The modeling system for the underlying stochastic variables is schematically summarized in a flowchart, presented in figure 5.7.

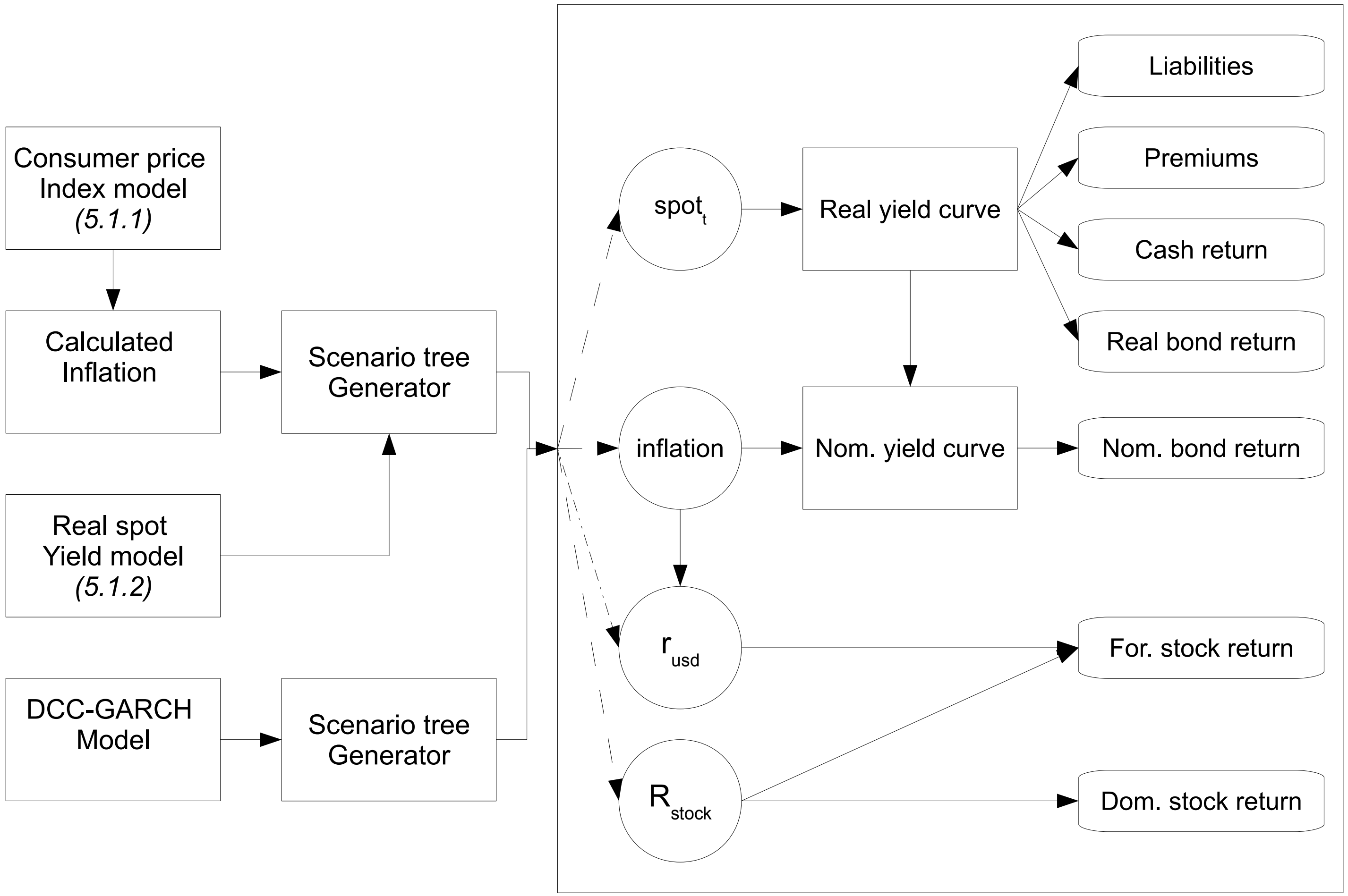

Figure 5.7. Flowchart for generating asset returns, liabilities and premiums. Simulated inflation, real spot yields and stock prices realizations are bundled together into a scenario tree using the K-means clustering algorithm. The dotted box represents one cluster in the scenario tree.

Chapter 6. Numerical results

In this chapter the asset liability management model[1] introduced in 3 is solved using the extended K-means scenario generation method described in chapter 4. The study is based on a hypothetical pension fund, where streams of future liabilities and premiums, provided by Arion Bank, are used. Given these estimated payments we are interested in optimizing the asset allocation for the next three years to achieve the maximum expected funding ratio at the horizon.

The chapter is organized as follows section 6.1 describes technical implementation of the stochastic optimization framework. In section 6.2 the setup of the study, such as parameter values are given. Section 6.3 describes computational experiments, which include in-sample stability analysis, comparison with fix-mix benchmark strategies and lastly sensitivity diagnostics.

6.1 Implementation

The models for the underlying stochastic variables, the economic variables and asset returns, from chapter 5, were written in MATLAB[2] and R. The models are used to simulate a fixed number of paths which were then fed into the extended K-means scenario tree generator (coded in MATLAB). In addition to the paths, the scenario tree generator takes as input the timescale and the branching structure of the scenario tree. The scenarios can be visually examined and restriction parameters adjusted until the scenario tree is satisfactory. Due to the nonanticipativity constraints, which are different among different scenario trees, the asset liability management model file is generated by a MATLAB program, in AMPL[3] format. The model and the scenario tree are then parsed by the AMPL program and the output fed to the MOSEK[4] solver. The solution details and statistics are imported back into MATLAB for visualization

[1] A summary of the model is presented in Appendix II.
[2] `www.mathworks.com`
[3] `www.ampl.com`
[4] `www.mosek.com`

and further analysis.

6.2 Model setup

The first of December 2010 was chosen as the beginning ($t = 0$) in the experiments with a horizon of three years was set. Every year decisions on portfolio rebalancing are made, which correspond to a yearly meeting of the pension fund board of directors. The horizon varies among the ALM studies in the literature, but is usually three to ten years. Table 6.1 presents the stages, the branching structure and number of scenarios of six published ALM studies.

Table 6.1. Scenario tree structure in six published asset liability management studies for pension funds.

Author(s)	Stages	Branching-structure	Scenarios
Kouwenberg (2001)	$\{1, 2, 3, 4, 5\}$	$\{10, 6, 6, 4, 4\}$	5,760
Hilli *et al.* (2007)	$\{1, 3, 6\}$	$\{25, 10, 10\}$	2,500
Geyer & Ziemba (2008)	$\{1, 2, 4, 6, 10\}$	$\{100, 5, 5, 2, 2\}$	10,000
Drijver (2005)	$\{1, 2, 3, 4\}$	$\{6, 6, 5, 5\}$	900
Dupacová & Polivka (2004)	$\{1, 2, 3\}$	$\{20, 8, 5\}/\{10, 8, 8\}$	800/640
Klein Haneveld *et al.* (2010)	$\{1, 2, 3\}$	$\{10, 10, 10\}$	1,000

The branching structure for the precent study was set equal to $\text{BS} = \{15, 15, 2\}$, which corresponds to $\{15, 225, 450\}$ clusters in each stage, and a total of 450 scenarios. The total number of scenarios is considerably less than published studies.

Five major asset classes were considered in the study and the proportion of the wealth held in each asset class at time at the beginning ($t = 0$) was

$$\begin{aligned} \mathcal{Q} &= \{\text{cash, domestic stocks, foreign stocks, real bonds, nominal bonds}\} \\ &= \{4.6\%,\ 6\%,\ 21\%,\ 47.9\%,\ 20.5\%\} \end{aligned}$$

where the pension fund's initial wealth is $W_0 = 19150$ thousand ISK. The fund is interested in the asset allocation that maximizes the funding ratio at the horizon. For comparison, a target wealth benchmark is constructed, based on the cash inflow from premiums and cash outflow from paid pension and a yearly return of 3.5 precent. The target wealth at each time from the beginnig to the horizon is

$$W^{\text{target}} = \{19150, 19790, 22527, 25242\}$$

thousand ISK.

In table 6.2 a summary of the scenario tree structure, restrictions and model parameters which were used in the study, is given. The values for the penalty parameters, λ_1 and λ_2 are somewhat arbitrarily chosen. Penalties in the objective are four times higher ($\lambda_2/\lambda_1 = 4$) for a funding ratio which falls below a minimum of -5%, compared to a funding ratio that does not achieve the pension fund's target of 8.5% at the horizon. Values for the portfolio parameters are based on Icelandic regulations and a upper bound of 20% on the maximum amount available for purchase, as a percentage of the wealth at each time, is set to prevent spurious asset switching over the next three years.

Table 6.2. Scenario tree structure, restriction and model parameters.

Tree structure	BS	=	{15,15,2}	$\mathcal{S}$	=	{1,...450}
	$\mathbb{I}$	=	{1,2,3}	$\mathcal{N}$	=	{1,...10,000}
	$\mathcal{T}$	=	{1,2,3}	$\mathcal{J}$	=	{1,...8}
Restriction parameters	$\pi^{\max}$	=	10.0	$\text{spot}_t^{\min}$	=	1.0%
	$\pi^{\min}$	=	1.0%	$\text{spot}_t^{\max}$	=	3.2%
Model and constraint parameters	λ_1	=	2.0	l^{cash}	=	1.0%
	λ_2	=	8.0	u^{cash}	=	5.0%
	F^{target}	=	8.5%	l^{bond}	=	50.0%
	$F^{\min}$	=	-5.0%	u^{bond}	=	90.0%
	$c_2 \dots c_6$	=	1.0%	l^{stock}	=	90.0%
	c_7, c_8	=	0.2%	u^{stock}	=	50.0%
	u^p	=	20.0%	u^{foreign}	=	50.0%

Summary of the asset liability model is presented in Appendix II. A sensitivity analysis is presented in section 6.3.3 to analyze the effects of the parameters' values on the objective value and the optimal asset allocation.

6.3 Computational experiments

The ALM model and scenario tree structure results in 64,414 constraints and 47,940 decision variables for the 450 scenarios. The optimization problem is linear and AMPL/MOESK solves the model in less than ten seconds on 2.26 GHz Intel Core 2 Duo, with 2GB memory. A breakdown of the total computation time is given in table 6.3.

Table 6.3. Breakdown of the computational time (in seconds)

Time to simulate paths	283.4
Clustering time	55.32
Solution time	9.44
Total time	348.16

The optimized expected asset allocation (the solution) is presented in table 6.4 where the expectation is calculated as $E[\cdot] = \sum_{s \in \mathcal{S}} p^s(\cdot)$, where p^s is the probability of scenario s. For comparison the target wealth benchmark is included.

Table 6.4. Optimized expected asset allocation, funding ratio and wealth.

Q	t=0	t=1	t=2	t=3
E[%Cash]	4.57	4.93	4.96	1.31
E[%Dom. Stocks]	6.00	9.52	7.03	9.5
E[%Foreign stocks]	21.00	7.96	19.33	27.90
E[%Real bonds]	47.90	48.26	42.26	38.47
E[%Nominal bonds]	20.53	29.34	26.42	22.81
E[%FR]		2.65	7.08	**8.67**
E[W]	19105	19525	22385	25238
W_{target}	N/A	19774	22072	24368

The target wealth benchmark is reached in the second year and the expected wealth exceeds the benchmark in the third year. The expected funding ratio at the horizon is 8.67% which is 0.17% above the target funding ratio. The expected asset allocation can be visualized in figure 6.8 which shows the proportion held in each asset class, as a percentage of the wealth.

The solution for the first year indicates an increase in the proportion of nominal bonds and domestic stocks at the cost of a reduction in foreign stocks. For years two and three a steady increase in foreign stocks is favored while cash, nominal and real bonds decrease.[5]

In figure 6.2, the expected funding ratio at the horizon is plotted against the penalty components for various paris of the penalty parameters, (λ_1, λ_2), in an efficient frontier setting.

[5]In later analysis, it is referred to as the solution.

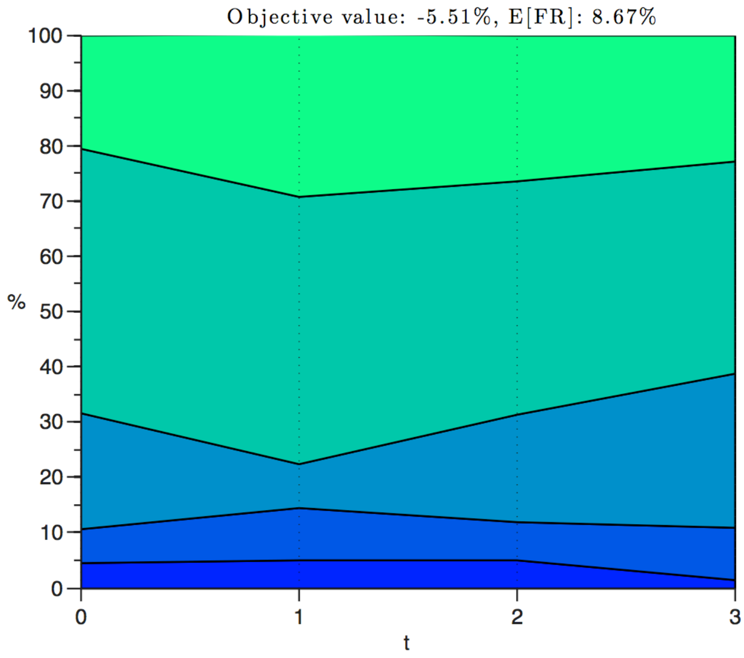

Figure 6.1. Optimized asset allocation with the following legend. (■) Cash, (■) domestic stocks, (■) foreign stocks, (■) real bonds and (■) nominal bonds.

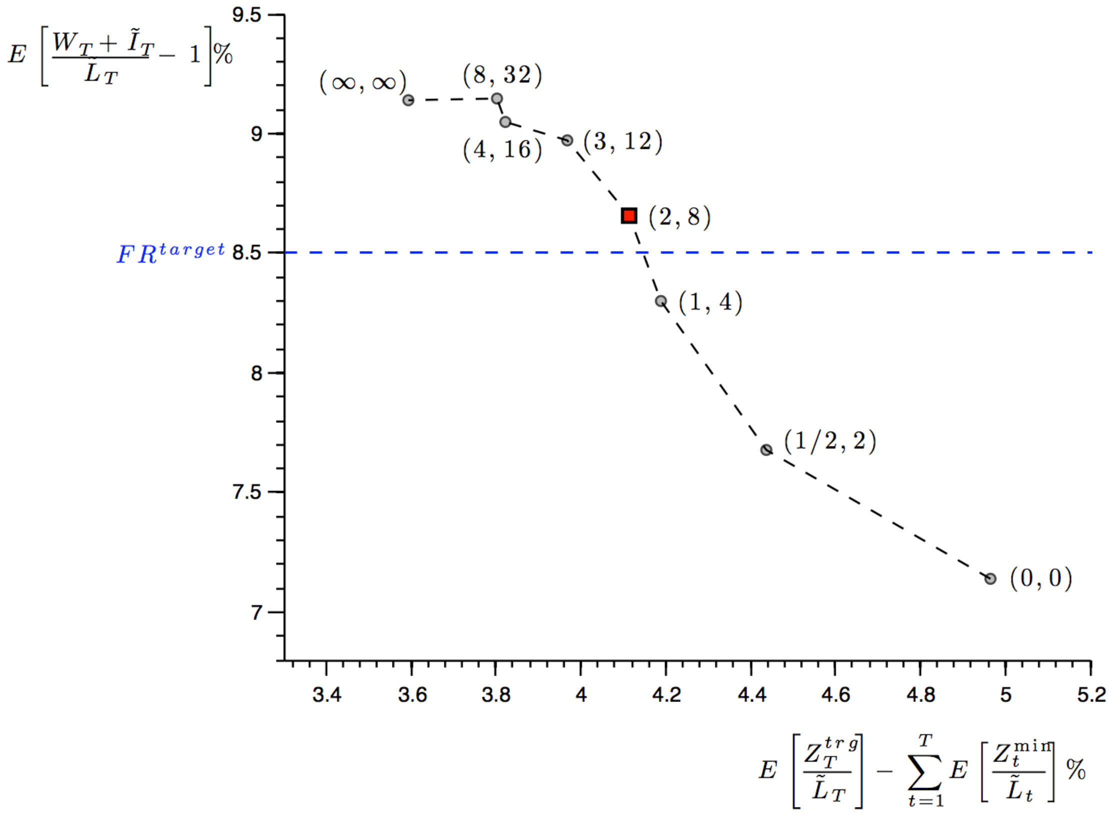

Figure 6.2. The efficient frontier for various pairs of the penalty parameters, (λ_1, λ_2). The black edged, red square corresponds to the solution.

From the efficient frontier it is apparent that an increase in the penalty parameters drives the expected funding ratio at the horizon upwards. Figure 6.3 shows the expected asset allocation for most extreme values of the penalty parameters. In the absence of penalty functions ($\lambda_1 = \lambda_2 = 0$) the holding in nominal bonds are further increased in favor of a decrease in other asset classes in the third year. On the other end, when ($\lambda_1 \to \infty$ and $\lambda_2 \to \infty$), holdings in real bonds are substantially higher in favor of holdings in nominal bonds for all years. Moreover holdings in foreign stocks are considerably lower in favor of domestic stocks in year one. The expected asset allocation for the other cases ($\lambda_1 = 1/2, \lambda_2 = 4$) and ($\lambda_1 = 8, \lambda_2 = 32$) falls in-between the two extremes.

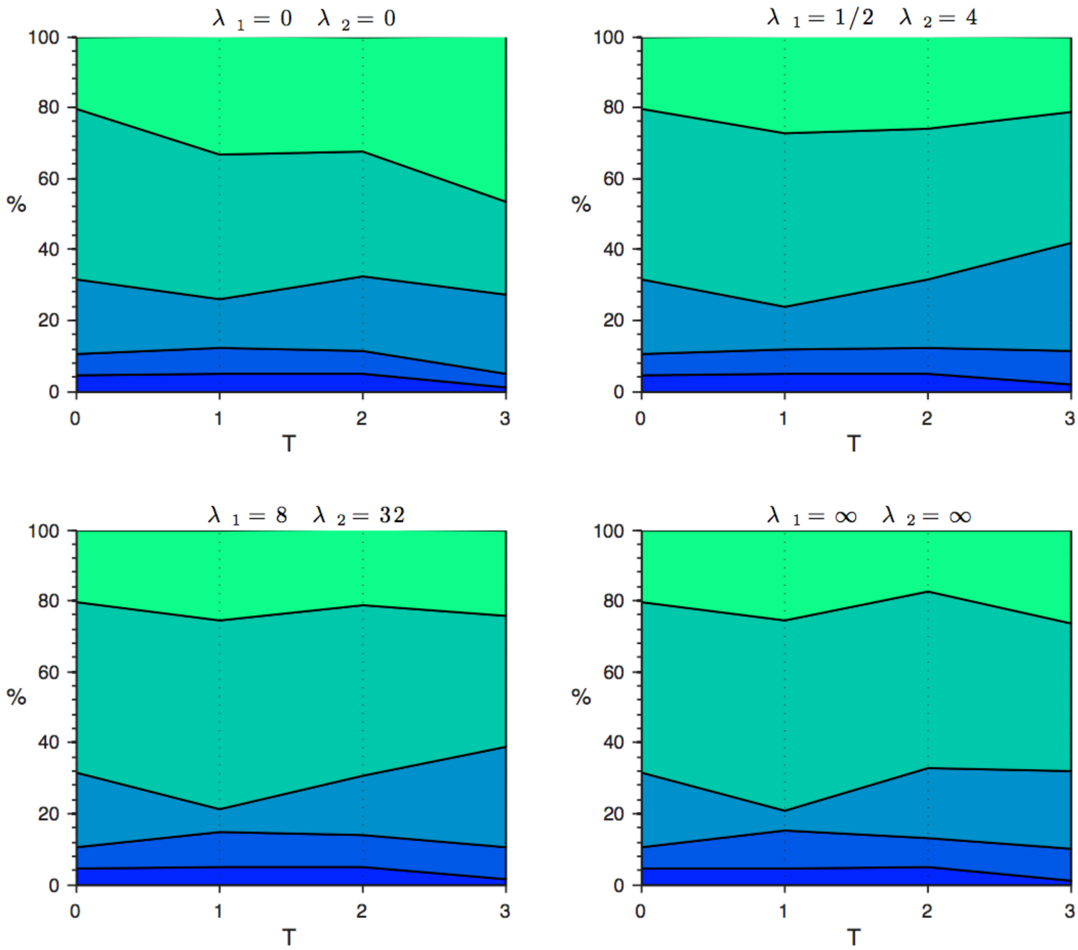

Figure 6.3. Expected asset allocation for extreme pairs of penalty parameters, (λ_1, λ_2) with the following legend. (■) Cash, (■) domestic stocks, (■) foreign stocks, (■) real bonds and (■) nominal bonds.

6.3.1 In-sample stability

To examine in-sample stability, the stochastic programming model is solved with scenario trees with an increasing number of scenarios. The scenario generation method is unable to produce a scenario tree with more than 600 scenarios. The main reason is the limited ability of the K-means method to generate trees with many scenarios, the reason being that all paths simulated before clustering. This is a major drawback since stability analysis in the literature is conducted on a much larger number of scenarios. The objective value, as a function of the total number of scenarios, is shown in figure (6.4) for branching structures BS $= \{15, i, 2\}$ for $i = 2, \ldots, 20$. The red circle corresponds to the solution.

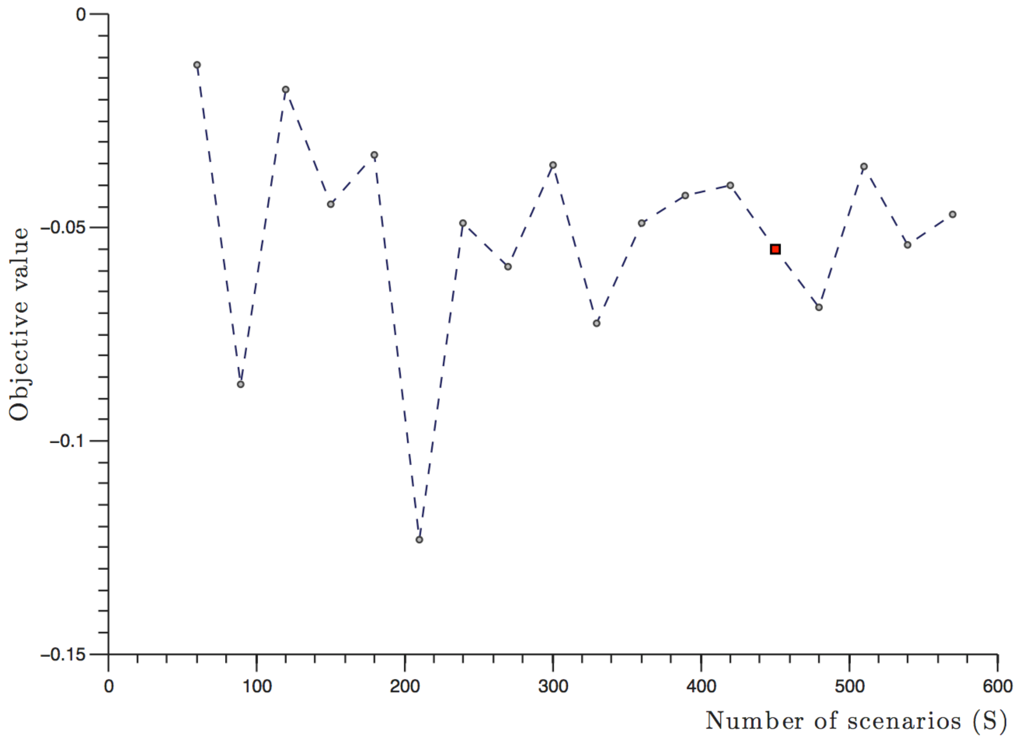

Figure 6.4. The objective value for branching structures BS $= \{15, i, 2\}$ where $i = 2, \ldots, 20$. The solution is denoted by (■)

It is evident that fluctuation in the objective decreases as the number of scenarios increases, but considerable fluctuations are still present for 400 to 600 scenarios. Therefore we were unable to conclude that the objective has reached has stabilized properly. This is not surprising in the light of the low number of scenarios. This indicates that the scenario generation method needs improvements. This is confirmed when the components of the objective function are analyzed. Figure 6.5 shows the fluctuations in the expected funding ratio at the horizon (panel a) along with the risk aversion components (panel b and c). Note that the fluctuations in the risk components include the risk parameters λ_1 and λ_2.

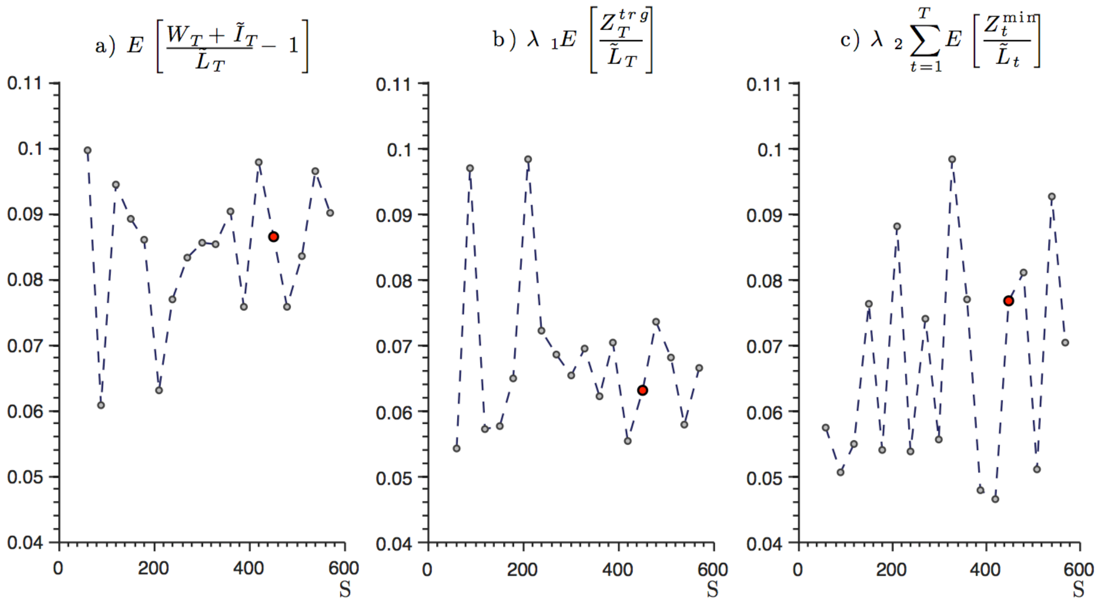

Figure 6.5. Stability analysis on the three components of the objective value for branching structures BS $= \{15, i, 2\}$,where $i = 2, \ldots, 20$.

It is, however, worth noting that no general accepted tolerance on fluctuation in the objective value exists for in-stability tests in the literature. This can lead to controversial in-sample stability conclusions. For example, fluctuations can be very large for a small number of scenarios and decrease as the number of scenarios increase but with unsatisfactory variance. This might create the illusion of stability while the still existent fluctuation fora larger number of scenarios might have considerable impact on the stability of the objective. This is, of course subject to the modeler's judgement.

Given the fluctuations in the objective value, it is interesting to examine the variation in the expected funding ratio at the horizon when the same branching structure is used to solve the optimization model. Seven different scenario trees were generated, all with branching structure BS $= \{15, 15, 2\}$ the model is subsequently solved. The expected funding ratio at the horizon is plotted as a function of the risk aversion components in figure 6.6. For comparison, seven scenario trees with branching structure BS $= \{15, 7, 2\}$ are also shown. It is clear that when the number of scenarios is increased, the variability in the expected funding ratio at the horizon becomes smaller. However considerable variability still is present for the $\{15, 15, 2\}$

branching structure. This further suggests that stability has not been reached.

The expected wealth, for the {15,15,2} branching structure, is compared to the target wealth benchmark in figure (6.7).

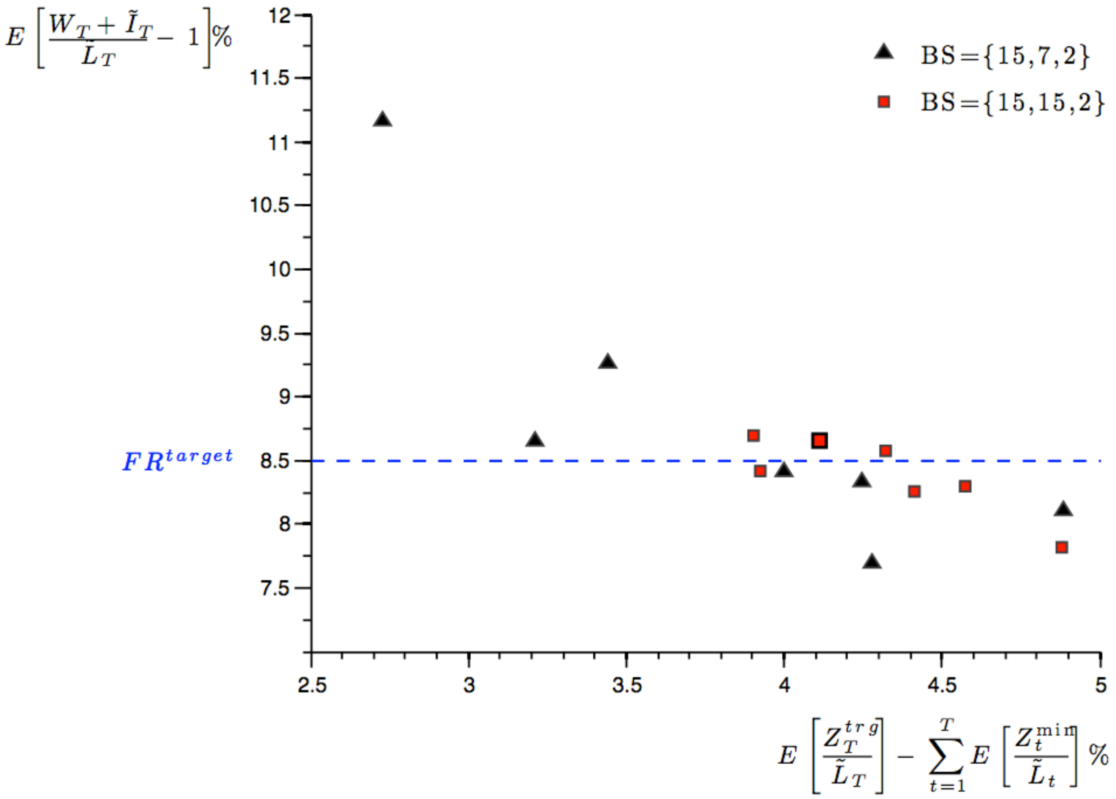

Figure 6.6. Variability in the expected funding ratio, as a function of the risk aversion components, for BS $= \{15, 7, 2\}$ (▲) and BS $= \{15, 15, 2\}$ (■). The solution is marked by red square with black edges.

In the first year, the target wealth is not reached in all cases, which indicates that the expected wealth does not yield a 3.5% real increase in the first year. However, in the third year the target is reached in all cases, with a difference of 580.21 thousand ISK between the highest and lowest expected wealth. This further strengthens the suspicion of non-acceptable fluctuations in the solutions, and we conclude that stability is not attained.

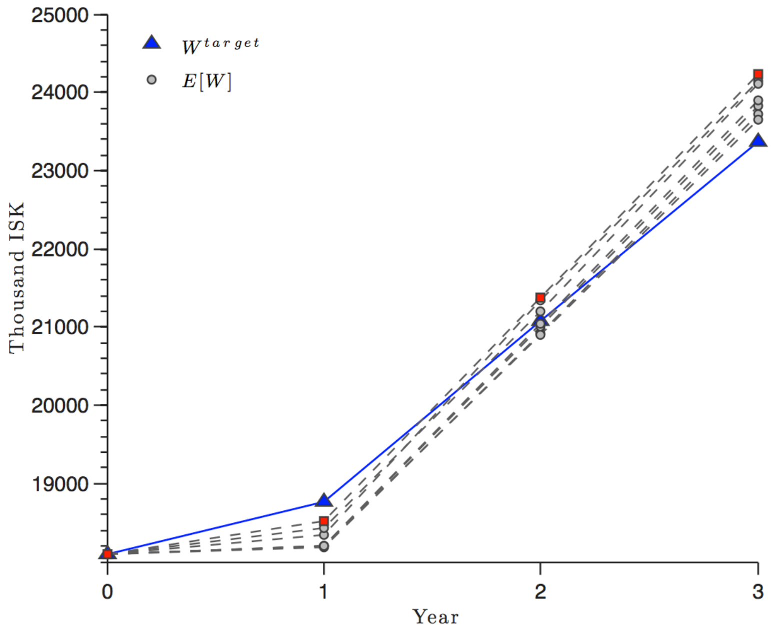

Figure 6.7. Evolution of the expected wealth for seven scenario trees with BS $= \{15, 15, 2\}$ denoted by (●) and the target wealth denoted by (▲). The solution is denoted by (■).

It is interesting to examine the difference between the expected asset allocation obtained by the six other scenario trees with BS $= \{15, 15, 2\}$, shown in figure 6.8. Surprisingly little attention has been paid to this in the literature despite its importance. The reason for the importance is that one might be able to obtain objective values with little variation but large variation in the expected asset allocation. In such cases, no sensible asset allocation decisions, with respect to the model, can be made.

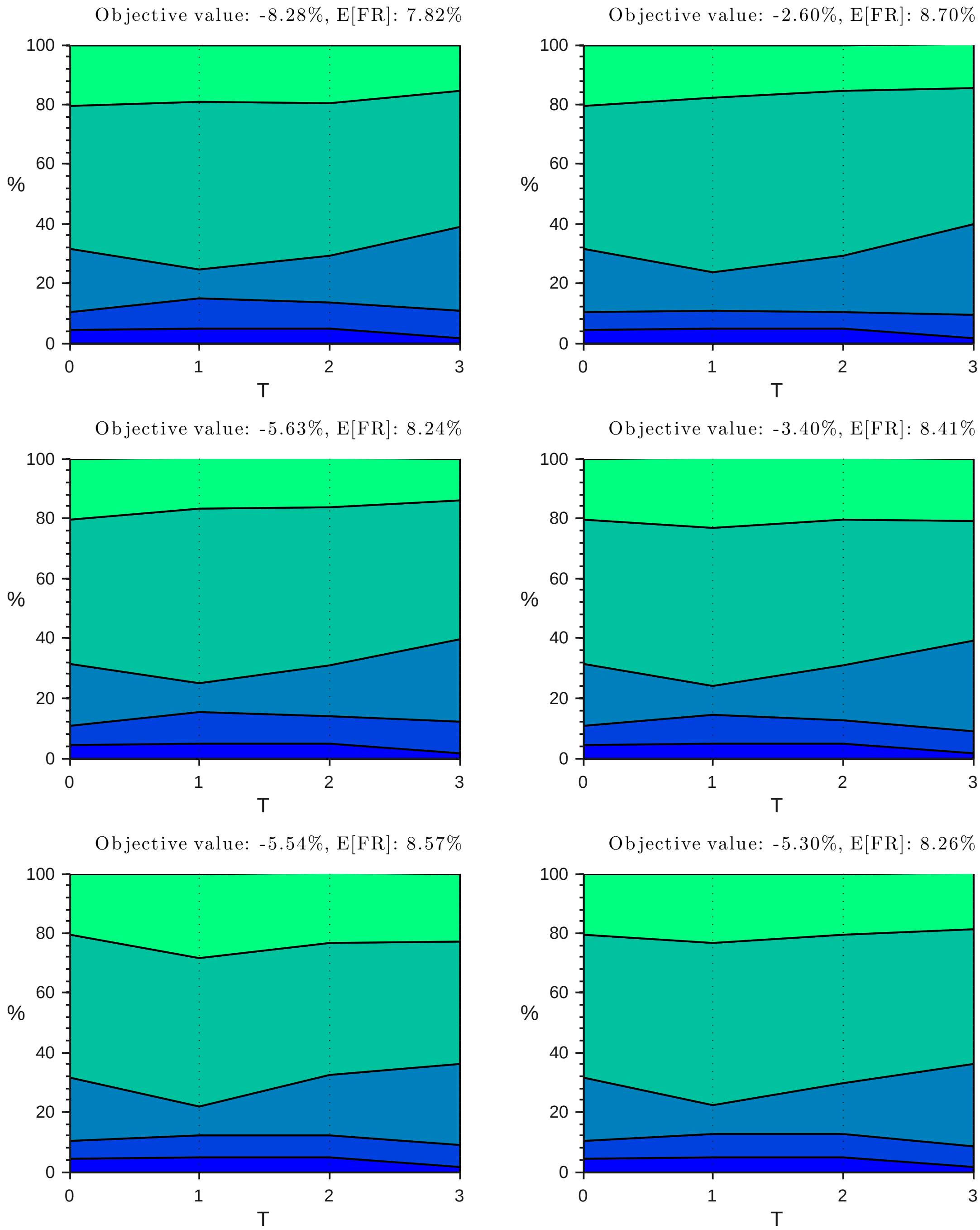

Figure 6.8. Expected asset allocation for the other six scenario trees, all with branching structure BS $= \{15, 15, 2\}$, with the following legend. (■) Cash, (■) domestic stocks, (■) foreign stocks, (■) real bonds and (■) nominal bonds

Compared to the solution, the expected asset allocation all follow mainly the same decreasing pattern in foreign stocks in the first year and then an increase in years two and three in favor of a decrease and increase in other asset classes respectively. Furthermore no dramatic changes are apparent.

6.3.2 Fixed-mix comparison

The performance of the multistage stochastic model is compared to three partly dynamic fixed-mix strategies. Fixed-mix strategies are simple investment decision rules where the asset portfolio is rebalanced to maintain fixed asset proportions. While such strategies are by no means realistic investment decision rules for pension funds, they are often for comparisons. The strategies are partly dynamic in the sense that the proportion of the wealth held in cash, stocks and bonds is fixed, but the proportion of the wealth invested in each sub class may vary. For clarification, the amount held in stocks is fixed but no restriction is made on the ratio between domestic and foreign stocks. The same holds for bonds and the ratio between real and nominal bonds. Three fixed-mix benchmark strategies are selected conservative, moderate and aggressive, with difference based on the proportion of the wealth held in bonds at each time. Ten scenario trees with branching structure BS $= \{15, 15, 2\}$ were used and the expected funding ratio at the horizon, as a function of the risk aversion components, is shown in figure 6.9 for all the strategies.

The asset liability management model outperforms the three benchmarks in terms of a higher funding ratio at the horizon for given value for the risk aversion components. Moreover no benchmark clearly outperforms the other. However, on average the conservative benchmark performs best, and the aggressive benchmark worst.

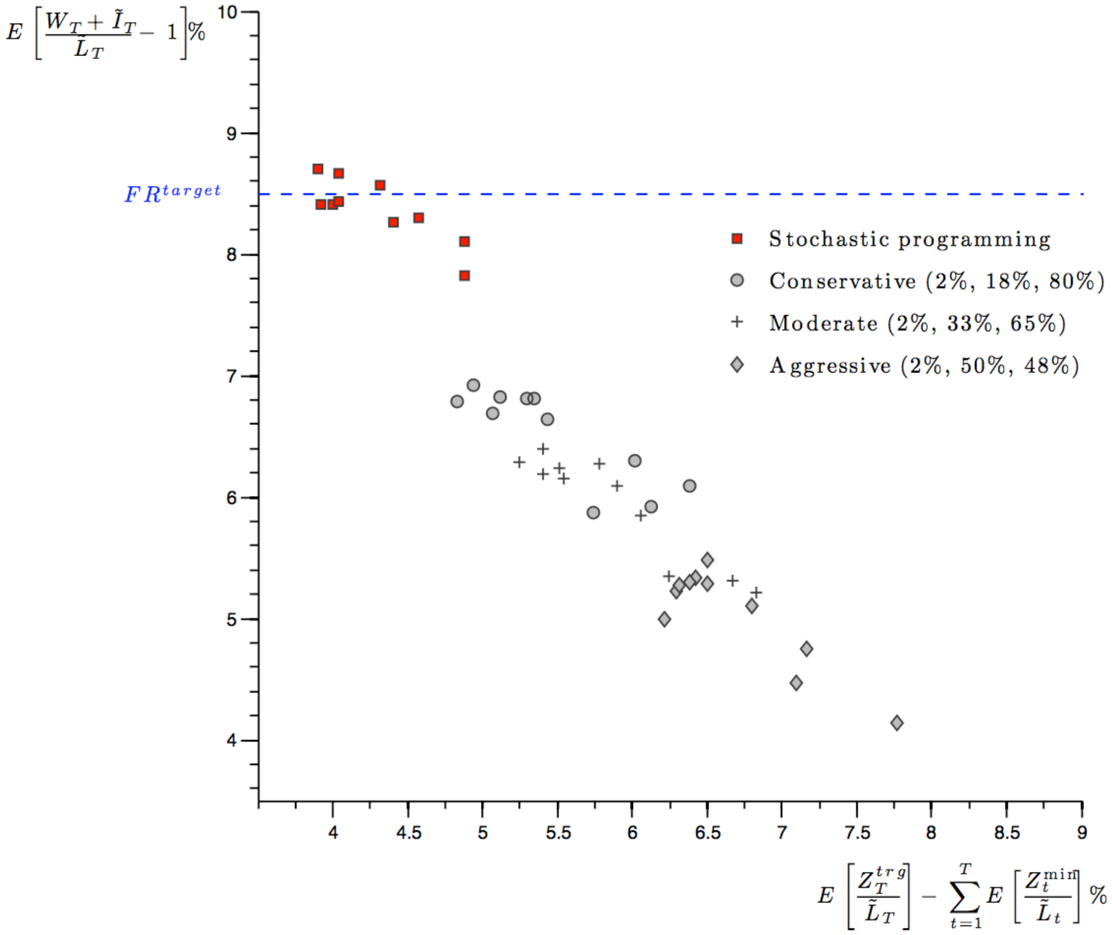

Figure 6.9. Performance of the mulistage stochastic model compared to three benchmark partly dynamic fixed-mix strategies, (%cash, % stocks, % bonds).

6.3.3 Sensitivity analysis

In this section the effects of altering parameters values in the asset liability management model, on the objective value and the asset allocation are analyzed. Due to the variation in the objective between data sets with the same tree structure, as shown in figures 6.6 and 6.9, only sensitivity analysis on the model and portfolio constraint parameters is conducted.

Figure 6.10 illustrates the effect, of altering the values of F^{target}, $F^{\min}$, u^p, λ_1 and λ_2 by $\pm 50\%$, on the expected funding ratio at the horizon. Lower values for the penalty parameters, λ_1 and λ_2, results in a lower funding ratio at the horizon. The reason for this is that lower values lead to smaller penalty for a funding ratio which falls below the minimum in each scenario and also for a funding ratio at the horizon which does not reach the target funding ratio at the horizon. This effect is reversed when the penalty parameters values are increased. Although the change in the funding ratio, due to changes in λ_1 and λ_2, at the horizon is not extreme[6], the values must be carefully estimated by the modeler. Furthermore the penalty parameter for a funding ratio which does not reach the target at the horizon, λ_2 has a greater effect on the expected funding ratio. Similar effects are revealed when the maximum amount for purchase, as a percentage of the wealth at each time (u^p), is altered. When lowered, which corresponds to more rigorous constraints on asset switch, the expected funding ratio at the horizon is lower. This is reversed when the maximum amount is increased. This underlines the necessity of constraints on asset switching, when they are absent, a to spurious funding ratio at the horizon and an overconfident solution may be obtained. The value, on u^p, should reflect the pension funds manager's will to change the short term investment policy. The effects of a smaller minimum funding ratio, $F^{\min}$, allowed without penalty in each scenario is a lower expected funding ratio at the horizon while an increase will lead to a higher expected funding ratio. Lastly, the effects of a higher target funding ratio has no effect on the expected funding ratio at the horizon which indicates that the stochastic programming model is unable to drive the expected funding ratio at the horizon further upwards in the presence of a more ambitious target.

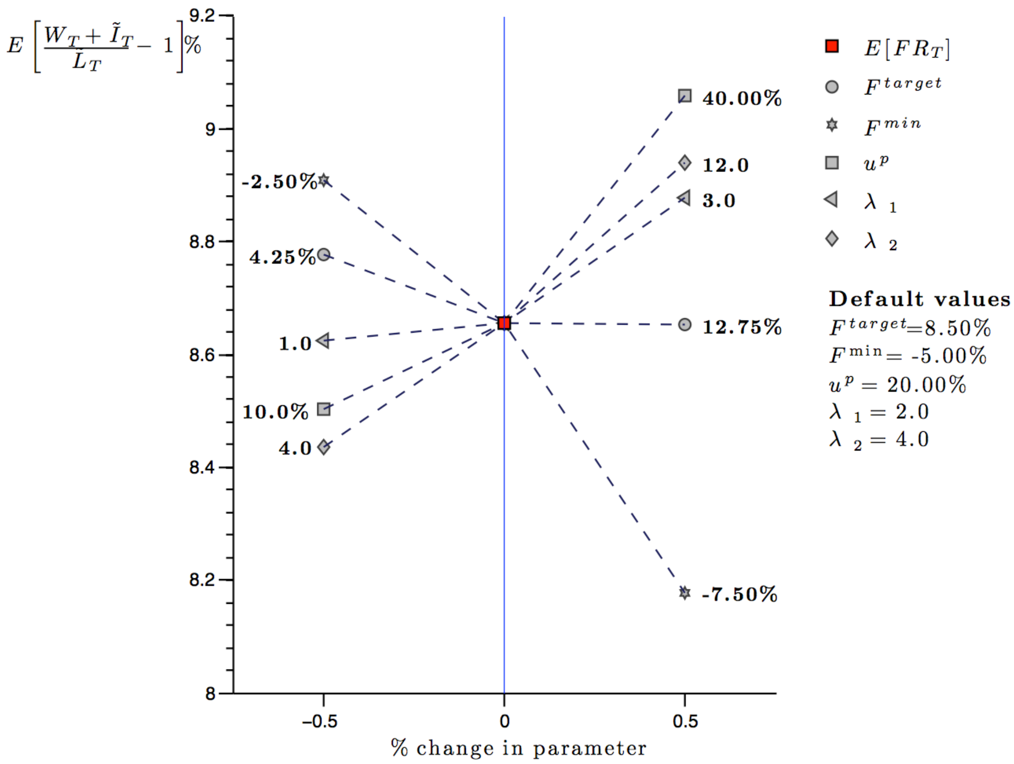

Figure 6.10. The effects, of altering the model parameters, on the expected funding ratio at the horizon.

[6]A maximum absolute difference of 0.27% in the funding ratio at the horizon.

The effects on the expected funding ratio at the horizon is not the only thing one needs to consider. We are also interested in analyzing the effects on the expected asset allocation when the values are altered. The effects, of altering values of the penalty parameters, on the expected asset allocation are presented in figure 6.11. When compared to figure (6.8), the expected asset allocation does not change dramatically. A decrease in foreign stocks is observed instead of an increase in nominal bonds in the first year and a increase in foreign stocks in the latter two years. In the case of λ_2 the effects are clearer in the third year.

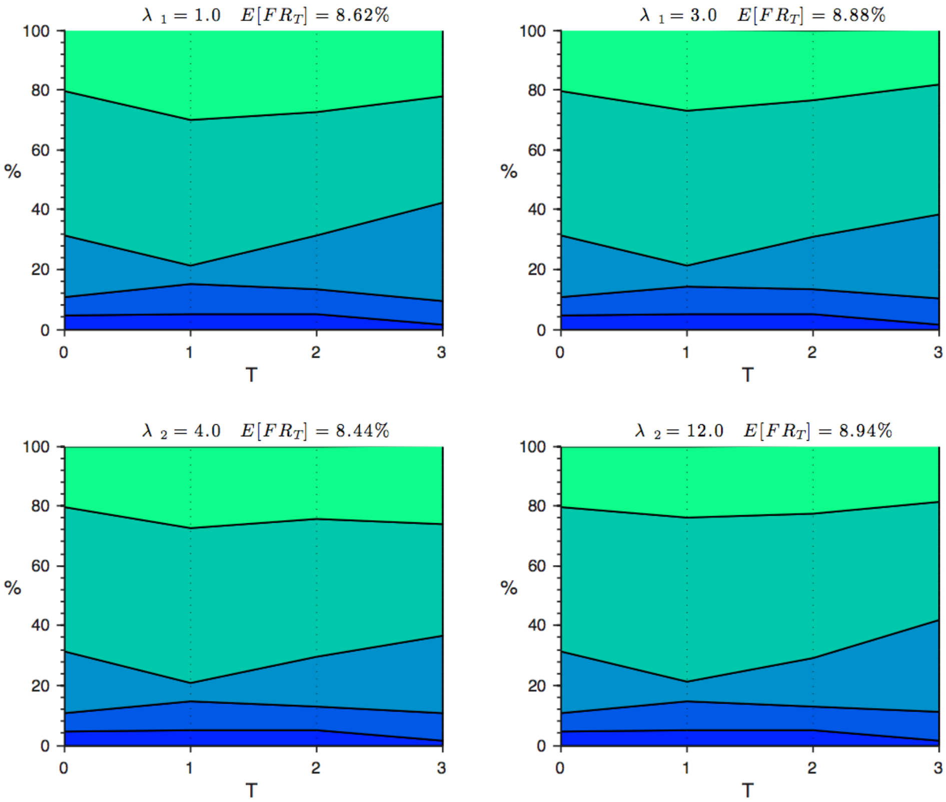

Figure 6.11. The effects, of altering values on penalty parameters, on the expected asset allocation, with the following legend. (■) Cash, (■) domestic stocks, (■) foreign stocks, (■) real bonds and (■) nominal bonds.

The effect of the other parameters on the expected asset allocation are displayed in figure 6.12. When compared to figure 6.8, only the values on the maximum allowed amount for purchase result in drastic changes in year three. This further strengthens the necessity for carefully chosen values for u^p. Furthermore this weakens the confidence in the optimal solution. However the drastic change appears when the maximum amount is set to 10% which might considered too conservative.

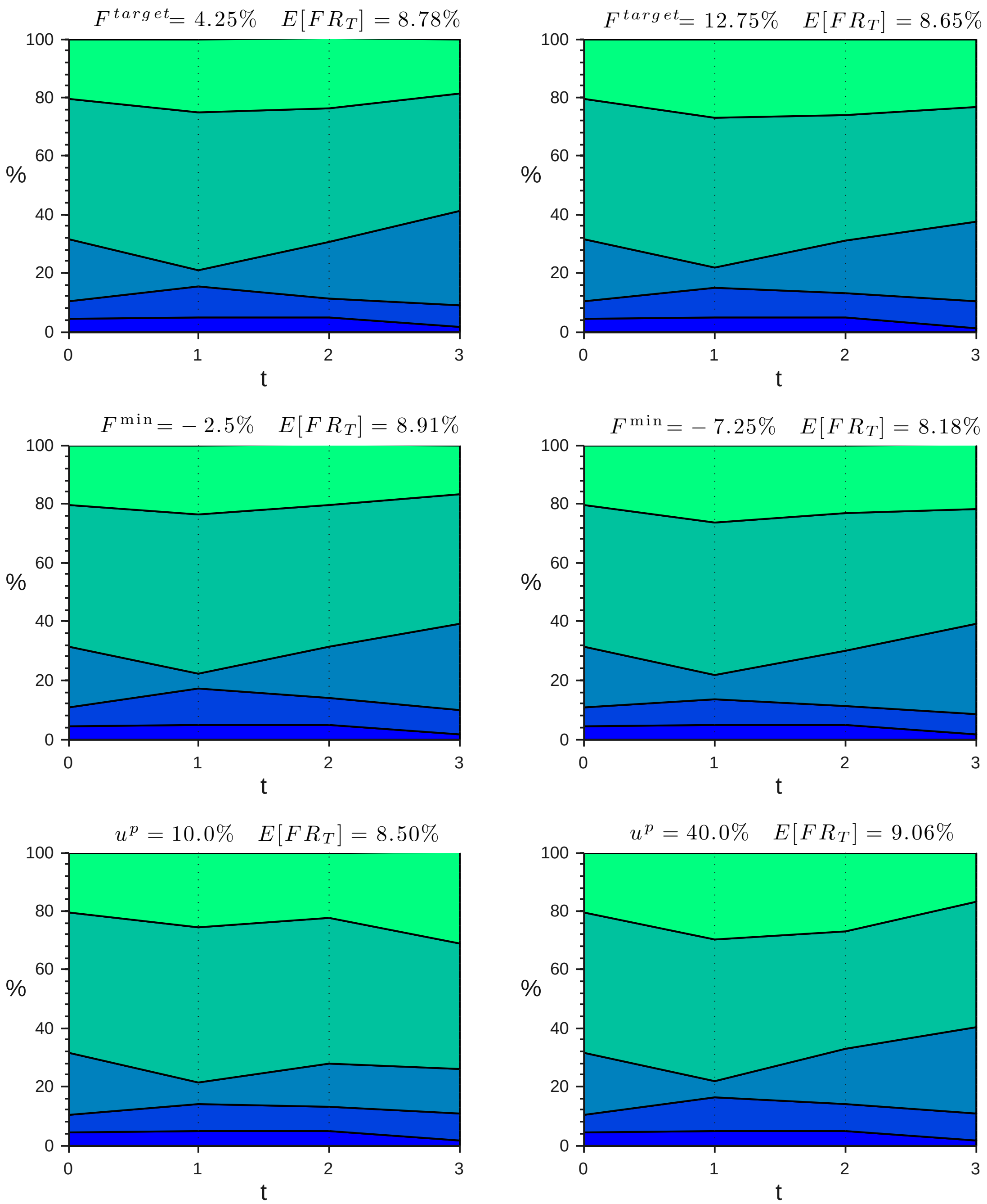

Figure 6.12. The effects, of altering values on $F^{\text{target}}, F^{\text{min}}, u^p$, on the expected asset allocation with the following legend. (■) Cash, (■) domestic stocks, (■) foreign stocks, (■) real bonds and (■) nominal bonds.

6.3.4 Fixed interest rate valuation

Computational experiments have been carried out using market valued liabilities and premiums. However, the Icelandic Financial Supervisory Authority (FME) base their monitoring on the total actuarial position (in %), where a fixed 3.5% interest rates are used to value liabilities, premiums and bonds. Icelandic pension funds which show poor actuarial position must reduce accrued pension rights to remedy the situation.

Under this supervisory framework, interest rate risk is not accounted for. To investigate the affects interest rate risk on the pension fund's asset allocation decisions, the asset liability management model was solved with resect to fixed interest rates. As previously discussed, the best performance in terms of the highest expected funding ratio at the horizon, was obtained when extremely high values for the penalty parameters were used. Therefore, the case where $(\lambda_1 \to \infty, \lambda_2 \to \infty)$ is included in the comparison. Figure 6.13 shows the resulting expected asset as well as the optimized expected wealth and expected funding ratio at the horizon, where panel (a) refers to the penalty values $(2, 8)$ and panel (b) refers to the extreme values (∞, ∞).

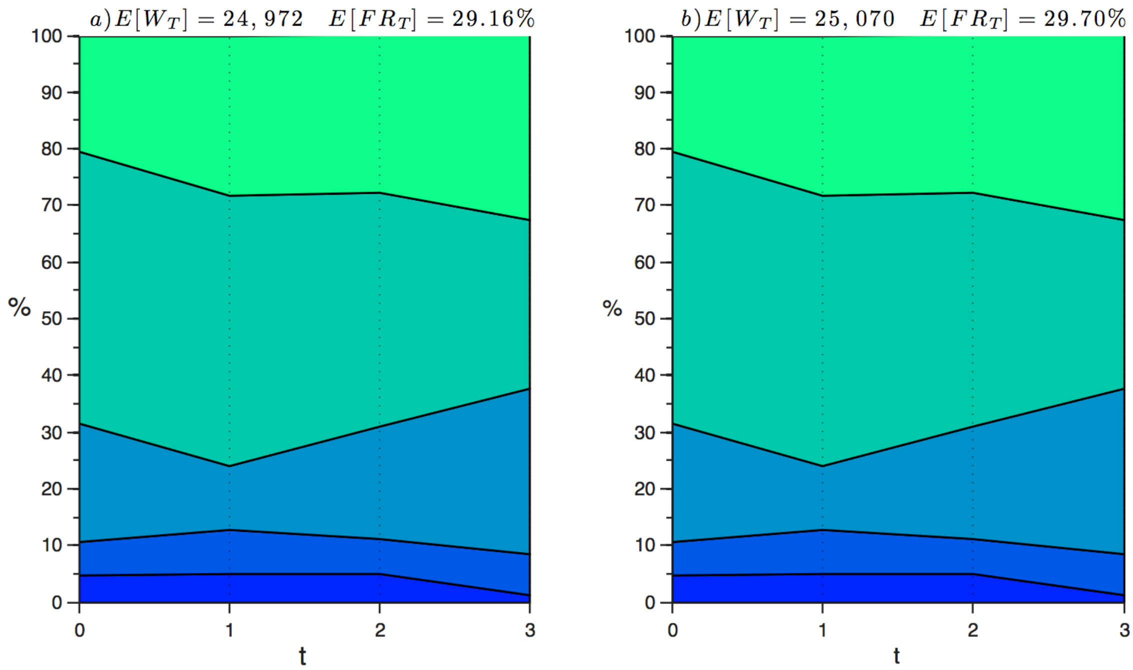

Figure 6.13. Optimized asset allocation as a result of fixed valued (at 3.5%) liabilities and premiums. Panel (a) corresponds to penalty parameters (2,8), whereas panel (b) corresponds to the extreme case (∞, ∞). $E[W_T]$ figures in thousand ISK. (■) Cash, (■) domestic stocks, (■) foreign stocks, (■) real bonds and (■) nominal bonds.

Compared to the solutions presented earlier, no tremendous change in terms of the optimized asset allocations takes place in the first two stages but in the third stage nominal bonds are favored at the cost of reduction in real bonds.

On the other hand, the funding ratio at the horizon are significantly higher. A staggering difference of 20.49% for penalty parameters $(2, 8)$ and 29.61% in the extreme case, (∞, ∞), is revealed. This raises concern on the credibility of the current risk measure used by the FME, especially since lower expected wealths were obtained in both cases when fixed rates were used. Therefore the significant increase in the funding ratio cannot be justified in terms of extensive growth in the pension funds wealth. The results indicates that future liabilities are underestimated in terms of interest rate risk. This underestimation leads to overconfidence in the pension fund's ability to meat their future liabilities, which have serious consequences for future retirees.

However, two important things are worth mentioning. The significantly higher expected funding ratios obtained at the horizon when fixed valuation is used could have neutralized the penalty functions. This might affect the resulting asset allocations. Secondly, in this comparison bonds were valued using the real yield curve in each scenarios whereas bond are currently valued with fixed interest rates. This affects the expected wealth at the horizon.

Chapter 7. Conclusions and future work

7.1 Conclusions

The extended K-means scenario generation method proved to be the weakest link in the modeling framework. Due to its limited ability to generate large scenario trees, in-sample stability analysis indicates that the objective value does not reach satisfactory stability when number of scenarios is increased to the maximum capability of the scenario generator. Although fluctuations in the objective decreased when the number of scenarios was increased, they still have considerable impact on the solution. Furthermore, the objective value also varied for different scenario trees (different data) with the same structure, although this variability is greatly reduced when the number of the scenarios is increased from 240 to 450.

The instability can result from one of two reasons. Either the scenario generator is unable to achieve satisfactory in-sample stability performance, or the limitations on the size of the scenario trees plays a major role. In the latter case, possible stability could be obtained if the scenario generation method could produce a tree with a considerably larger number of scenarios. However, it is difficult to favor one reason over the other at this stage.

Despite the relatively small number of scenarios, the computational experiments proposed revealed important results. They showed that the optimized asset allocations outperformed the three partly dynamic fixed-mix investment strategies. The basis for the comparison was the funding ratio at the horizon which measures the ability of the fund to meet future pension payments (liabilities). The three fixed-mix strategies used for comparison served as a benchmark for conservative, moderate and aggressive investment strategies, in respect to the amount held in domestic government bonds.

Analysis of the credibility of the optimized asset allocation suggests that no spurious allocation changes occur between stages. This attribute is essential if asset allocation decisions are to be made with regard to the optimal asset allocation suggested by the model. In the absence of this property, the confidence in the optimized asset allocations is severely weakened. However, given its importance, surprisingly little attention has been to paid to such diagnostics in the literature.

From a modeling point of view, the best performance in terms of the highest funding ratio was obtained when extremely high values for the penalty parameters were used. The high values correspond to low tolerance of violation of the pension fund risk aversion. The risk aversion of the fund was modeled with two penalty functions, one penalizing for low funding ratios and the other for funding ratios that do not reach a pre-specified target at the horizon. In addition, sensitivity diagnostics revealed that the most important constraint in the optimization model was the maximum amount allowed on purchases. This was measured by altering the parameter and examining the resulting change in the expected funding ratio at the horizon. This underlines the necessity of a carefully chosen value for the parameter. In theory, the value should reflect the tolerance and the will of the pension fund manager to change the investment strategy. A more conservative tolerance for change results in more rigorous constraints.

A Comparison between market-valued liabilities and premiums as opposed to valuation using fixed interest rates was made. This comparison revealed that when fixed interest rates are used, an overconfident funding ratio at the horizon is obtained. The fixed valued funding ratio differs from the one currently used by the Icelandic Financial Supervisory Authority. The difference is that bonds are currently valued at fixed interest whereas market-valued bonds were used. However the results suggest that current risk measures severely underestimate interest rate risk. Although that pension fund managers are aware of interest rate risk, the current supervisory framework is based on the funding ratio obtained when liabilities, premiums and bonds are valued at fixed interest rates. This might cause an imbalance since pension funds are required to reduce pension payments to current pensioners if the funding ratio falls below -10% in one year or -5% in five consecutive years. As a result of an overestimated funding ratio, current pension payments are not reduced as much as they should be. This might suggest that current pensioners are receiving too high pension payments as a result of bad risk measures.

Finally, it is worth emphasizing that the conclusions are based on the fact that market liquidity is readily available. Therefore it is assumed that the pension fund can at all times alter its allocation in all asset classes.

7.2 Future work

Despite the extensions made to the modified K-means scenario generation method, future work and further modifications are required. One suggestion is to break the method into two phases: a simulation phase and a clustering phase. The two phases are then executed in an alternating manner. In the first stage all paths are simulated from the root cluster and then bundled into cluster according to the branching structure, as before. At the second stage, the time series models are reset in accordance with each cluster and thereafter paths are simulated from each cluster one cluster at a time. After the paths have been simulated from the first cluster in the second stage, the paths are immediately bundled into clusters which form the first part of the third stage clustering. This is then done for all clusters which completes the clustering in the third stage. The alternating procedure is then repeated until the horizon. This enables the scenario generation algorithm to construct scenario trees of unlimited size. In addition to modifications, a comparison with other scenario generation methods in the literature in terms of computational efficiency and stability would be most interesting.

If improvements on in-sample stability can be obtained, a more detailed sensitivity analysis could be performed. This would be beneficial for analyzing the effects of altering the values of estimated parameters in the models for the underlying stochastic variables, on the optimal expected funding ratio at the horizon. For example, analysis on the real yield curve model which plays a major role in the study could be performed.

Further research on possible underestimation of interest rate risk on the current actuarial position valuation used by the Icelandic Financial Supervisory Authority is required. Modifications to model the current valuation method can be made by incorporate fixed rate valued bonds.

A systematic comparison between the scenario based recourse formulation of stochastic programs and dynamic programming methods needs to be undertaken. Little attention has been paid to such a comparison in the literature and therefore there is room for such a study.

Appendix I

An example of the evolution of the modified K-means method is given in figure 7.1 for construction of a scenario tree with branching structure $\mathbb{B} = \{1, 4, 2, 2\}$ and four stages, $\mathcal{T} = \{0, 2, 4, 8\}$. In panel (a) a simulated one dimensional scenario fan with $N = 50$ individual paths is shown. In panel (b) the result of the first stage clustering is shown, where the clusters are marked with red circles. Panel (c) and (d) show the result of the second and third stage clustering.

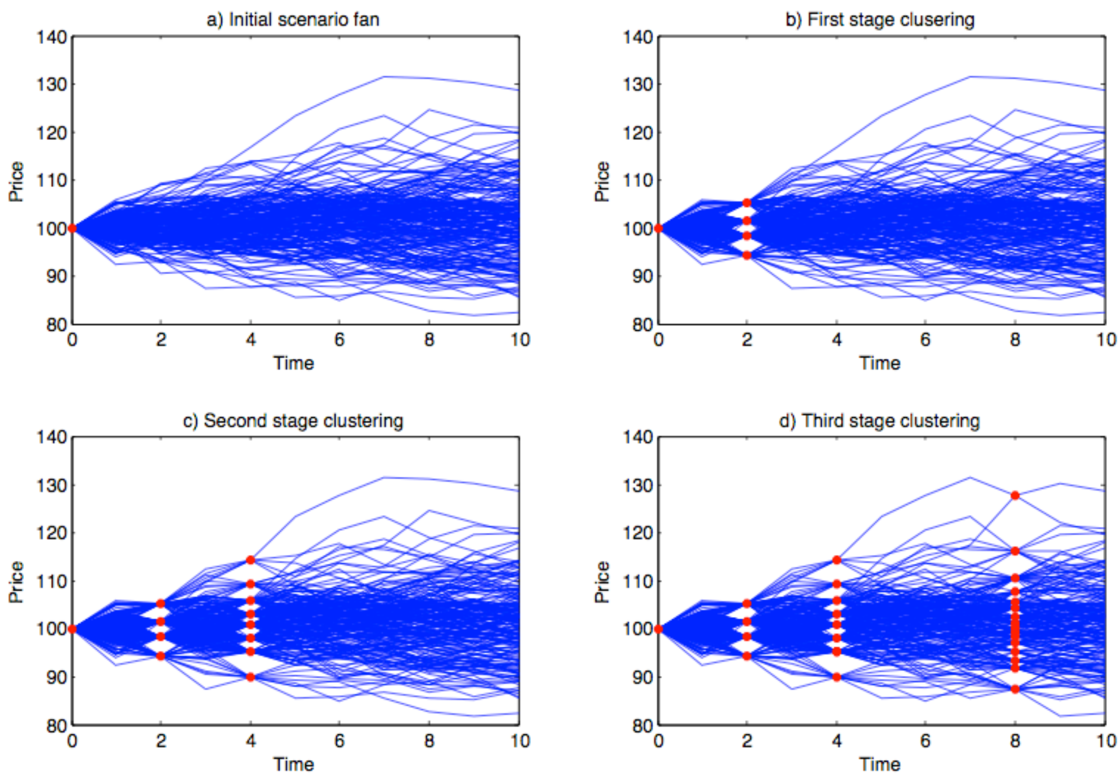

Figure 7.1. Evolution of the modified K-means clustering method.

In figure 7.2 the resulting scenario tree is represented, where the red branches denote the connection between each cluster and its sub custers.

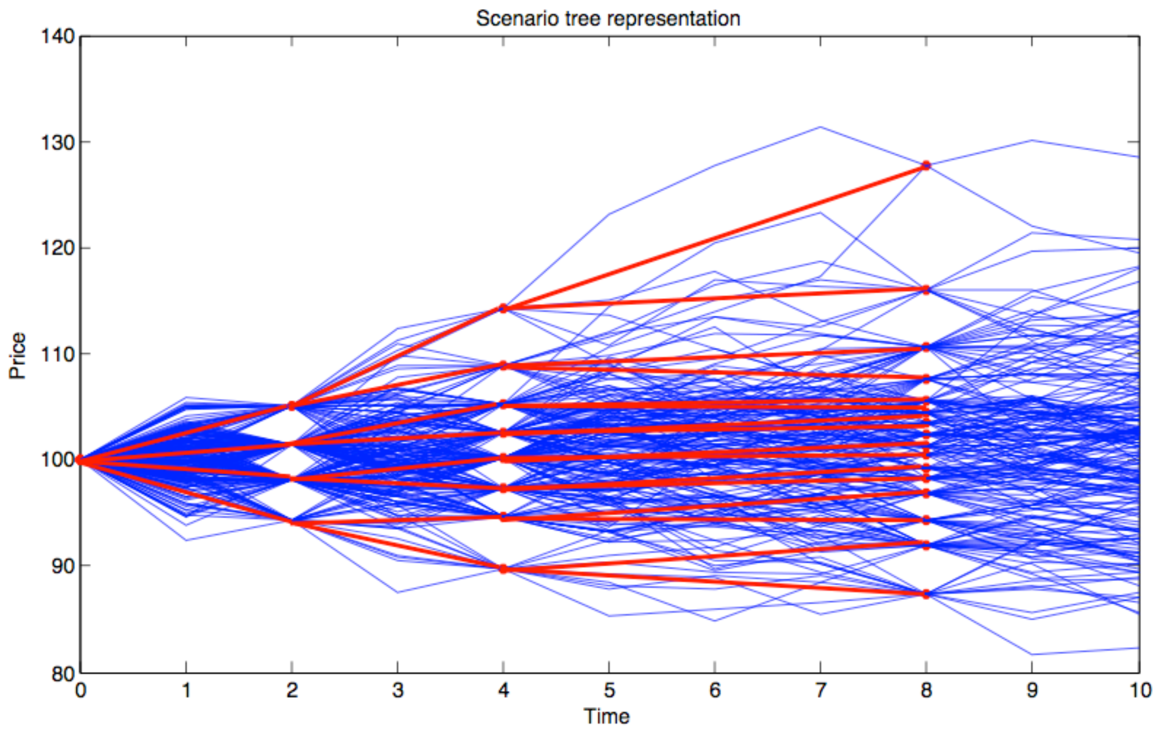

Figure 7.2. Scenario tree representation.

Appendix II

Summary of the asset liability model introduced in chapter 3.

Parameters:

h_j^0 : Known initial holdings in asset j in stage $t = 0$.

$R_{t,j}$: Return of asset j in stage t.

c^q : Transaction cost for asset class q.

L_t: Liability at time t.

I_t: Premium at time t.

$\tilde{L}_t$: Present value of future liability payments in stage t.

$\tilde{I}_t$: Present value of future premiums in stage t.

l^q : Lower bound for holdings in asset class q.

u^q : Upper bound for holdings in asset class q.

u^p : Upper bound for purchase.

Decision variables

$h_{t,j}$: Holdings in asset j in stage t.

$p_{t,j}$: Purchase in asset j in stage t.

$s_{t,j}$: Sales in asset j in stage t.

W_t : Wealth in stage t.

Z_T^{target} : Penalty for deficit on the funding ratio at the horizon.

$Z_t^{\min}$: Penalty for funding ratio below the minimum in stage t.

$$
\begin{aligned}
\max_{\boldsymbol{h},\boldsymbol{p},\boldsymbol{s},\boldsymbol{W},\boldsymbol{Z}} \quad & \mathbb{E}\left[\left(\frac{W_{T,s}+\tilde{I}_{T,s}}{\tilde{L}_{T,s}}-1\right)-\lambda_1\left(\frac{Z_{T,s}^{\text{target}}}{\tilde{L}_{T,s}}\right)-\lambda_2\sum_{t=1}^{T}\left(\frac{Z_{t,s}^{\text{min}}}{\tilde{L}_{t,s}}\right)\right] \\
\text{subject to} \quad & \\
& h_{t,j} = h_{t-1,j}(1+R_{t,j}) + p_{t,j} - s_{t_i,j} \\
& \sum_{j\in\mathcal{J}}(1+c^q)p_{0,j} = \sum_{j\in\mathcal{J}}(1-c^q)s_{0,j} \\
& \sum_{j\in\mathcal{J}}(1+c^q)p_{t,j} + L_t = \sum_{j\in\mathcal{J}}(1-c^q)s_{t,j} + I_t \\
& l_j\sum_{j\in\mathcal{J}}h_{t,j} \le h_{t,j} \le u_j\sum_{j\in\mathcal{J}}h_{t,j} \\
& p_{t,j} \le u^p\sum_{j\in\mathcal{J}}h_{t,j} \\
& W_t = W_{t-1} + \sum_{j\in\mathcal{J}}(1+R_{j,t})h_{t,j-1} \\
& (W_T+\tilde{I}_T) \ge (1+FR^{\text{target}})\tilde{L}_T - Z_T^{\text{target}} \\
& (W_t+\tilde{I}_t) \ge (1+FR^{\text{min}})\tilde{L}_t - Z_t^{\text{min}} \\
& h_{t,j}^{\lfloor s\rfloor} = h_{t,j}^{\lfloor s'\rfloor}, \qquad p_{t,j}^{\lfloor s\rfloor} = p_{t,j}^{\lfloor s'\rfloor}, \qquad s_{t,j}^{\lfloor s\rfloor} = s_{t,j}^{\lfloor s'\rfloor} \\
& h_0^j \ge 0, h_{t,j} \ge 0, \quad , p_{t,j} \ge 0, \quad , s_{t,j} \ge 0, \quad Z_T^{\text{target}} \ge 0, Z_T^{\text{min}} \ge 0 \\
& \text{for} \quad j\in\mathcal{J}, \quad \{0,t,T\}\in\mathcal{T}, \quad q\in\mathcal{Q} \quad \text{and} \quad s=s'
\end{aligned}
$$

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