On a generalization of Pfister forms

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ON A GENERALIZATION OF PFISTER FORMS

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Magister Scientiarum degree in Mathematics

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Abstract

If K is a field of characteristic 2, then the Pfister forms over K have a certain connection with the groups $H_2^{n+1}(K)$. We will introduce a natural generalization of these Pfister forms to forms of degree p over fields of characteristic p, and see whether these generalizations have a similar connection with the groups $H_p^{n+1}(K)$. It turns out that this is not the case and we provide a counterexample at the end. There remains the question whether our interpretation of "similar connection" is wrong or whether our generalization of Pfister forms is not the right one to find such a connection. Then there remains the possibility that there is no such generalization.

Also, since $H_p^2(K)$ relates to the central simple algebras $[a, b)_K$, we will do some calculations of the reduced norm of $[a, b)_K$.

Útdráttur

Ef K er kroppur með kennitölu 2, þá hafa Pfister formin yfir K ákveðna tengingu við grúpurnar $H_2^{n+1}(K)$. Við munum innleiða náttúrlega útvíkkun á þessum Pfister formum í form af gráðu p yfir kroppa með kennitölu p, og sjá hvort þessar útvíkkanir hafa svipaða tengingu við grúpurnar $H_p^{n+1}(K)$. Svo reynist ekki vera og við setjum fram mótdæmi í lok ritgerðarinnar. Enn er þó ósvarað hvort okkar túlkun á „svipaðri tengingu" er röng eða hvort okkar útvíkkun á Pfister formum er ekki sú rétta til að finna slíka tengingu. Síðan er eftir sá möguleiki að engin slík útvíkkun sé til.

Þar sem $H_p^2(K)$ tengjast miðlægu einföldu algebrunum $[a, b)_K$ munum við einnig framkvæma nokkra útreikninga á smækkaða normi $[a, b)_K$.

Contents

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1 Introduction

1.1 Pfister forms

In the study of quadratic forms over a field K, certain forms, the Pfister forms, play a special role. An n-fold Pfister form is a particular kind of quadratic form in 2^n variables over K. If the characteristic of K does not equal 2 then an n-fold Pfister form with variables $x_{\varepsilon_1,\ldots,\varepsilon_n}$ can be described as a sum $\sum_{\varepsilon_1,\ldots,\varepsilon_n=0}^{1} a_1^{\varepsilon_1}\cdots a_n^{\varepsilon_n} x_{\varepsilon_1,\ldots,\varepsilon_n}^2$, where $a_i \in K^*$. These Pfister forms relate to, amongst other things, Galois cohomology.

In the case that the characteristic of K does equal 2, an $(n+1)$-fold Pfister form can be described as a direct sum $\bigoplus_{\varepsilon_1,\ldots,\varepsilon_n=0}^{1} b_1^{\varepsilon_1}\cdots b_n^{\varepsilon_n}[1,a]$, where $a \in K$, $b_i \in K^*$, and $[1,a]$ is the form mapping $(x,y) \in K \times K$ to $x^2 + xy + ay^2$. Note that $[1,a]$ can also be described as the norm form $N_a : K[\tau] \to K$, where τ is the image of X in $K[X]/(X^2 - X - a)$. In this case Pfister forms have a relation with the groups $H_2^{n+1}(K)$, which we will try to generalize to arbitrary positive characteristic.

In what follows K is always a field of characteristic $p > 0$. For $a \in K$ we let τ be the image of X in $K[X]/(X^p - X - a)$ and denote by N_a the norm $K[\tau] \to K$. For $b_1, \ldots, b_n \in K^*$ we then define the generalized Pfister form $[[a, b_1, \ldots b_n\rangle\rangle$ as

$$\bigoplus_{\varepsilon_1,\ldots,\varepsilon_n=0}^{p-1} b_1^{\varepsilon_1}\cdots b_n^{\varepsilon_n} N_a$$

Note that $[[a, b_1, \ldots b_n\rangle\rangle$ is a form of degree p in p^{n+1} variables.

1.2 The groups $H_p^{n+1}(K)$

To define the groups $H_p^{n+1}(K)$ we use differential forms over K. One can define the space Ω_K^1 of 1-fold differential forms by considering the free K-module generated by the symbols dx, with $x \in K$, and dividing out the submodule generated by elements of the form dz, $d(x_1+x_2)-dx_1-dx_2$ and $d(x_1x_2)-x_1dx_2-x_2dx_1$, with $z \in \mathbb{F}_p$ and $x_1, x_2 \in K$. One can then define the space Ω_K^n of n-fold differential forms over K by taking the n-fold wedge product of Ω_K^1. We also define Ω_K^0 to be K. Given these spaces of differential forms we have a K-linear map $d : \Omega_K^n \to \Omega_K^{n+1}$, called the exterior derivative, that satisfies the following conditions: $d(x) = dx$ for any $x \in K$, $d^2 = 0$ and $d(\alpha \wedge \beta) = d\alpha \wedge \beta + (-1)^k(\alpha \wedge d\beta)$ if α is a k-form and β is an m-form.

So we let Ω_K^n be the space of n-fold differential forms over K. Then there is a well-defined Artin-Schreier operator $\wp : \Omega_K^n \to \Omega_K^n / d\Omega_K^{n-1}$ mapping the form $x\frac{dx_1}{x_1} \wedge \cdots \wedge \frac{dx_n}{x_n}$ to the class of the form $(x^p - x)\frac{dx_1}{x_1} \wedge \cdots \wedge \frac{dx_n}{x_n}$ (cf. [Mi], §1). The group $H_p^{n+1}(K)$ is defined as the cokernel of this operator. We shall denote by $\left[x\frac{dx_1}{x_1} \wedge \cdots \wedge \frac{dx_n}{x_n}\right]$ the class of $x\frac{dx_1}{x_1} \wedge \cdots \wedge \frac{dx_n}{x_n}$ in $H_p^{n+1}(K)$.

Kato has done much work on these groups, for instance in [Ka]. Also, in [Iz] Izhboldin interprets the groups $H_p^{n+1}(K)$ as the p-part of the Galois cohomology of K. The groups $H_p^{n+1}(K)$ can then serve as substitutes for certain Galois cohomology groups.

2 A problem

We have the following theorem, proven in [AB]:

Theorem: Let $a \in K$ and $b_1, \dots, b_n \in K^*$ be given. Let E be the set of all $\underline{\varepsilon} = (\varepsilon_1, \dots, \varepsilon_n)$ with $0 \le \varepsilon_1, \dots, \varepsilon_n \le p-1$ and write $\underline{0} = (0, \dots, 0) \in E$. Then $\left[a\frac{db_1}{b_1} \wedge \cdots \wedge \frac{db_n}{b_n}\right] = 0$ in $H_p^{n+1}(K)$ if and only if the form

$$\sum_{\underline{\varepsilon} \in E} b_1^{\varepsilon_1} \cdots b_n^{\varepsilon_n} x_{\underline{\varepsilon}}^p - x_{\underline{0}} y^{p-1} - ay^p \tag{2.1}$$

of degree p in the $p^n + 1$ variables $(x_{\underline{\varepsilon}})_{\underline{\varepsilon} \in E}$ and y has a nontrivial K-rational zero. $\square$

To see how this relates to $[[a, b_1, \dots b_n\rangle\rangle$ we calculate $N_a(x + y\tau)$ for $x, y \in K$. This is equal to the determinant

$$\begin{aligned}
\begin{vmatrix} x & 0 & 0 & \cdots & 0 & ay \\ y & x & 0 & \cdots & 0 & y \\ 0 & y & x & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & x & 0 \\ 0 & 0 & 0 & \cdots & y & x \end{vmatrix}
&= x \begin{vmatrix} x & 0 & \cdots & 0 & y \\ y & x & \ddots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & x & 0 \\ 0 & 0 & \cdots & y & x \end{vmatrix} + ay \begin{vmatrix} y & x & \cdots & 0 & 0 \\ 0 & y & \ddots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & y & x \\ 0 & 0 & \cdots & 0 & y \end{vmatrix} \\
&= x^2 \begin{vmatrix} x & 0 & \cdots & 0 \\ y & x & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & x \end{vmatrix} - xy \begin{vmatrix} y & x & \cdots & 0 \\ 0 & y & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & y \end{vmatrix} + ay^p \\
&= x^p - xy^{p-1} + ay^p
\end{aligned}$$

So $N_a(x + y\tau) = x^p - xy^{p-1} + ay^p$, and more specifically $N_a(x) = x^p$, which means that the form (2.1) can be written as

$$\sum_{\underline{\varepsilon} \in E, \underline{\varepsilon} \neq \underline{0}} b_1^{\varepsilon_1} \cdots b_n^{\varepsilon_n} N_a(x_{\underline{\varepsilon}}) + N_a(x_{\underline{0}} - y\tau)$$

We see that this form is a subform of the form $[[a, b_1, \dots b_n\rangle\rangle$, since written with variables, $[[a, b_1, \dots b_n\rangle\rangle$ is

$$\sum_{\varepsilon \in E} b_1^{\varepsilon_1} \cdots b_n^{\varepsilon_n} N_a(x_{\varepsilon,0} + x_{\varepsilon,1}\tau + \cdots + x_{\varepsilon,p-1}\tau^{p-1})$$

If the subform is isotropic (and equivalently $\left[a\frac{db_1}{b_1} \wedge \cdots \wedge \frac{db_n}{b_n}\right] = 0$) then, of course, the original form $[[a, b_1, \dots b_n\rangle\rangle$ is also isotropic (a form being isotropic means that it has a nontrivial zero). The question is whether the converse holds. In [Ka] Kato proves that this is true when $p = 2$. In this essay we will see whether this holds for arbitrary p or not.

2.1 The case $n = 0$

We start by considering the case $n = 0$. Then we just have $N_a(x_0 - y\tau)$ and $N_a(x_0 + x_1\tau + \cdots + x_{p-1}\tau^{p-1})$. Now if $N_a(x_0 + x_1\tau + \cdots + x_{p-1}\tau^{p-1})$ has a nontrivial zero, then $K[\tau]$ is not a field, which means that $X^p - X - a$ is reducible. But then, if we let s be a root of $X^p - X - a$, we see that $s + 1, \dots, s + p - 1$ are also roots, so $X^p - X - a$ factors linearly over $K(s)$. So if f is a factor of $X^p - X - a$ that has degree k, $0 < k < p$, then the coefficient of X^{k-1} in f is $-ks - (u_1 + \cdots + u_k)$, with $\{u_1, \dots, u_k\} \subset \{0, 1, \dots, p-1\} \subseteq K$, so we get that $s \in K$. But then we have that $N_a(s - \tau) = s^p - s - a = 0$. This proves our conjecture in the case $n = 0$.

The next thing to consider is the case $n = 1$. In this case we have a connection with central simple algebras.

3 The case $n = 1$ and central simple algebras

We recall that a K-algebra A is said to be *simple* if it has no two-sided ideals other than 0 and A. Furthermore A is said to be *central* if its center equals K. Central simple algebras have many interesting attributes, especially the reduced norm function, which we will do some calculations of in this chapter. For more on central simple algebras we refer to [Ja] or [GS].

3.1 The central simple algebra $[a, b)_K$

Elements $a \in K$ and $b \in K^*$ determine a central simple algebra $[a, b)_K$ of degree p over K. As a K-algebra it is generated by two elements x and y with the defining relations $x^p - x = a$, $y^p = b$ and $yx = (x + 1)y$. Note that $L = K[x]$ is isomorphic to $K[\tau]$ and is a commutative K-algebra. It is in fact a cyclic étale over K, a generator ρ of the Galois group being determined by $x \mapsto x + 1$ (we recall that a cyclic étale is a cyclic Galois extension of K if it is a field, otherwise it is isomorphic to $K[X]/(X^p)$). Using L and ρ the algebra $[a, b)_K$ can be described as the direct sum $L \oplus Ly \oplus \cdots \oplus Ly^{p-1}$, where $y^p = b$ and $yw = \rho(w)y$ for every $w \in L$.

These cyclic p-algebras relate to our problem in the case $n = 1$ since the group $H^2_p(K)$ is isomorphic to $\mathrm{Br}_p(K)$, the subgroup of elements of order 1 or p in the Brauer group of K (a group of certain equivalance classes of central simple algebras over K, cf. [Ja] or [GS]), where the elements $\left[a\frac{db}{b}\right]$ of $H^2_p(K)$ correspond to the classes of the cyclic p-algebras $[a, b)_K$ over K [Ka]. So instead of our initial question we could ask whether $[[a, b\rangle\rangle$ being isotropic is equivalent to $[a, b)_K$ not being a division algebra. An effective tool for deciding whether a central simple algebra A over K is a division algebra or not is the reduced norm $N_A : A \to K$ on A. N_A being isotropic is equivalent to A not being a division algebra. We will show one way to compute the reduced norm, but for more detail we refer to [GS].

Let $N_a : L \to K$ be the norm form on L. In the case $p = 2$ the reduced norm on $[a, b)_K$ is given by $[[a, b\rangle\rangle = N_a \oplus bN_a$, corresponding to the decomposition $[a, b)_K =$

$L \oplus Ly$. This shows that our conjecture is true in this case. One can ask whether for arbitrary p the reduced norm on $[a, b)_K$ is given by $[[a, b\rangle\rangle = N_a \oplus bN_a \oplus \cdots \oplus b^{p-1} N_a$, corresponding to the decomposition $[a, b)_K = L \oplus Ly \oplus \cdots \oplus Ly^{p-1}$. If this was true it would solve our problem in the case $n = 1$.

One way to compute the reduced norm on $[a, b)_K$ is as follows: Choose a field extension K' of K such that $[a, b)_{K'} = K' \otimes_K [a, b)_K$ splits. Choose an isomorphism $[a, b)_{K'} \cong M_p(K')$ of K'-algebras, where $M_p(K')$ is the algebra of $p \times p$ matrices over K'. Then the composition of the reduced norm on $[a, b)_K$ and the inclusion $K \to K'$ equals the composition of the inclusion $[a, b)_K \to [a, b)_{K'}$, the isomorphism $[a, b)_{K'} \cong M_p(K')$ and the determinant $M_p(K') \to K'$.

We can, for example, let $K' = K(\alpha)$, with α a root of $t^p - t - a$. We then get an isomorphism $[a, b)_{K'} \cong M_p(K')$ by mapping x to the diagonal matrix X with diagonal elements $\alpha, \alpha + 1, \ldots, \alpha + p + 1$ and by mapping y to the transpose

$$Y = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ 0 & 0 & \cdots & 0 & 1 \\ b & 0 & \cdots & 0 & 0 \end{bmatrix}$$

of the companion matrix of the polynomial $t^p - b$. In this case, $L' = K'(x)$ corresponds to the subalgebra $\mathcal{D}'$ of diagonal matrices in $M_p(K')$ and the norm form $N_a : L' \to K'$ corresponds to the restriction of the determinant. As $\det(Y) = b$, it follows that the question of whether the reduced norm on $[a, b)_K$ is given by $[[a, b\rangle\rangle$ means whether in general

$$\begin{aligned} & \det(D_0 + D_1 Y + \cdots + D_{p-1} Y^{p-1}) \\ = \; & \det(D_0) + \det(D_1)\det(Y) + \cdots + \det(D_{p-1})\det(Y)^{p-1} \\ = \; & \det(D_0) + b\det(D_1) + \cdots + b^{p-1}\det(D_{p-1}) \end{aligned}$$

for any diagonal matrices $D_0, D_1, \ldots, D_{p-1}$.

In the case $p = 3$ easy computations show that the determinant of the matrix

$$D(u_1, u_2, u_3) + D(v_1, v_2, v_3)Y + D(w_1, w_2, w_3)Y^2$$

equals

$$u_1 u_2 u_3 + b v_1 v_2 v_3 + b^2 w_1 w_2 w_3 - b(u_1 v_3 w_2 + u_2 v_1 w_3 + u_3 v_2 w_1)$$

Here we have denoted by $D(x_1, x_2, x_3)$ the diagonal matrix with diagonal elements x_1, x_2, x_3. We see that in the case $p = 3$, the reduced norm on $[a, b)_K$ is not given by $N_a \oplus bN_a \oplus b^2 N_a = [[a, b\rangle\rangle$. To get a specific example over K, we look at the reduced norm of the element $x^2 + y + y^2$ in $[a, b)_K$. Then we have $u_1 = \alpha^2, u_2 = (\alpha+1)^2, u_3 = (\alpha-1)^2$ and $v_i, w_i = 1$, so we get

$$\begin{aligned}
& \alpha^2(\alpha+1)^2(\alpha-1)^2 + b + b^2 - b(\alpha^2 + (\alpha+1)^2 + (\alpha-1)^2) \\
= \; & (\alpha^3 - \alpha)^2 + b + b^2 - b(\alpha^2 + \alpha^2 + 2\alpha + 1 + \alpha^2 - 2\alpha + 1) \\
= \; & a^2 + b + b^2 - 2b \\
= \; & a^2 - b + b^2
\end{aligned}$$

However $(N_a \oplus bN_a \oplus b^2N_a)(x^2 + y + y^2) = a^2 + b + b^2$.

This does not give a negative answer to our original question, since $[[a, b\rangle\rangle$ being isotropic could still be equivalent to $N_{[a,b)_K}$ being isotropic, even if they are not the same. To further investigate on this let's try to find a more explicit formula for the reduced norm of $[a, b)_K$ in some special cases. By σ we denote the automorphism of K' over K mapping α to $\alpha + 1$. Recall that

$$X = \begin{bmatrix} \alpha & 0 & \cdots & 0 \\ 0 & \sigma(\alpha) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \sigma^{p-1}(\alpha) \end{bmatrix} \quad \text{and} \quad Y = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ 0 & 0 & \cdots & 0 & 1 \\ b & 0 & \cdots & 0 & 0 \end{bmatrix}$$

We see that our map $[a, b)_K = L \oplus Ly \oplus \cdots \oplus Ly^{p-1} \to M_p(K')$ is given by

$$r_0 + r_1 y + \cdots + r_{p-1} y^{p-1}$$

$$\downarrow$$

$$\begin{bmatrix} r'_0 & r'_1 & r'_2 & \cdots & r'_{p-1} \\ b\sigma(r'_{p-1}) & \sigma(r'_0) & \sigma(r'_1) & \cdots & \sigma(r'_{p-2}) \\ b\sigma^2(r'_{p-2}) & b\sigma^2(r'_{p-1}) & \sigma^2(r'_0) & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & \sigma^{p-2}(r'_1) \\ b\sigma^{p-1}(r'_1) & b\sigma^{p-1}(r'_2) & \cdots & b\sigma^{p-1}(r'_{p-1}) & \sigma^{p-1}(r'_0) \end{bmatrix} \tag{3.1}$$

where $r \mapsto r'$ denotes the K-linear map $L \to K'$ mapping x to α. The reduced norm of $r_0 + r_1 y + \cdots + r_{p-1} y^{p-1}$ is the determinant of this matrix.

3.2 Subform of the reduced norm

First let's use (3.1) to see that the form (2.1) is a subform of the reduced norm, or more specifically that it is the reduced norm of $r_0 + r_1 y + \cdots + r_{p-1}y^{p-1}$, with $r_0 = s - tx$ and $s, t, r_1, \ldots, r_{p-1} \in K$. To do that we have to show that the determinant of

$$\begin{bmatrix} s-t\alpha & r_1 & r_2 & \cdots & r_{p-1} \\ br_{p-1} & s-t(\alpha+1) & r_1 & \cdots & r_{p-2} \\ br_{p-2} & br_{p-1} & s-t(\alpha+2) & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & r_1 \\ br_1 & br_2 & \cdots & br_{p-1} & s-t(\alpha+p-1) \end{bmatrix}$$

is $s^p + br_1^p + \cdots + b^{p-1}r_{p-1}^p - st^{p-1} - at^p$. To do that we use the following formula, proven in A.7 of [GS]:

Proposition: Let A be a commutative ring of characteristic p and let D be a derivation on A (i.e. $D(cd) = D(c)d + cD(d)$ for $c, d \in A$). For any element $c \in A$ denote by $M_c : A \to A$ the multiplication by c. Then

$$(M_c + D)^p = M_c^p + D^p + M_{D^{p-1}(c)} \tag{3.2}$$

for every $c \in A$. □

Now we let $A = K(t_0, \ldots, t_{p-1}, v)[y]$, where $y^p = b$ and $t_0, \ldots, t_{p-1}, v$ are variables, and let $c = t_0 + t_1 y + \cdots + t_{p-1}y^{p-1}$. Now, looking at A as a vector space with basis $1, y, \ldots, y^{p-1}$, let D be the linear operator on A mapping $y^i \mapsto iy^i$ and let S be the linear operator given by $S = M_c - vD$. It is trivial to check that vD is a derivation on A, so using the formula (3.2) and the identities $D^p = D$ and $D^{p-1}(c) = c - t_0$, we get that

$$\begin{aligned} S^p &= M_c^p - v^p D^p + M_{v^{p-1}D^{p-1}(c)} \\ &= M_{c^p} - v^{p-1}(vD) + v^{p-1}M_c - M_{v^{p-1}t_0} \\ &= v^{p-1}S + M_{c^p - v^{p-1}t_0} \end{aligned}$$

Now $c^p - v^{p-1}t_0 \in K(t_0, \ldots, t_{p-1}, v)$, so we get that S is a zero of the polynomial $T^p - v^{p-1}T - (c^p - v^{p-1}t_0)I$, where I is the identity operator. Now if $X^p - v^{p-1}X - (c^p - v^{p-1}t_0)$ is reducible, let q be a zero of it. Then $q + v, \ldots, q + (p-1)v$ are also zeros and using the same kind of argument we used to prove our conjecture in the case $n = 0$, we get that $q \in K(t_0, \ldots, t_{p-1}, v)$. But considering the equation $q^p - v^{p-1}q - (c^p - v^{p-1}t_0) = 0$ we first see that q would have to be integral, i.e. in $K[t_0, \ldots, t_{p-1}, v]$. Then, by considering powers of t_1 in q, we see that this is actually impossible. Firstly, q would have to have some term with a power of t_1, or else the bt_1^p term of c^p wouldn't cancel out. Now if any terms of q had a power of t_1 greater than 1, then the corresponding terms of q^p wouldn't cancel out. So q must have terms with t_1 to the power of 1. But then the corresponding terms of $v^{p-1}q$ wouldn't cancel out, so this is in fact impossible.

So $X^p - v^{p-1}X - (c^p - v^{p-1}t_0)$ is irreducible and it is therefore the minimal polynomial of S. Then by Cayley-Hamilton it is also the characteristic polynomial of S, so the determinant of S is $c^p - v^{p-1}t_0$. Now letting

$$v = t, t_0 = s - t\alpha, t_1 = r_1, \ldots, t_{p-1} = r_{p-1}$$

we get that the determinant of (the transpose of) our original matrix is $s^p + br_1^p + \cdots + b^{p-1}r_{p-1}^p - st^{p-1} - at^p$, which is what we wanted to show.

3.3 Reduced norm of $r_k y^k + r_m y^m$

Now let's consider the special case $r_k y^k + r_m y^m$, where $0 \leq k < m \leq p-1$. Then (3.1) gives us a matrix $(c_{i,j})$, $i, j \in \{0, 1, \ldots, p-1\}$, which has $c_{i,j} = 0$, unless $j - i \equiv k$ or m (mod p). Using the Leibniz formula for the determinant, $\det(A) = \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) \prod_{i=1}^n a_{i,\sigma(i)}$, we know that the determinant is a sum of terms that are multiples of p entries of the matrix, one from each row and column. Now let's see which of these terms can be nonzero. Such a term is the multiple of p entries $c_{i,j}$ such that $j - i \equiv k$ or m (mod p). Now if we have d pairs (i, j) such that $j - i \equiv k$, and then $p - d$ pairs such that $j - i \equiv m$, then adding all these up gives

$$\begin{aligned} dk + (p-d)m &\equiv (j_0 - i_0) + \cdots + (j_{p-1} - i_{p-1}) \\ dk - dm &\equiv (j_0 + \cdots + j_{p-1}) - (i_0 + \cdots + i_{p-1}) \\ d(k-m) &\equiv (0 + 1 + \cdots + p - 1) - (0 + 1 + \cdots + p - 1) \\ d(k-m) &\equiv 0 \end{aligned}$$

So we can only have $d = 0$ or $d = p$, so the only terms of the determinant that can be nonzero are $r'_k \cdots \sigma^{p-k-1}(r'_k) b \sigma^{p-k}(r'_k) \cdots b\sigma^{p-1}(r'_k) = b^k r'_k \cdots \sigma^{p-1}(r'_k) = b^k N_a(r_k)$ and $b^m r'_m \cdots \sigma^{p-1}(r'_m) = b^m N_a(r_m)$. We also see that both corresponding permutations have sign $+$, so the reduced norm of $r_k y^k + r_m y^m$ is $b^k N_a(r_k) + b^m N_a(r_m)$. So, although the forms $N_{[a,b)_K}$ and $[[a, b\rangle\rangle$ are not the same, they agree on elements of this special type.

3.4 Reduced norm when $p = 3$

Now let's look at the special case $p = 3$. Using (3.1) we see that our map is then given by

$$r + sy + ty^2 \mapsto \begin{bmatrix} r' & s' & t' \\ b\sigma(t') & \sigma(r') & \sigma(s') \\ b\sigma^2(s') & b\sigma^2(t') & \sigma^2(r') \end{bmatrix}$$

The determinant of this matrix is easily seen to be

$$\begin{array}{rl} & r'\sigma(r')\sigma^2(r') + bs'\sigma(s')\sigma^2(s') + b^2t'\sigma(t')\sigma^2(t') \\ - & br'\sigma(s')\sigma^2(t') - bs'\sigma(t')\sigma^2(r') - bt'\sigma(r')\sigma^2(s') \\ = & N_a(r) + bN_a(s) + b^2N_a(t) - b\mathrm{Tr}_{K'/K}(r'\sigma(s')\sigma^2(t')) \end{array}$$

Computations of $\mathrm{Tr}_{K'/K}(r'\sigma(s')\sigma^2(t'))$ show that, writing $r = r_0 + r_1x + r_2x^2$, $s = s_0 + s_1x + s_2x^2$, $t = t_0 + t_1x + t_2x^2$ with coefficients in K, the reduced norm of $r + sy + ty^2$ is given by:

$$\begin{array}{rl} & N_a(r) + bN_a(s) + b^2N_a(t) \\ + & br_0s_0t_2 + br_0s_1t_1 - br_0s_1t_2 + br_0s_2t_0 + br_0s_2t_1 - br_0s_2t_2 \\ + & br_1s_0t_1 + br_1s_0t_2 + br_1s_1t_0 - br_1s_2t_0 + abr_1s_2t_2 + br_2s_0t_0 \\ - & br_2s_0t_1 - br_2s_0t_2 + br_2s_1t_0 + abr_2s_1t_2 - br_2s_2t_0 + abr_2s_2t_1 \end{array} \tag{3.3}$$

4 Solution attempts for $n = 1$ and $p = 3$

Let's try to solve the main problem posed in Chapter 2 in the case $p = 3$. Then

$$\begin{aligned} N_a(x + y\tau + z\tau^2) &= \begin{vmatrix} x & az & ay \\ y & x+z & y+az \\ z & y & x+z \end{vmatrix} \\ &= x^3 + 2x^2z - xy^2 + xz^2 + ay^3 - ayz^2 + a^2z^3 \end{aligned}$$

By definition, the form $[[a, b\rangle\rangle$ is therefore given by

$$\begin{aligned} & x_0^3 + 2x_0^2z_0 - x_0y_0^2 + x_0z_0^2 + ay_0^3 - ay_0z_0^2 + a^2z_0^3 \\ + \quad & b(x_1^3 + 2x_1^2z_1 - x_1y_1^2 + x_1z_1^2 + ay_1^3 - ay_1z_1^2 + a^2z_1^3) \\ + \quad & b^2(x_2^3 + 2x_2^2z_2 - x_2y_2^2 + x_2z_2^2 + ay_2^3 - ay_2z_2^2 + a^2z_2^3) \end{aligned}$$

The subform (2.1), however is given by

$$x_0^3 - x_0y_0^2 - ay_0^3 + bx_1^3 + b^2x_2^3$$

4.1 First attempt

In the previous chapter we showed that $N_{[a,b)_K} \neq [[a, b\rangle\rangle$ by noting that the two forms did not agree on the element $x^2 + y + y^2$. Our first attempt at solving the problem is inspired by this example. So we assume that $z_0 = x_1 = x_2 = 1$, other variables $= 0$, is a zero of $[[a, b\rangle\rangle$. Then we get the equation $a^2 + b + b^2 = 0$. Now if we let $K = K_0(t)$, where K_0 is some field of characteristic 3, and let $a = \frac{2t}{t^2+1}$, $b = \frac{2t^2}{t^2+1}$, then this equation is satisfied. The isotropy of the form (2.1) then gives the following functional equation

$$f_0^3 - f_0h^2 - \frac{2t}{t^2+1}h^3 + \frac{2t^2}{t^2+1}f_1^3 + 4t^4(t^2+1)g_2^3 = 0$$

where $h, f_0, f_1, g_2 \in K$. If we could show that this functional equation has no solution we would disprove our conjecture. However, after some speculations we see that $a^2 + b + b^2 = 0$ would indeed mean that the form (2.1) was isotropic, since

$$\begin{aligned}
&(a+b)^3 - (a+b)(-1-b)^2 - a(-1-b)^3 + b(1+a)^3 + b^2(-1)^3 \\
&= a^3 + b^3 - a - 2ab - ab^2 - b - 2b^2 - b^3 + a + ab^3 + b + a^3b - b^2 \\
&= a^3(1+b) + a(-2b - b^2 + b^3) \\
&= a^3(1+b) + ab(1 + 2b + b^2) \\
&= a^3(1+b) + ab(1+b)^2 \\
&= a(1+b)(a^2 + b + b^2) = 0
\end{aligned}$$

And by letting $f_0 = 2t(t+1)$, $h = -1$, $f_1 = (t+1)^2$, $g_2 = -1$, we get a solution to the functional equation. While this is a result in its own right, it does not solve our initial problem.

4.2 The connection with central simple algebras

Now let's look at the connection with central simple algebras. We can play around with the formula (3.3), and for instance we get that

$$N_a(r) + bs_0^3 + b^2t_0^3 + br_2s_0t_0$$

is the reduced norm of $r + s_0y + t_0y^2$ with $r = r_0 + r_1x + r_2x^2$ and $s_0, t_0 \in K$. Let $s_0 = \delta r_2$, $t_0 = \varepsilon r_2$. Then we get

$$N_a(r) + (b\delta^3 + b^2\varepsilon^3 + b\delta\varepsilon)r_2^3$$

Let $\varepsilon = 1$ and get

$$N_a(r) + (b(\delta^3 + \delta) + b^2)r_2^3$$

Now let $\delta = -1$. Then we get

$$N_a(r) + (b(-2) + b^2)r_2^3 = N_a(r) + (b + b^2)r_2^3$$

This means that the reduced norm of $r - r_2y + r_2y^2$ is $N_a(r) + (b + b^2)r_2^3$. In particular, the reduced norm of $x^2 - y + y^2$ is $a^2 + b + b^2$, which gives an easier way to see that if $a^2 + b + b^2 = 0$, then $\left[a\frac{db}{b}\right] = 0$. Furthermore this considerably widens the range of zeros that $[[a, b\rangle\rangle$ can have to give the same result. Now any nontrivial zero that has $y_1, z_1, y_2, z_2 = 0$ and $z_0 = x_1 = x_2$ will give $\left[a\frac{db}{b}\right] = 0$.

We also have that if $0 \le i < j \le 2$ and $u, v \in L$ then the reduced norm of $uy^i + vy^j$ equals $b^iN_a(u) + b^jN_a(v)$, so any nontrivial zero of $[[a, b\rangle\rangle$ that has $x_i, y_i, z_i = 0$ for some $i \in \{0, 1, 2\}$ will give $\left[a\frac{db}{b}\right] = 0$.

4.3 Solving the problem

The connection with central simple algebras is interesting in its own right; however, our next serious approach at solving our problem is similar to the first one, except instead of letting $z_0 = 1$, we let $z_0 = r$, where r is a variable. This gives the equation

$$r^3a^2 + b + b^2 = 0$$

We can solve the equation for a and b and write both as functions of t and r. Let $t = b/a$, $b = at$. Then we get

$$r^3a^2 + at + a^2t^2 = 0$$

This gives $a = \frac{2t}{t^2+r^3}$, and then $b = \frac{2t^2}{t^2+r^3}$, with indeterminate r and t. So now we let K be the rational function field $K_0(r,t)$, where K_0 is some field of characteristic 3, and we let $a = \frac{2t}{t^2+r^3}$ and $b = \frac{2t^2}{t^2+r^3}$. This gives a counterexample to the original question, since now $[[a, b\rangle\rangle$ is isotropic, but the form (2.1) is not. To see that there are no $f_0, f_1, f_2, h \in K$, not all 0, such that

$$f_0^3 - f_0h^2 - \frac{2t}{t^2+r^3}h^3 + \frac{2t^2}{t^2+r^3}f_1^3 + \frac{4t^4}{(t^2+r^3)^2}f_2^3 = 0$$

we assume on the contrary that this is the case. Using a common denominator we may assume that f_0, f_1, f_2, h are all polynomials in r and t and that they have no common factor. It then follows at once that $f_2 = (t^2 + r^3)g_2$ with $g_2 \in K_0[r, t]$. We then get the equation

$$f_0^3 - f_0h^2 - \frac{2t}{t^2+r^3}h^3 + \frac{2t^2}{t^2+r^3}f_1^3 + 4t^4(t^2+r^3)g_2^3 = 0$$

From this equation we see that h can't be divisible by $t^2 + r^3$, because that would imply that f_1, and then f_0 also, would be divisible by $t^2 + r^3$. But that would contradict our assumption of no common factor. Now multiplying the equation by t we get

$$tf_0^3 - tf_0h^2 - \frac{2t^2}{t^2+r^3}h^3 + \frac{2t^3}{t^2+r^3}f_1^3 + 4t^5(t^2+r^3)g_2^3 = 0$$

Writing $2t^2 = 2(t^2 + r^3) - 2r^3$ we get

$$tf_0^3 - tf_0h^2 - 2h^3 + \frac{2r^3}{t^2+r^3}h^3 + \frac{2t^3}{t^2+r^3}f_1^3 + 4t^5(t^2+r^3)g_2^3 = 0$$

This can be written as

$$tf_0^3 - tf_0h^2 + h^3 + \frac{2}{t^2+r^3}(rh + tf_1)^3 + 4t^5(t^2+r^3)g_2^3 = 0$$

It follows that $rh + tf_1$ is divisible by $t^2 + r^3$. But then $\frac{2}{t^2+r^3}(rh + tf_1)^3$ is integral and divisible by $(t^2+r^3)^2$. It then follows that $tf_0^3 - tf_0h^2 + h^3$ is divisible by t^2+r^3. We do know that h is not divisible by $t^2 + r^3$. We now work modulo $t^2 + r^3$, i.e. in the field $K_0(r)(\gamma)$, where $\gamma = \sqrt{-r^3}$. We denote by ξ the class of $-\frac{f_0}{h}$ in this field. Then the fact that $tf_0^3 - tf_0h^2 + h^3$ is divisible by $t^2 + r^3$ means that $\gamma\xi^3 - \gamma\xi - 1 = 0$, i.e. $\xi^3 - \xi = \frac{1}{\gamma}$, i.e. $\xi^3 - \xi = -\frac{1}{r^3}\gamma$. Writing $\xi = \zeta + \eta\gamma$, with ζ and η in $K_0(r)$, this means that $-r^3\eta^3 - \eta = -\frac{1}{r^3}$, i.e. $r^6\eta^3 + r^3\eta = 1$. Of course $\eta \neq 0$. Writing $\omega = \frac{1}{\eta}$ we can rewrite this equation as $\omega^3 - r^3\omega^2 - r^6 = 0$. It follows that ω must lie in $K_0[r]$, and in fact be divisible by r^2, say $\omega = r^2\omega_2$. For ω_2 we then get the equation $\omega_2^3 - r\omega_2^2 - 1 = 0$. But this is easily seen to be impossible.

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