# A Supply Curve Analysis for the Icelandic Housing Financing Fund Bond Market

Hannes Árdal

Faculty of Industrial Engineering,
Mechanical Engineering and
Computer Science
University of Iceland
2013

# A Supply Curve Analysis for the Icelandic Housing Financing Bond Market

Hannes Árdal

30 ECTS thesis submitted in partial fulfillment of a
*Magister Scientiarum* degree in Financial Engineering

Advisors
Birgir Örn Arnarson
Haraldur Óskar Haraldsson
Ólafur Pétur Pálsson

External examiner
Eggert Þröstur Þórarinsson

Faculty of Industrial Engineering, Mechanical Engineering and Computer Science
School of Engineering and Natural Sciences
University of Iceland
Reykjavik, September 2013

A Supply Curve Analysis for the Icelandic Housing Financing Fund Bond Market
A Supply Curve Analysis for the HFF Bond Market
30 ECTS thesis submitted in partial fulfilment of a *Magister Scientiarum* degree in Financial Engineering



Faculty of Industrial Engineering, Mechanical Engineering and Computer Science
School of Engineering and Natural Sciences
University of Iceland
VRII Hjardarhagi 2-6
107, Reykjavik
Iceland

Telephone: 525 4000

Bibliographic information:
Hannes Árdal, 2013, A Supply Curve Analysis for the Icelandic Housing Financing Fund Bond Market, Master's thesis, Faculty of Industrial Engineering, Mechanical Engineering and Computer Science, University of Iceland, pp. 39.

Printing: Háskólaprent ehf.
Reykjavik, Iceland, September 2013

# Abstract

Risk management deals with three types of financial risks: market risk, credit risk and liquidity risk. Much has been written about both market and credit risk but substantially less has been written about liquidity risk. Liquidity risk has been modelled for securities as a supply curve where the price obtained in a trade of a given security is reflected not only in the time of the trade but also in the size and direction (buy or sell orders). In this thesis, data provided by the Icelandic Stock Exchange is used to examine the supply curve, and thus the liquidity, of three bonds traded in Iceland. Four different models for the supply curve will be presented and fitted to actual data. The supply curve is shown to exist and therefore it is shown that size and direction of a trade does affect the price obtained in a trade. In conclusion, the model most likely to be the best to describe the liquidity for the given bonds is selected. The model that best fitted the provided data was the *S*-shaped *logarithmic* model where the rather high bid-ask spread and market participants' tendency to place their orders alongside other orders were well captured.

# Útdráttur

Áhættustýring á fjármálamarkaði fæst aðalega við þrjár mismunandi áhættur; markaðsáhættu, endurgreiðsluáhættur og seljanleikaáhættu. Töluvert hefur verið ritað um fyrri tvær áhætturnar en mun minna um seljanleikaáhættu. Seljanleikaáhættu á verðbréfi hefur verið lýst með framboðskúrfu (e. Supply Curve) en í því felst að verð í viðskiptum með viðkomandi verðbréf er ekki einungis háð tímasetningu viðskiptanna heldur einnig magni í viðskiptunum og því hvort um kaup eða sölu á verðbréfinu sé að ræða, þ.e. átt viðskiptanna. Í þessari ritgerð eru notuð gögn frá Kauphöll Íslands til að rannsaka framboðskúrfu, og þar með seljanleika, þriggja skuldabréfa í Kauphöll Íslands. Fjögur mismunandi líkön fyrir framboðskúrfuna eru sett fram og aðhvarfsgreiningu er beitt til að aðlaga líkönin að raunverulegum gögnum. Sýnt er fram á að framboðskúrfan sé raunverulega til og þar með að magn og átt viðskiptanna hefur áhrif á verðið sem viðskiptin verða á. Að endingu er það líkan valið sem er líklegast til að lýsa framboðskúrfunni, og þar með seljanleikanum, best. Líkanið sem lýsti gögnunum best var *logra* líkan og reyndist þar muna miklu um að *S*-lögun þess fangaði hið háa verðbil vel auk þess sem það hentar vel til að lýsa því hvernig markaðsaðilar eiga það til að setja tilboð sín við hliðina á öðrum tilboðum í tilboðsbókunum.

*I lovingly dedicate this thesis to my fiancée and wife-to-be, Patricia Anna Þormar, who supported me each step of the way.*

# Table of Contents

# List of Figures

# List of Tables

# Acknowledgements

First and foremost, I would like to thank Professor Ólafur Pétur Pálsson, the Head of Faculty of Industrial Engineering, Mechanical Engineering and Computer Science, for his assistance and inputs. Secondly I would like to thank my advisors, Dr. Birgir Örn Arnarson and Dr. Haraldur Óskar Haraldsson, for their support, insight and advice. Thirdly I would like to thank Mr. Finnbogi Jónsson and other staff at the Icelandic Stock Exchange for providing me with the necessary data to do this research. Finally, I am grateful to my family, friends and colleagues for the support they provided me with while working on this thesis.

# 1 Introduction

Risk management deals with three types of financial risks: market risk, credit risk and liquidity risk. Market risk is the risk of price fluctuation in financial securities due to changes in interest rates, trading prices, commodities prices or foreign exchange rates. Credit risk is the risk of price fluctuation in financial securities due to defaults. Much has been written about both market and credit risk in abstract theory and both are well known (see Duffie [1] and Bielecki and Rutkowski [2]). Implementations of various risk measures for market and credit risk have also been addressed (see e.g. Jorion [3]). Much less has been written about the third type of risk, that is, liquidity risk. Liquidity risk is the risk of price fluctuation in financial securities due to the impact a trade's size can have on the price obtained when supply/demand is limited. Thus liquidity risk is the risk that a given security cannot be traded quickly enough in the market to prevent a loss, i.e. the additional risk due to the timing and size of trades.

Liquidity risk was first modulated as a convenience yield (see Jarrow [4] and Jarrow and Turnbull [5]). This solution successfully captures the part of liquidity risk due to inventory considerations and also retains the price taking condition so classical arbitrage theory can still be applied. However, this solution does not capture the impact a trade size can have on the price.

Cetin et al [6] have successfully overcome this by providing a general methodology for modelling liquidity risk. Their approach hypothesizes the existence of a stochastic supply curve for a financial security as a function of transaction size. They characterize conditions on the supply curve analogous to the conditions imposed by Heath et al [7] on the term structures of interest rates, for the supply curve to be arbitrage free. Given the arbitrage free environment they characterize the conditions for a complete market and further more study the pricing of derivatives. This model has become the most popular way of examining and characterizing liquidity risk.

In this thesis order book data from the Icelandic Stock Exchange will be used to examine and analyse the supply curve postulated by Cetin et al [6] for three different bonds trading at the exchange. For each bond, four different models for the supply curve will be examined. The Icelandic bond market is not a very liquid one with just about dozen participants of which about four to six act as market makers for the bonds under consideration. The issuer of these three bonds, The Housing Financing Fund, has signed a Market Making Agreement with the market makers where they have specified a certain bid ask spread for each bond that the market maker must hold while the market is open for trading. Other market participants tend to place their orders alongside the market makers and therefore the bid-ask spread holds even as more and more orders are placed in the market. All this results in rather high bid ask spreads, low trading volume and rather high volatility.

An outline for this thesis is as follows. In Chapter 2, the model set forth by Cetin et al [6] is described. Chapter 3 describes the supply curve models that are examined for each of the bond and it is also argued why these models were chosen. In Chapter 4, the data needed to examine the supply curves is described and also the data set obtained from the Icelandic Stock Exchange. The reasons for choosing these three bonds are also listed. Chapter 5 presents the results of the examinations of the different supply curve models where it is firstly shown that the supply curves really do exist and secondly it is examined which of the model best fits the data and is therefore the best model to describe the supply curve for the Icelandic Housing Financing Fund bond market. Chapter 6 concludes the thesis with a discussion and summarisation of the results.

# 2 Background Theory

This chapter presents the model set forth by Cetin et al [6]. Blais and Protter [8] followed their work and their short version is followed here. The interested reader is referred to these two papers for further reading. A filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F})_{0\leq t\leq T}, \mathbb{P})$ is given that satisfies the usual conditions where $T$ is a fixed time. Let $\mathbb{P}$ stand for the statistical probability measure for this space. It is also assumed that $\mathcal{F}_0 = \{\emptyset, \Omega\}$, i.e. $\mathcal{F}_0$ is trivial. A security that initially is assumed to have no cash flow associated with it is considered and also that there exists a market for that given security where it can be traded. A money market account is also traded in this market and it accumulates value at the sport rate of interest. It is assumed, without loss of generality, that the spot rate of interest is zero. Thus the money market account has unit value for all times. This restriction on the spot rate of interest can of course be removed without much effort.

## 2.1 The Supply Curve

Now an arbitrary price taking trader is considered. This trader acts as a price taking trader with respect to a given supply curve for units bought and sold of the given security. More formally the securities unit price at time $t \in [0, T]$ that the trader pays (or receives) for an order of size $x \in \mathbb{R}$, given the state $\omega \in \Omega$, is represented by $S(t, x, \omega)$. A buy order is represented with a positive sign on $x$, i.e. $x > 0$, a sell order is represented with a negative sign on $x$, i.e. $x < 0$ while the marginal trade corresponds to a zero order, i.e. $x = 0$.

In the classical theory the trader faces a horizontal supply curve where the same price is given for any order size, regardless of the direction of the order, i.e. a buy order or a sell order. Now, on the other hand, the trader faces a supply curve that depends not only on his order size but also on its direction.[1] The supply curve is otherwise independent of the trader's past actions, risk aversion, or beliefs as the trader is a price taking trader. Thus the investor's historical trades have no lasting impact on the securities price process. This restriction distinguishes this economy from the situation where the supply curve also depends on the entire history of the trader's trades.

The following structure was imposed on the supply curve by Cetin et al [6].

**Assumptions** (*For the Supply curve*)

1. *$S(t, x, \cdot)$ is $\mathcal{F}_t$-measurable and non-negative.*
2. *$x \mapsto S(t, x, \omega)$ is a.e. t non-decreasing in x, a.s. (i.e. $x \leq y$ implies $S(t, x, \omega) \leq S(t, y, \omega)$ a.s. $\mathbb{P}$, a.e. t).*
3. *$S$ is $C^2$ in its second argument, $\partial S(t, x)/\partial x$ is continuous in t, and $\partial^2 S(t, x)/\partial x^2$ is continuous in t.*
4. *$S(\cdot, 0)$ is a semi-martingale.*
5. *$S(\cdot, x)$ has a continuous sample paths (including time 0) for all x.*

[1] Note however that trader is assumed to have no impact the money market account with his activity.

Assumption 1 says that the securities price can be observed from historical information and that the price is always non-negative. Assumption 2 states that the larger the purchase (or sale), the larger the price impact that occurs on the securities price is. This is usually the case that traders face in asset pricing markets, where the quantity impact on the price is due to either information effects or supply/demand imbalances (see Kyle [8]; Glosten and Milgrom [9]; Grossman and Miller [11]). This excludes the more familiar situation in consumer products where there are quantity discounts for large orders. It includes however, as a special case, the horizontal supply curves from the classical theory. Assumption 3 and 5 ensure the smoothness of the supply curve and its partial derivatives. Assumption 4 says that the securities price process can be decomposed as $S(t,0) = A(t) + M(t)$ where $A(t)$ is a càdlàg[2] adapted process of locally bounded variation and $M(t)$ is a local martingale[3]. It is worth mentioning that all of the above assumptions, except assumption 2, are technical in nature.

## 2.2 Trading Strategies

Cetin et al [6] define an investor's trading strategy by the following.

> ***Definition 2.1.*** *A trading strategy is a triplet $((X_t, Y_t : t \in [0,T]), \tau)$ where $X_t$ represents the trader's aggregated securities holding at time t (in units of the security), $Y_t$ represents the trader's aggregated money market account position at time t (units of the money market account), and $\tau$ represents the liquidation time of the security position, subject to the following restrictions: (a) $X_t$ and $Y_t$ are predictable and optional processes, respectively, with $X_{0-} \equiv Y_{0-} \equiv 0$, and (b) $X_T = 0$ and $\tau$ is a predictable $(\mathcal{F}_t : 0 \leq t \leq T)$ stopping time with $\tau \leq T$ and $X = H1_{[0,\tau)}$ for some predictable process $H(t,\omega)$.*

A particular type of trading strategy, self-financing strategy, is of interest. A self-financing trading strategy does not generate any cash flow while it is in place, i.e. with $t \in [0,T)$. This means that a purchase of the given security is always funded with a borrowing from the money market account. Also, if the security is sold the entire amount is invested in the money market account. This means that at all times $Y_t$ is fully determined by $(X_t, \tau)$. The goal is then to define condition for this trading strategy, $Y_t$, given an arbitrary securities holding $(X_t, \tau)$ that make it a self-financing trading strategy. Cetin et al [6] do this in the following way.

---

[2] A càdlàg is a function defined on a set of real numbers that is everywhere right-continuous and has left limits everywhere.

[3] A process X is a local martingale if it is a càdlàg and there exists a sequence of stopping times $\tau_n$ increasing to infinity, such that $1_{\{\tau_n > 0\}} X^{\tau_n}$ is a martingale for each *n*. A martingale is of course a bounded zero-drift stochastic process where knowledge of past events does not help in predicting the next value and the expected next value is equal to the latest observed value.

***Definition 2.2.*** *A self-financing trading strategy (s.f.t.s.) is a trading strategy* $((X_t, Y_t : t \in [0,T]), \tau)$ *where (a)* $X_t$ *is càdlàg if* $\partial S(t,0)/\partial x \equiv 0$ *for all t, and* $X_t$ *is càdlàg with finite quadratic variation* ($[X,X]_T < \infty$) *otherwise, (b)* $Y_0 = -X_0 S(0, X_0)$*, and (c) for* $0 < t \leq T$,

$$Y_t = Y_0 + X_0 S(0, X_0) + \int_0^t X_{u-} dS(u,0) - X_t S(t,0)$$
$$- \sum_{0 \leq u \leq t} \Delta X_u [S(u, \Delta X_u) - S(u,0)] - \int_0^t \frac{\partial S}{\partial x}(u,0) d[X,X]_u^c \quad (2.1)$$

Acceptable trading strategies classes are restricted by condition (a). Under the hypotheses that $X_t$ is càdlàg and of finite quadratic variation, the right side of equation (2.1) is always well-defined although the last two terms (never being positive) may be negative infinity. This restriction is not needed in the classical theory, where markets are frictionless and competitive. Blais and Protter [8] take an example of a trading strategy that is allowed in the classical theory, but disallowed here. They set $X_t = 1_{\{S(t,0)>K\}}$ for some constant $K > 0$ where $S(t,0)$ follows a Brownian motion. Under the Brownian motion hypothesis this is a discontinuous trading strategy that jumps infinitely often immediately after $S(t,0) = K$ (the jumps are not square summable), and hence $Y_t$ is undefined. Condition (b) simply states that the strategy requires zero initial investment at time 0 as all investments/sells are borrowed/invested in the money market account. Condition (c) is the actual self-financing condition at time *t*. The money market account is its value at time 0, added by the accumulated trading gains (the marginal trade is used to evaluate this), subtracting the cost of attaining the current position, subtracting the price impact costs of discrete changes in the securities holdings. This expression is in fact an extension of the classical self-financing condition when the supply curve is horizontal. This is easily shown using condition (b) with equation (2.1) to give the self-financing condition the following simplified form:

$$Y_t + X_t S(t,0) = \int_0^t X_{u-} dS(u,0) - \sum_{0 \leq u \leq t} \Delta X_u [S(u, \Delta X_u) - S(u,0)]$$
$$- \int_0^t \frac{\partial S}{\partial x}(u,0) d[X,X]_u^c \quad \text{for } 0 \leq t \leq T \qquad (2.2)$$

The classical portfolio value at time 0 is represented by the left side of equation (2.2). Decomposition into various components is represented on the right side. The right side's first term is the classical "accumulated gains/losses" of the portfolio's value. The last two terms, both entering with a negative sign, capture the impact of illiquidity.

## 2.3 The Marked-to-Market Value of a Self-Financing Trading Strategy and its Liquidity Cost

This section again follows the work of Cetin et al [6] where they define the marked-to-market value of a trading strategy and its liquidity cost. Prior to liquidation, the trading strategy or portfolio has no unique value and actually it is possible to use any price on the supply curve in valuing the portfolio. Cetin et al [6] point out at least three economically meaningful possibilities that can be identified:

1. *The immediate liquidation value (assuming that $X_t > 0$ gives $Y_t + X_t S(t, -X_t)$)*
2. *The accumulated cost of forming the portfolio ($Y_t$)*
3. *The portfolio evaluated at the marginal trade $Y_t + X_t S(t, 0)$).*[4]

The *market-to-market value* of the self-financing trading strategy $(X, Y, \tau)$ is defined to be the last possibility, 3. It represents the value of the portfolio under the classical price taking condition.

Motivated by equation (2.2), Cetin et al [6] define the liquidity cost as the difference between the accumulated gains/losses to the portfolio, compounded as if all trades are executed at the marginal trade price $S(t, 0)$, and the marked-to-market value of the portfolio. Their definition is as follows:

***Definition 2.3.*** *The liquidity cost of a s.f.t.s. $(X, Y, \tau)$ is*

$$L_t \equiv \int_0^t X_{u-} dS(u, 0) - [Y_t + X_t S(t, 0)].$$

The following lemma is then set forth by Cetin et al [6] following from the preceding definition.

***Lemma 2.1. (Equivalent Characterization of the Liquidity Cost).***

$$L_t = \sum_{0 \le u \le t} \Delta X_u [S(u, \Delta X_u) - S(u, 0)] + \int_0^t \frac{\partial S}{\partial x}(u, 0) d[X, X]_u^c \ge 0$$

*where $L_{0-} = 0$, $L_0 = X_0[S(0, X_0) - S(0,0)]$ and $L_t$ is non-decreasing in t.*

The interested reader is refereed to Jarrow and Protter [11] for a (simple) proof.

Cetin et al [6] then go on and say that it can be seen that the liquidity cost is non-negative and non-decreasing in $t$ and that it consists of two components. The first component is due to discontinuous changes in the securities holdings, i.e. due to trading. The second is due to the continuous component, i.e. due to price changes in the securities holdings at each time. This expression is quite intuitive as one would assume in advance that this would be the case. Note that because $X_{0-} = Y_{0-} = 0$ it is possible to have $\Delta L_0 = L_0 - L_{0-} = L_0 > 0$.

[4] Cetin et al [6] also point out that these three valuations are (in general) distinct except at one date, the liquidation date. At the liquidation time $\tau$, the value of the portfolio under each of these three cases are equal because $X_\tau = 0$.

# 3 The Supply Curve Models

In the classical asset pricing theory the trading price of a stock follows a geometric Brownian motion process often referred to as $S(t)$ where the drift and volatility terms are considered to be constants. Trading price of a bond[5] is, however, often referred to as $P(t, T)$ representing the cash price of a zero-coupon bond with maturity $T$ at time $t$. To simplify the notation, the price of a specific zero-coupon bond is often referred to as $P(t)$, as the maturity $T$ is known for the given bond. The geometric Brownian motions used for stocks cannot be used for bonds, as volatility is not a constant for bonds.[6] To price zero-coupon bonds, a stochastic process for short interest rates is defined and from that a term structure for interest rates is built[7]. By use of this term structure the zero-coupon bond price $P(t)$ can be derived. By using the prices of zero-coupon bonds, other bond scan be priced as coupon paying bonds and instalments on bonds are simply sums of zero-coupon bonds. For further reading see Björk [13]. From now on $S(t)$ will be used to represent the trading price of a security, both stock and bond.

This price process, $S(t)$, is in fact the same process as the process of the marginal trade $S(t, 0)$ from the previous chapter. The most convenient way of defining a supply curve $S(t, x)$ for a security is to set $S(t, x) = M(x)S(t)$. The first term, $M(x)$, is a function of the trade size $x$ that captures the quantity impact of a trade on the price obtained for that trade. The second term, $S(t)$, is the price process mentioned above. Different supply curve models therefore differ in the form of the function $M(x)$.

The two most popular forms for $M(x)$ have been linear and exponential. Blais and Protter [8] examined a linear function $M(x)$ for a few highly liquid stocks trading at the New York Stock Exchange. They also introduced a *jump-linear* supply curve for illiquid ones. Cetin et al [6][8], have formulated an exponential form for $M(x)$ to use in option pricing theory with liquidity risk. Later on in this chapter two more models that should prove good in these studies will be presented.

It is now necessary to examine the form of the supply curves for bonds issued by the Icelandic Housing Financing Fund, but first, a better understanding of the Icelandic bond market is necessary. Icelandic bonds are traded on screen in the Icelandic Exchange so all trades are visible to all market participants immediately after they are executed. This also means that the full depth of all order books for bonds are visible to all participants at all times. Many issuers of Exchange traded bonds, i.e. the Icelandic government and the Housing Financing Fund, have signed a market making agreement with some market participants (usually traders within the larger banks in Iceland) who then act as market makers for that particular issuer. The market making agreements usually have the following terms:

---

[5] Bond trading prices are usually clean prices, i.e. cash prices without accrued interest. They therefore represent a trading price for the bond where the cash flow is ignored and can therefore be used in the context of Chapter 2.

[6] The reason for this is the pull-to-pair feature of a bond. The price volatiltiy will decrease as the time to maturity decreases because for all bond we have the no-arbitrage restriction that $P(T, T) = 1$.

[7] It is also possible to define a stocastic process for forward interest rates.

[8] Many others have used the same form in their studies of liquidity risk.

1. The market making orders must hold a given bid-ask spread during trading hours.
2. The market making orders must be of a given size in volume.
3. The market maker can abandon his obligations if his total turnover during the day exceeds a given volume.
4. The issuer pays a yearly commission to the market makers for his liquidity service.

The market makers place their orders in the market before opening each day and all orders are immediately seen by other market makers as well as all participants in the market. This visibility of orders has a peculiar end result. Most of the market makers place their orders in the market at the same bid and ask prices as all other market makers. This in turn makes other participants place their orders very close to the market makers and usually with no more than a few basis points price change from the market making orders. This means that during trading hours the bid-ask spread in the market is rather wide[9] while traders who act as price takers can buy large amounts in the market at a price very close to the offer side (or sell very close to the buy side). This means that price volatility is high, and, as to be expected in a market with wide bid ask spreads and high volatility, the trading volume is low.

These insights from above lead to a form for the supply curve for the Icelandic bond market that (a) captures the high bid-ask spread in the market and (b) allows for large size trades to be executed on a small spread to the bid ask prices at the time of trade. This is possible by using the form of the square root function or the natural logarithmic function with the sign function.[10]

Therefore four different types of forms for the function $M(x)$ will be presented in this chapter and thus four different models for the supply curve. These function forms are:

$$
\begin{aligned}
&M_1(x) = \alpha_1 \cdot x + 1\\
&M_2(x) = e^{\alpha_2 \cdot x}\\
&M_3(x) = \alpha_3 \cdot \mathrm{sign}(x)\sqrt{|x|} + 1\\
&M_4(x) = \alpha_4 \cdot \mathrm{sign}(x)\ln(1 + |x|) + 1
\end{aligned}
$$

And thus the supply curve models are given by:

$$
\begin{aligned}
&S_1(t,x) = M_1(x)S(t) = (\alpha_1 \cdot x + 1)S(t,0)\\
&S_2(t,x) = M_2(x)S(t) = e^{\alpha_2 \cdot x}S(t,0)\\
&S_3(t,x) = M_3(x)S(t) = (\alpha_3 \cdot \mathrm{sign}(x)\sqrt{|x|} + 1)S(t,0)\\
&S_4(t,x) = M_4(x)S(t) = (\alpha_4 \cdot \mathrm{sign}(x)\ln(1 + |x|) + 1)S(t,0)
\end{aligned}
$$

From now on these models will be referred to as *linear*, *exponential*, *root* and *logarithmic* models respectively. The parameter $\alpha$ is here written as a constant but obviously this constant will differ between models and can even be set up as time dependant, $\alpha(t)$, in each model. In all models the same applies, i.e. the higher the value of $\alpha$ the less liquidity and therefore higher liquidity risk. Also note that if $\alpha = 0$ then all models represent the same trivial price processes mentioned at the beginning of this chapter where the supply cure is horizontal at all times.

---

[9] Most of the market making agrrements allow for the bid-ask spread to be between 0,4% to 1% based on the bonds duration or maturity.

[10] The sign function has the following properties: $\mathrm{sign}(x) = \begin{cases} 1 \text{ if } x \geq 0 \\ -1 \text{ if } x < 0 \end{cases}$

# 4 The Data

In previous chapters, the theory of supply curves was sketched, the possible structure of the supply curve was discussed and four different models for the supply curve were presented. The next step would be to show that the supply curve actually does exist but is not just a trivial horizontal line. A horizontal supply curve would render all the work in previous chapters meaningless and in fact mean that the size and direction of a trade does not affect the price of that trade. However, if it can be shown that the supply curve actually does exist, different structures can be modelled for it and then analysed to see if some models are better than others to describe actual data.

## 4.1 Trade Tick Data vs. Order Book Data

The best way to check if a supply curve exists for a given security, and if so, to describe its structure, is to use trade tick data or order book data gathered from a stock exchange or other data source. At first glance tick data would be preferred as tick data contains both the size and price of actual trades executed in the stock exchange. This would match the supply curve models very well as the information matches the supply curves output perfectly. However, with tick data there is no information on whether the trade was initiated by a buyer or a seller and therefore it is impossible to determine with accuracy if the trade size should have a positive or a negative sign in the data set. This problem has long been known in the liquidity literature and many have proposed ways to resolve it. The best known algorithm is probably from Lee and Ready [14] which is thought to have about 74% to 85% accuracy, although others have found it to be as low as 61% (see Blais and Protter [15]). The basic idea in the Lee and Ready algorithm is that if the price increases between two trades, the trade must be buyer initiated and vice versa. The accuracy of the Lee and Ready algorithm is too low to base the supply curve analysis on and therefore tick data is rendered not useful in these studies. The attention will therefore be reverted to order book data.

Order book data contains information on limit orders that have been placed into the market but have not been executed yet. A limit order placed into a market contains information such as direction (buy or sell order), size and price and therefore represents the willingness of a trader to buy or sell a certain amount of a given security at a given price. By gathering all these limit orders it is possible to build what is called an *order book*. Order book data for a given security therefore contains information from all market participants on how much of that security they are willing to buy and sell and at what prices. By gathering all the buy orders it is possible to build the *buy-side* of the order book while the sell orders represent the *sell-side*. Both the buy- and sell-side are ordered by prices, descending on the buy-side while ascending on the sell-side and market participants can see the full depth of all limit orders. The difference between the best offer on the sell-side and the best bid on the buy-side is called the *bid-ask spread*. When a price taking trader enters the market with an order to buy at market he or she starts buying up the best offers, i.e. the lowest orders, on the sell-side until the buy order is filled or there are no more offers that match the buy order. The same applies for price taking traders with orders to sell but they will, obviously, match the buy-side. Note that there might be some market participants that do not show their bids or offers in the market although they would be willing to do trades either inside or outside the bid-ask spread. Therefore liquidity could easily be higher than observed from the order book data.

The main problem with using order book data is that it is hard to obtain as it is usually not stored by the exchange. Stock exchanges do however store historical information on all limit orders placed in the market, their modifications and cancellations along with trade tick data. With some programming it is possible to use this historical limit order information to reconstruct the order book at any given time. This programming is in fact a playback of the market where the user can see the flow of orders coming into the market, being modified, cancelled or turning into trades. It is therefore possible to examine the full debt of the order book at any given time and therefore see what sizes the market is willing to buy and sell at different prices. This is a huge improvement from standard tick data as demand and supply for the given security can be seen and with that actual points on the supply curve can be constructed.

## 4.2 Constructing the Order Book Data

By taking a closer look at the bid side of the order book it is possible to construct the actual points on the supply curve. At a fixed time $t_0$ suppose the highest bid for a security is at the price of $P$ and that there are $N$ traders in the market willing to buy $x_i$ shares each at the price of $P$ where $i \in \{1, \dots, N\}$. This implies that the top bid in the market is for $\sum_{i=1}^{N} x_i$ nominal units at the price of $P$. This is called the *aggregated* bid at the price of $P$. If a price taking seller comes into the market with $X$ shares to sell the trade will occur at the price of $P$ as long as $X \leq \sum_{i=1}^{N} x_i$. The point $\left(-\sum_{i=1}^{N} x_i\,, P\right)$ therefore corresponds to an actual point on the supply curve as this bid entry enables price taking sellers to sell up to $X$ shares in the market at the price of $P$. Notice that the negative sign on the size represents a potential seller-initiated trade.

It is now supposed in general that at a fixed time $t_0$ there are $N$ aggregated bids in the market at prices $P_i$ for $x_i$ shares each where $i \in \{1, \dots, N\}$. If now a price taking seller comes to the market at time $t_0$ with $K$ shares to sell, the first $x_1$ shares will be sold at the price of $P_1$, next $x_2$ shares will be sold at the price of $P_2$ and so forth until all the $K$ shares have been sold. The average price the seller receives at time $t_0$ per share can then be derived as $\frac{\sum_{i=1}^{n-1} x_i P_i + X_n P_n}{\sum_{i=1}^{n-1} x_i + X_n}$ where $n = \inf\left\{k : \sum_{i=1}^{k-1} x_i \leq K \leq \sum_{i=1}^{k} x_i\right\}$ and $X_n = K - \sum_{i=1}^{n-1} x_i$. By defining $K_m = \sum_{i=1}^{m} x_i$ and $P_m = \frac{\sum_{i=1}^{m} x_i P_i}{\sum_{i=1}^{m} x_i}$ , with $m \in \{1, \dots, N\}$, then the points $(-K_m, P_m)$ represent actual points on the supply curve. Note again that the negative sign on the trade size is to indicate that the trade would be seller initiated. The ask-side of the order book can now been used similarly to construct points on the supply curve that would represent buyer initiated trades.

One setback of analysing the supply curve based on points constructed from the order book data is that there are no data points inside the bid-ask spread. At first this sounds like huge disadvantage as many trades are actually executed inside the bid-ask spread. These trades are, however, usually executed by brokers, who are acting on behalf of two or more clients, but not price taking traders so this setback should not be consider being too serious. Also, though there are no actual data points inside the bid-ask spread, the supply curves are considered to be continuous and differential so there are no gaps or jumps on the curve around, or inside, the bid-ask spread. The model will therefore allow for trades to happen inside the spread as well as outside but it must be pointed out that because trades can be done with brokers, rather than executed at the market, the liquidity can be even higher than the model states.

## 4.3 The Data Set

The Icelandic Stock Exchange was gracious enough to provide historical limit order data for three bonds issued by the Housing Financing Fund in Iceland. The three bonds are HFF150224, HFF150434 and HFF150644, from now on these bonds will be referred to as HFF24, HFF34, and HFF44, respectively. These bonds are all CPI linked annuities[11] with maturity in 2024, 2034 and 2044, respectively. The data originates from the period of July 8th 2004 until February 5th 2010. The reasons these bonds where picked to be the basis of the analysis on the supply curve are mainly three. Firstly, the Icelandic bond market has long been keener on trading government guaranteed CPI linked bonds than the standard Treasury notes. The volume traded has been much higher than with the standard nominal denoted Treasury notes and the issues have been much larger in scale then the Treasury notes. Secondly, these bonds traded in the market for the duration of the entire period as their maturity is quite long while none of the nominal denoted treasury notes had maturity in the same magnitude. For the purpose of these studies it is also very convenient to have continuous order book data for a given security to analyse as otherwise it would have been necessary to merge the data from two or more securities. Thirdly, it is worth mentioning that although the maturity of the bonds got shorter during the data period, the duration of the bonds changed less as these are all annuities and therefore the pull-to-par effect did not affect liquidity or any barriers in market making agreements[12].

It would be possible to use other securities to analyse the supply curve but none of the securities traded in Iceland during this period did fulfil the requirements. All the stocks traded before the crisis either where taken off the market or the market making was permanently stopped after the crisis. None of the municipalities bonds traded had good market making agreements, with more than just one to three market makers, so the order books where mostly empty. The HFF bond maturing in 2014, HFF150914, and the Treasury note maturing in 2013, RIKB 13 0517, could have been used. These two bonds did trade during the whole period and there where good market making agreements with these bonds. However, the maturity of the bonds got a lot shorter during the period and therefore the bid-ask spread got lower with time. This would result in the $\alpha$ values of the models getting lower with time and that would be interpreted as a sign of better liquidity as the maturity gets closer. This is not a desirable attribute and therefore these two bonds were not used.

To construct the order book data a program had to been written to convert the limit order data from the Icelandic Stock Exchange into aggregated order book data using the method outlined above. The program was written in `Matlab` and the basic flow of the program is as follows. Each time an action is made in the limit order system, where one of the market participants puts in a new order, makes modifications to an old one or cancels an existing one, the program picks up the current order book and makes modifications to it according to the action just taken by the trader. This results in a new status of the order book and the process is executed again when the next action occurs. This way it is possible to examine the order book at any given time during trading hours and run whatever regression models or statistical analysis is desired based on the order book data. Figure 1 shows a flowchart for the basic flow of this program. Note that for simplicity reasons this flow chart of the program does not show the algorithm used to follow the Stock Exchanges rules on order priority and order book modifications.

[11] The bonds are all quoted on clean prices and they trade on the orginal ISK face value (nominal value).
[12] The allowed bid-ask spread by a market making agrrement will get lower as the maturity of bonds gets shorter and this would affect the parameters of the supply curve models.

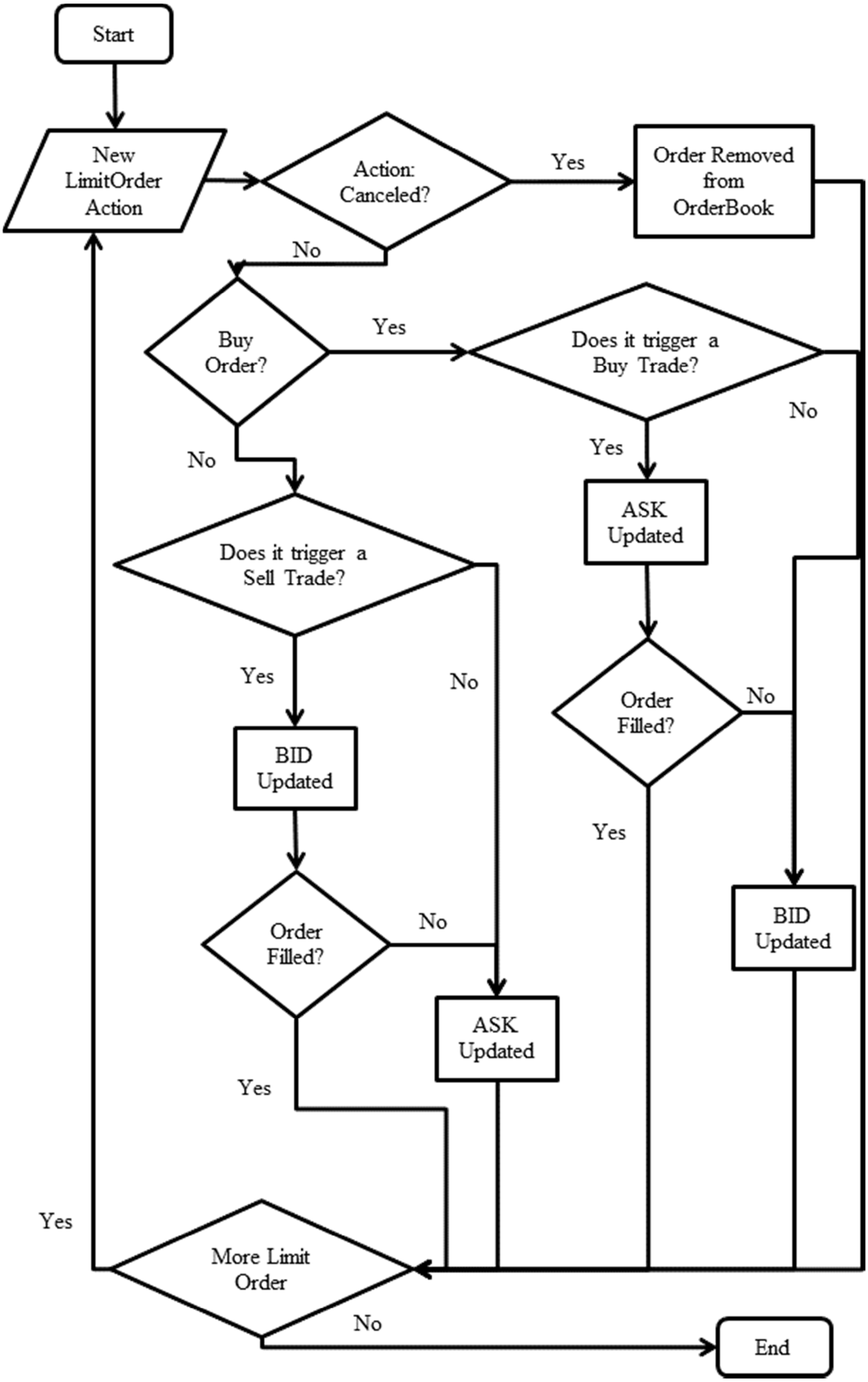


*Figure 1: Flowchart showing how limit order actions update the Order Book data*

By running this program it is possible to use the limit order data to derive the order book data at any given time. Table 1 shows the order book for HFF34 at 12:00pm on March 12th 2007. The buy side of the order book contains six buy orders amounting to a total of 700 million nominal value and the prices are the quoted clean prices for HFF34. The best bid price is 90.52 while the worst bid is 90.43. The sell side also contains six sell orders that also amount to a total of 700 million nominal value. The best offer is 90.89 while the worst offer is 91.40. Table 2 shows the order book for HFF34 one hour later, at 1:00pm on March 12th 2007. Market participants have made some modifications to their limit orders resulting in the best bid to be up to 90.72 and the best offer up to 90.92. To construct the actual data points on the supply curve the method from the previous section is used. Table 3 shows the aggregated order book for HFF34 at 1:00pm on March the 12th 2007. The data provided in this table is plotted in Figure 2 to show the actual data points for the supply curve.

*Table 1: The order book for HFF34 at 12:00pm on March 12th 2007*

| Buy Side | | Sell Side | |
|---|---|---|---|
| Quantity [m] | Price | Price | Quantity [m] |
| 100 | 90.52 | 90.89 | 100 |
| 100 | 90.51 | 90.92 | 100 |
| 200 | 90.50 | 90.95 | 200 |
| 100 | 90.48 | 90.97 | 100 |
| 100 | 90.46 | 91.20 | 100 |
| 100 | 90.43 | 91.40 | 100 |

*Table 2: The order book for HFF34 at 1:00pm on March 12th 2007*

| Buy Side | | Sell Side | |
|---|---|---|---|
| Quantity [m] | Price | Price | Quantity [m] |
| 100 | 90.72 | 90.92 | 100 |
| 100 | 90.52 | 90.95 | 200 |
| 100 | 90.51 | 90.97 | 100 |
| 200 | 90.50 | 91.17 | 100 |
| 100 | 90.48 | 91.20 | 100 |
| 100 | 90.47 | 91.40 | 100 |

*Table 3: The aggregated order book for HFF34 at 1:00pm on March 12th 2007*

| Buy Side | | Sell Side | |
|---|---|---|---|
| Quantity [m] | Price | Price | Quantity [m] |
| 100 | 90.7200 | 90.9200 | 100 |
| 200 | 90.6200 | 90.9400 | 300 |
| 300 | 90.5833 | 90.9463 | 400 |
| 500 | 90.5500 | 90.9920 | 500 |
| 600 | 90.5383 | 91.0267 | 600 |
| 700 | 90.5279 | 91.0800 | 700 |

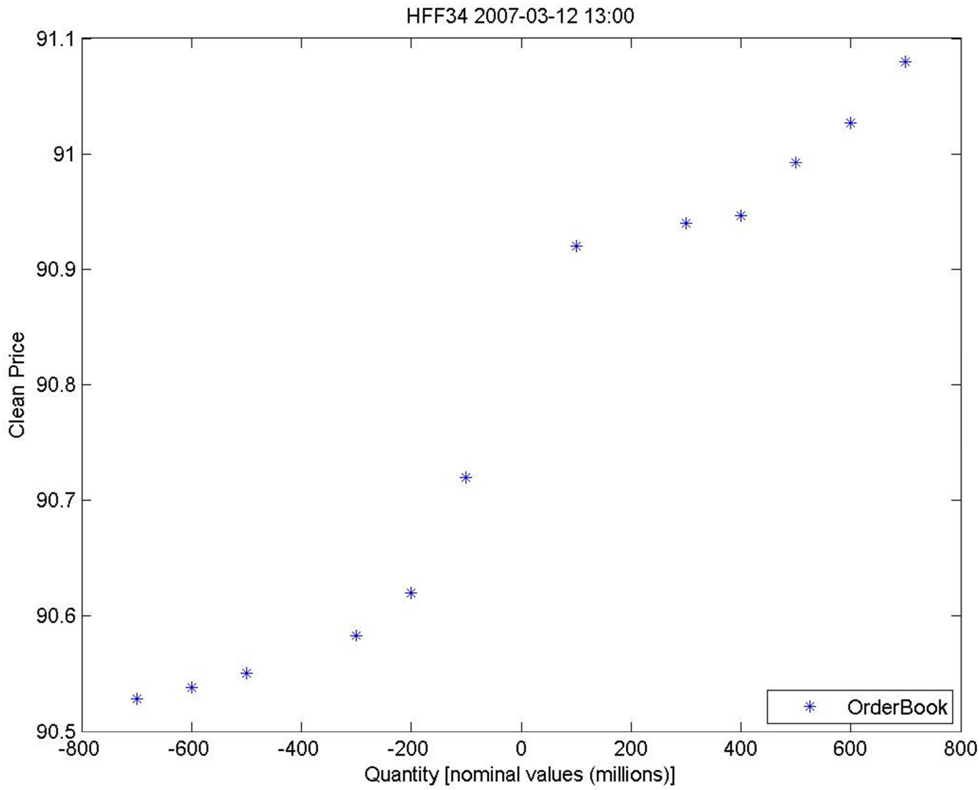


*Figure 2: Actual supply curve points for HFF34 at 1:00pm on March 12th 2007*

It is worth mentioning before continuing to the next chapter for the results of the analysis that the time period covered by the data contains the financial crisis in Iceland in 2008 and some of its after effects. During the peak of the crisis, market makers abandoned their posts in the Housing Financing Fund bond market so no market making orders were in the order books and liquidity mostly dried up during that time. Shortly after the peak of the crisis a very weak market making agreement between The Housing Financing Fund and the newly established banks was signed and liquidity improved. A few months into the crisis The Housing Financing Fund signed a market making agreement with traders within the larger banks in Iceland. This agreement was similar to the ones before the crisis and liquidity returned to normal. Therefore the data in the analysis will be broken down to four parts based on the market making situation in the market and then it will be shown that the parameters in the supply curve models are not the same in all market situations. These four parts will be known as; *before the crisis*, *during the crisis*, *weak market making* and *normal market making*.

# 5 Results

Before the results from the analysis will be looked at, a short description of the regression method used to derive α-values for the models outlined in Chapter 3 by using order book data with the same format as described in Section 4.2 is in order. The standard approach of *least squares* is used to fit the models to the order book data. The best fit in the *least squares* sense minimises the sum of squared residuals where a residual is the difference between an observed order book value and the value provided by the model.

## 5.1 The Regression

As stated earlier, in Section 4.3, the handling of the data provided by the Icelandic Stock Exchange allows the order book to be examined at any given time during the data period. However, due to the length of the data period and the huge number of actions taken by traders in the market, the order book data is only examined five times each day. The observations where made hourly from 11:00am to 3:00pm. For each observation a regression method is used to fit all four models to the order book data. Regressions where limited to observations with at least three buy and sell orders in the order book to prevent extreme cases where there only were a hand full of orders in the market[13].

By using the order book data for HFF34 at 1:00pm on March 12th 2007, the same data as shown in Table 2 and plotted in Figure 2, the following model parameters resulted from the regression.

$$
\begin{aligned}
S_1(t,x) &= M_1(x)S(t) = (0.00000477 \cdot x + 1) \cdot 90.780 \\
S_2(t,x) &= M_2(x)S(t) = e^{0.00000477 \cdot x} \cdot 90.780 \\
S_3(t,x) &= M_3(x)S(t) = (0.0001106 \cdot \operatorname{sign}(x)\sqrt{|x|} + 1) \cdot 90.782 \\
S_4(t,x) &= M_4(x)S(t) = (0.0003776 \cdot \operatorname{sign}(x)\ln(1 + |x|) + 1) \cdot 90.785
\end{aligned}
$$

In Figure 3 the observations taken from the order book of HFF34 at 1:00pm on March 12th 2007 are plotted as well as the four different models that show how they fit the order book data in different ways[14].

The regression above shows that all models return a similar estimate for the value $S(t)$. There is little more than half a basis point between the highest and lowest values. This is in line with what was expected and discussed at the beginning of Chapter 3 where $S(t)$ was described as the marginal trade price process. Thus for the marginal trade, where $x = 0$, all models give virtually the same result, i.e. the clean marginal trading price of HFF34 at 1:00pm on March 12th 2007 was about 90.78.

From now on the focus will be on the estimated α-values as the marginal trade is of little interest while the shape and form of the supply curve is more interesting.

---

[13] These cases could come up i.e. when a price taking buyer had just entered the market with a large order and hit all the offers. This also eliminates some cases durign the peak of the crisis due to few orders in the order book.

[14] Note that there is no visible difference between the *linear* and *exponential* models due to the fact that for small values of α it is known that $S_1(x,t) \approx S_2(x,t)$ around the origin.

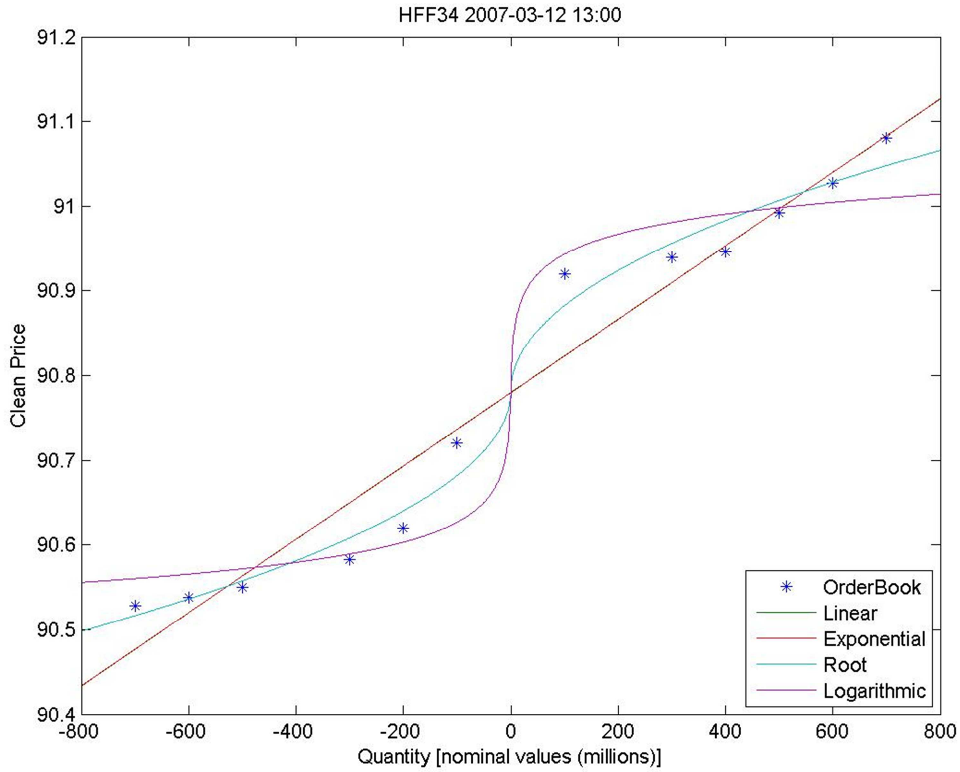


*Figure 3: Order book data for HFF34 from 1:00pm on March 12th 2007 with fitted models*

## 5.2 The Time Series of α-values

Turning the attention to the four different time periods mentioned in Section 4.3 and by visually examining the time series of α-values from our regression it can be seen that liquidity obviously varies between time periods for all bonds. Recall from Section 4.3 that there are four different time periods; *before the crisis*, *during the crisis*, *weak market making* and *normal market making*. Thus the dates that divide the periods down to four are firstly the day when market makers abandoned their posts in the market, October 7th 2008, secondly the day the weak market making agreement was signed, December 16th 2008 and finally the day when the new normal market making agreement was signed and liquidity returned to normal, July 1st 2009.

As mentioned in Section 5.1 regressions were limited to observations where there were no fewer than three buy and sell orders in the order book. The total number of order book observations used, denoted by $N$, were 13,309.[15] Table 4 shows how many order book observations were used for each bond along with break downs for each of the time periods. Four regressions were performed for each of the observations, one for each model, so the total number of regressions was 53,236. Each of this 53,236 regressions gave estimation to two parameters of the model, the value of α and the value of $S(t)$.

[15] The total number of skipped observation was 1,723. It is worth mentioning that the main results from this thesis are the same even if this restriction is changed. The only thing that chances is that the number of observations decreases and the variance of the α-values decreases with higher restriction on the number of observations in the order book as that is a sign for better liquidity.

*Table 4: The number of observation used for regression*

| Bond | Number of Observations (*N*) | Breakdown of number of Observations to Data Periods | | | |
|---|---|---|---|---|---|
| | | Before Crisis | During Crisis | Weak MM | Normal MM |
| HFF24 | 4433 | 3399 | 98 | 420 | 516 |
| HFF34 | 4460 | 3415 | 95 | 428 | 522 |
| HFF44 | 4416 | 3365 | 109 | 422 | 520 |

By plotting the time series of the α-values from these regressions it is easily shown that there is obvious difference in the magnitude of α-values between the periods mentioned above. Figure 4 shows the time series of α-values derived from the *linear* model for HFF24. The dates that divide the time period down to the four different periods are marked with vertical lines. There is clear evidence from this figure that the different state of market making agreements with HFF24 bonds affects the α-values of the *linear* model and thus the liquidity of the bonds. In Section 5.3 this will be examined in further details where it will be shown that the α-values derived from the *linear* model for different periods differ statistically between periods.

Figure 5 and Figure 6 show similar results as Figure 4 for HFF34 and HFF44, respectively. Figure 7 to Figure 15 in Appendix A show similar results for all three bonds for the *exponential*, *root* and *logarithmic* models. As with the *linear* model for HFF24 the difference of the α-values between periods will be examined in further details in Section 5.3 and the results will be the same as with the *linear* model for HFF24.

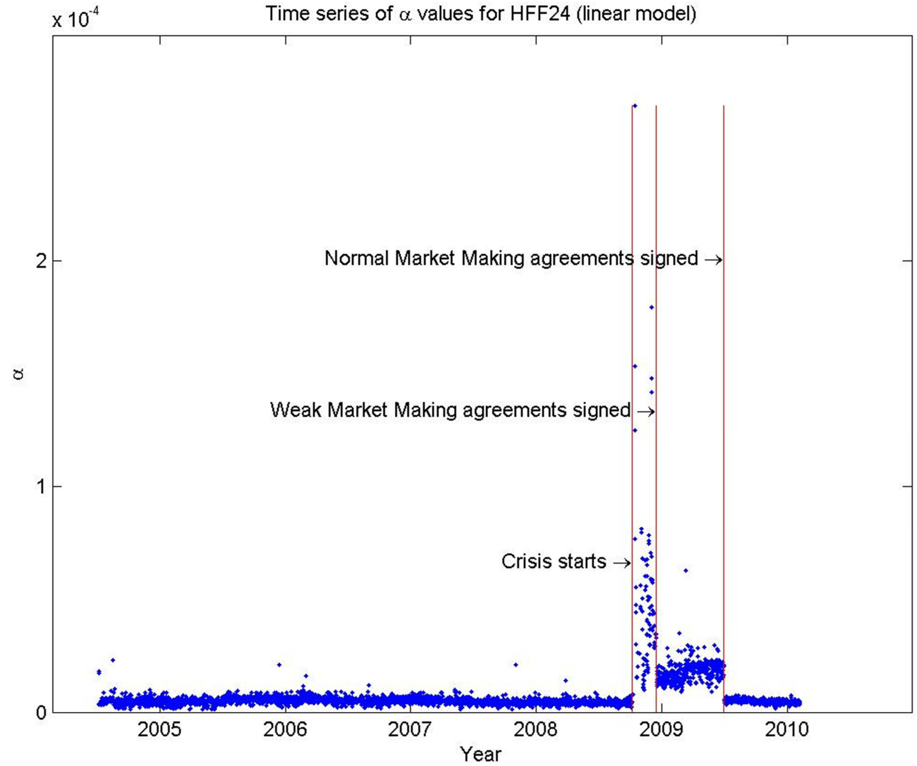


*Figure 4: Time series of α-values for HFF24 (linear model)*

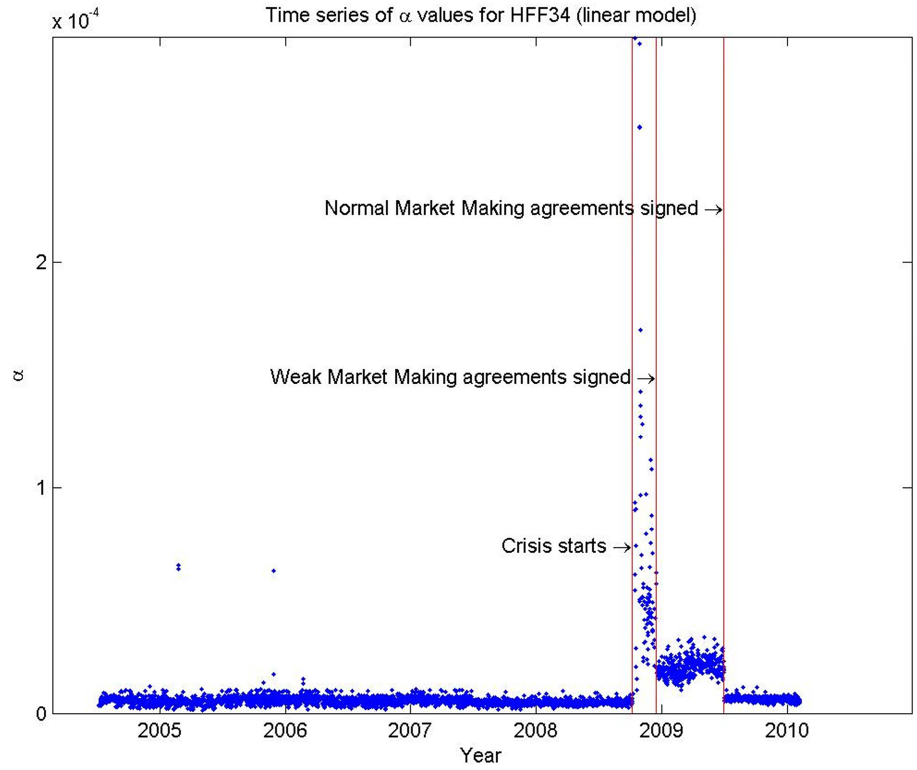


*Figure 5: Time series of α-values for HFF34 (linear model)*

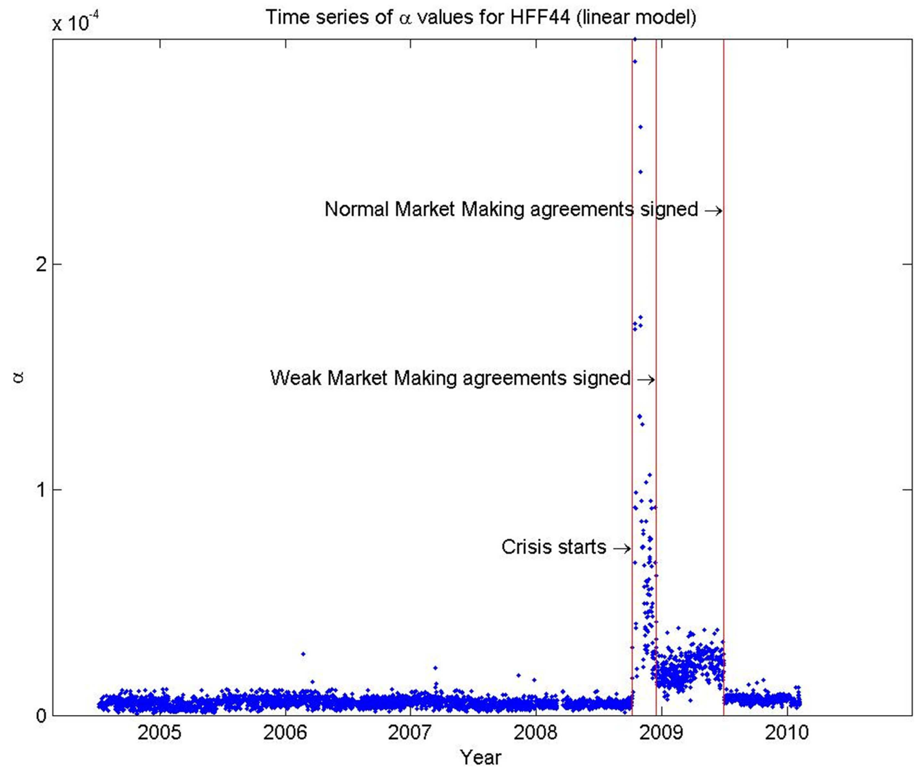


*Figure 6: Time series of α-values for HFF44 (linear model)*

## 5.3 Estimating the Supply Curve Parameters

In the previous section time series of α-values where examined and visual confirmation used to show that there is clear difference in the magnitude of the α-values between periods. Generally the α-values in the first and last periods, *before the crisis* and *normal market making*, seem to be the same. The α-values during the third period, *weak market making*, seem to be about two to three times larger than during the first and last periods while the α-values during the second period, *during the crisis*, are much higher. This shows that liquidity dried up during the crisis, increased shortly after it and then returned to normal again with the new market making agreements signed on July 1st 2009. Of particular interest is to examine the α-values within each time period and show that the α-values are statistically different from zero. If this were not the case, i.e. if $\alpha = 0$, then the supply curve would simply be a horizontal line and that would render the theory vacuous as mentioned in Chapter 4. It would therefore be good to be able to reject the classical case that $S(t,x) = S(t,0)$ and by so show that size and direction of the trades actually do affect the price obtained in the trade.

Now one of the four models from Chapter 3 is assumed, although this applies to all of them equally. The parameter α in the model is of concern. From the regression *n* supply curves $\{\hat{S}_i\}_{i=1}^{n}$ are observed and thus *n* values for the parameter α represented by $\{\alpha_i\}_{i=1}^{n}$.[16] Table 4 shows the number of regressions performed in each period and thus the value of *n*. It is now assumed that the parameters residuals have a mean zero and are normally distributed[17], i.e. $\bar{\alpha} - \alpha_i \sim N(0,\sigma^2)$ and therefore that $Z = \frac{\alpha_i - \alpha}{(\sigma_{\alpha_i - \alpha})/\sqrt{n}} \sim N(0,1)$. To show that the supply curve is non-trivial the null hypothesis $H_0\colon \alpha = 0$ is tested against the alternative $H_1\colon \alpha \neq 0$. Therefore it applies that $Z = \frac{\alpha_i}{(\sigma_{\alpha_i})/\sqrt{n}} \sim N(0,1)$ and the test statistic will be $t = \frac{\bar{\alpha} - 0}{S/\sqrt{n}} = \frac{\bar{\alpha}}{S/\sqrt{n}}$ where $S$ is the sample variance of the observed $\alpha_i$. The variance is estimated so this test statistic has the *t*-distribution with $n - 1$ degrees of freedom.

By assuming the *linear* model with the null hypothesis $H_0\colon \alpha = 0$ and the alternative $H_1\colon \alpha \neq 0$, for HFF24 during the first period, *before the crisis*, the null hypothesis will be rejected at the 95% confidence level. The estimated value of α is 0.0000049 and the 95% confidence interval is [0.0000028 ; 0.0000071]. The *linear* supply curve model for HFF24 before the crisis is thus estimated to be as follows:

$$S_1(t,x) = M_1(x)S(t) = (0.0000049 \cdot x + 1) \cdot S(t)$$

The result of this test strongly indicates that a non-trivial supply curve does exist for the HFF24 before the crisis based on the *linear* model. This will now be shown to be the case for all the bonds during all periods and based on all four models except during the crisis. The main result of our work is as follows:

Assuming any of the supply curve models from Chapter 3, with the null hypothesis $\mathrm{H}_0\colon \alpha = 0$ and the alternative $\mathrm{H}_1\colon \alpha \neq 0$, $\mathrm{H}_0$ is rejected at the 95% confidence level for all the bonds during all of the data periods except during the crisis. Moreover, only during the crisis periods do the *p*-values from the *t*-test turn out to be different from zero.

[16] The regression also provides *n* values for the marginal trade price of the underlying bond but as stated earlier this is of little interest in this contest, although it is possible to use these marginal trade prices to examine the classical trading price process for the underlying bond.
[17] In Appendix B this is investigated further.

The results of this test strongly indicate that a non-trivial supply curve does exist for these three bonds during all the data periods. Although it is not possible to reject the null hypothesis during the crisis period this should not be taken as a serious defect on the models or the theory presented in Chapter 2. The main reason for this result during the crisis is the low number of observations during the crisis and the very high standard deviation of the α values during the crisis.

Table 5 shows the estimated values of the parameter α for the *linear* supply curve model for all bonds and all periods and also the upper and lower limit of the 95% confidence interval. Table 6, Table 7 and Table 8 show the estimated values of the parameter α for the *exponential*, *root* and *logarithmic* models respectively along with the upper and lower limit of the 95% confidence interval for each estimate of α.

By using the data provided in the tables below it is possible to write out each of the estimated supply curve models for these three bonds during each of the time periods. For example, the *logarithmic* model for HFF34 during the weak market making is:

$$S_4(t,x) = M_4(x)S(t) = (0.000763 \cdot \operatorname{sign}(x)\ln(1+|x|) + 1) \cdot S(t)$$

*Table 5: Supply curve parameters for the linear model*

| Time Period | Series | Model | α | Lower Limit | Upper Limit |
|---|---|---|---|---|---|
| Before the Crisis | HFF24 | Linear | 0.0000049 | 0.0000028 | 0.0000071 |
| During the Crisis | HFF24 | Linear | 0.0000487 | -0.0000145 | 0.0000112 |
| Weak Market Making | HFF24 | Linear | 0.0000175 | 0.0000113 | 0.0000238 |
| Normal Market Making | HFF24 | Linear | 0.0000049 | 0.0000034 | 0.0000064 |
| Before the Crisis | HFF34 | Linear | 0.0000054 | 0.0000030 | 0.0000079 |
| During the Crisis | HFF34 | Linear | 0.0000681 | -0.0000230 | 0.0001592 |
| Weak Market Making | HFF34 | Linear | 0.0000203 | 0.0000138 | 0.0000270 |
| Normal Market Making | HFF34 | Linear | 0.0000065 | 0.0000049 | 0.0000082 |
| Before the Crisis | HFF44 | Linear | 0.0000055 | 0.0000024 | 0.0000085 |
| During the Crisis | HFF44 | Linear | 0.0000676 | -0.0000240 | 0.0001591 |
| Weak Market Making | HFF44 | Linear | 0.0000208 | 0.0000115 | 0.0000300 |
| Normal Market Making | HFF44 | Linear | 0.0000073 | 0.0000049 | 0.0000096 |

*Table 6: Supply curve parameters for the exponential model*

| Time Period | Series | Model | α | Lower Limit | Upper Limit |
|---|---|---|---|---|---|
| Before the Crisis | HFF24 | Exponential | 0.0000049 | 0.0000028 | 0.0000071 |
| During the Crisis | HFF24 | Exponential | 0.0000485 | -0.0000133 | 0.0001105 |
| Weak Market Making | HFF24 | Exponential | 0.0000175 | 0.0000113 | 0.0000238 |
| Normal Market Making | HFF24 | Exponential | 0.0000049 | 0.0000034 | 0.0000064 |
| Before the Crisis | HFF34 | Exponential | 0.0000054 | 0.0000030 | 0.0000079 |
| During the Crisis | HFF34 | Exponential | 0.0000681 | -0.0000218 | 0.0001580 |
| Weak Market Making | HFF34 | Exponential | 0.0000203 | 0.0000138 | 0.0000269 |
| Normal Market Making | HFF34 | Exponential | 0.0000065 | 0.0000049 | 0.0000082 |
| Before the Crisis | HFF44 | Exponential | 0.0000055 | 0.0000024 | 0.0000085 |
| During the Crisis | HFF44 | Exponential | 0.0000674 | -0.0000215 | 0.0001562 |
| Weak Market Making | HFF44 | Exponential | 0.0000208 | 0.0000115 | 0.0000300 |
| Normal Market Making | HFF44 | Exponential | 0.0000073 | 0.0000049 | 0.0000096 |

*Table 7: Supply curve parameters for the root model*

| Time Period | Series | Model | α | Lower Limit | Upper Limit |
|---|---|---|---|---|---|
| Before the Crisis | HFF24 | Root | 0.000107 | 0.000077 | 0.000136 |
| During the Crisis | HFF24 | Root | 0.000692 | -0.000299 | 0.001684 |
| Weak Market Making | HFF24 | Root | 0.000255 | 0.000197 | 0.000312 |
| Normal Market Making | HFF24 | Root | 0.000101 | 0.000079 | 0.000122 |
| Before the Crisis | HFF34 | Root | 0.000118 | 0.000087 | 0.000150 |
| During the Crisis | HFF34 | Root | 0.001084 | -0.000744 | 0.002912 |
| Weak Market Making | HFF34 | Root | 0.000296 | 0.000232 | 0.000260 |
| Normal Market Making | HFF34 | Root | 0.000130 | 0.000109 | 0.000151 |
| Before the Crisis | HFF44 | Root | 0.000124 | 0.000080 | 0.000168 |
| During the Crisis | HFF44 | Root | 0.001200 | -0.000972 | 0.003373 |
| Weak Market Making | HFF44 | Root | 0.000319 | 0.000238 | 0.000400 |
| Normal Market Making | HFF44 | Root | 0.000152 | 0.000123 | 0.000182 |

*Table 8: Supply curve parameters for the logarithmic model*

| Time Period | Series | Model | α | Lower Limit | Upper Limit |
|---|---|---|---|---|---|
| Before the Crisis | HFF24 | Logarithm | 0.000351 | 0.000271 | 0.000432 |
| During the Crisis | HFF24 | Logarithm | 0.001746 | -0.000955 | 0.004447 |
| Weak Market Making | HFF24 | Logarithm | 0.000656 | 0.000528 | 0.000783 |
| Normal Market Making | HFF24 | Logarithm | 0.000322 | 0.000255 | 0.000388 |
| Before the Crisis | HFF34 | Logarithm | 0.000391 | 0.000311 | 0.000472 |
| During the Crisis | HFF34 | Logarithm | 0.002967 | -0.003034 | 0.008968 |
| Weak Market Making | HFF34 | Logarithm | 0.000763 | 0.000608 | 0.000917 |
| Normal Market Making | HFF34 | Logarithm | 0.000408 | 0.000350 | 0.000467 |
| Before the Crisis | HFF44 | Logarithm | 0.000422 | 0.000304 | 0.000541 |
| During the Crisis | HFF44 | Logarithm | 0.003485 | -0.004029 | 0.011000 |
| Weak Market Making | HFF44 | Logarithm | 0.000847 | 0.000666 | 0.001029 |
| Normal Market Making | HFF44 | Logarithm | 0.000491 | 0.000406 | 0.000577 |

From the information provided in Table 5 it can be shown that there is statistical difference in the estimated value of α in the linear model between periods for each of the bonds based on the 95% confidence intervals. This can be seen by comparing the upper and lower limits of the confidence intervals between periods and confirming that they do not collide. The upper limit of the confidence interval of α for the HFF24 before the crisis is 0.0000071 while the lower limit during the weak market making is 0.0000113. This confirms that the value of α differs statistically before the crisis and during the weak market making. After normal market making the upper limit is 0.0000064 and thus lower than the lower limit during the weak market making. Therefore the value of α differs statistically during the weak market making and after the normal market making agreements were signed. Finally it can be seen from the data in Table 5 that there is no statistical difference in the value, and thereby the liquidity, of the α value before the crisis and after the normal market making agreement was signed.[18] Similar results as the one above can be derived from Table 6, Table 7 and Table 8 for the *exponential*, *root* and *logarithmic* models, respectively, showing how the parameters of the models vary between periods.

Comparing the confidence intervals for the estimated α values of the models between the first and the last period, i.e. *before the crisis* and in *normal market making*, should show if there is statistical difference between these two periods or not. This would then show if liquidity possibly changed permanently during the crisis and its after-effects or not. Interestingly, comparing the confidence intervals shows that there is no statistical difference between liquidity before the crisis and after the new market making agreement was signed.

The values from the tables above can be used to show how the size of a trade did affect the price obtained in the trade during different periods. Consider a price taking trader entering into the market *before the crisis* with 250 million nominal value of HFF34 to sell. If the trader used the *logarithmic* model to describe his supply curve he would be facing this model:

$$S_4(t,x) = M_4(x)S(t) = (0.000391 \cdot \text{sign}(x)\ln(1+|x|) + 1) \cdot S(t)$$

[18] There is no point in discussing the statistical difference between *during the crisis* values of α and other time periods as the high variance of α during that time results in a parameters estimate that is not statistically different from zero.

For simplicity reasons it is assumed that the bond was trading at par value, i.e. the marginal trading price was 100.00. The trader would then, on average, obtain the following price for his 250 million nominal of HFF34 at this time:

$$S_4(t, -250) = (0.000391 \cdot \mathrm{sign}(-250)\ln(1 + |-250|) + 1) \cdot 100 = 99.784$$

The trader thus gets a price that is on average 0.216% lower than the marginal trade price at the given time. Using the upper and lower limit of the 95% confidence interval for the value of α as an input in the model gives the upper and lower limit for the price obtained. The limit prices are 99.739 and 99.828 where 99.739 represents the upper limit of the confidence interval, i.e. less liquidity, while 99.828 represents the lower limit of the confidence interval and thus more liquidity. The price the trader can expect to get can therefore be presented in error terms as $99.78 \pm 0.05$.

If now this same trader would have entered the market to sell his 250 million nominal in HFF34 during the *weak market making* period he would have been faced with this supply curve model:

$$S_4(t, x) = M_4(x)S(t) = (0.000763 \cdot \mathrm{sign}(x)\ln(1 + |x|) + 1) \cdot S(t)$$

Again, it is assumed that the marginal trade price is 100.00. The trader would then, on average, have obtained the following price for his 250 million nominal of HFF34:

$$S_4(t, -250) = (0.000738 \cdot \mathrm{sign}(-250)\ln(1 + |-250|) + 1) \cdot 100 = 99.592$$

The trader now gets a price that is on average 0.408% lower than the marginal trade price at the given time. Therefore the trader receives a price that is twice as far from the marginal price as it was before the crisis. By using the limits from the confidence interval the price can be presented in error terms as $99.59 \pm 0.09$. This shows that there is statistical difference in the price obtained *before the crisis* and during the *weak market making*.

By looking at the parameters from the *crisis* period it can be shown that this trader would, on average, receive the following price for his sale of 250 million nominal in HFF34 based on the *logarithmic* model:

$$S_4(t, -250) = (0.002967 \cdot \mathrm{sign}(-250)\ln(1 + |-250|) + 1) \cdot 100 = 98.356$$

The trader now gets a price that is on average 1.64% lower than the marginal trade price at the given time. The trader thus receives a price that is almost 8 times further from the marginal trade price than he would have received during a period with normal market making and liquidity.

It has now been shown in this section that there is strong evidence showing that the supply curve is non-trivial and that all the models point to the same conclusion. It has, however, not yet been mentioned which of the four models is best to describe the supply curve for The Icelandic Housing Financing Fund bonds.

## 5.4 Model Comparison

The next step is to find out which of the models is the best one to describe the supply curve, i.e. the one that fits the data best. It must be pointed out here, that there is not necessarily one best model for all three bonds and all four time periods so the results can vary between bonds and periods. To compare the models the regression results for each bond will be examined within one time period at a time. For each regression the model that best fits the data is selected and the number of selections for each model is aggregated within the time period. The model that best fits the data most often is selected as the best model for that bond within the specified time period. If then, it turns out that one model always, or at least more often than others, is the best, then that model is most likely the best to describe the supply curve.

The most commonly used method to compare the performance of different models is to compare the coefficient of determination, denoted by $R^2$. The model that has the highest coefficient of determination is the one that best fits the data and is thus considered to be the best performing model. One of the drawbacks of using $R^2$ to compare models performances is that in least squares regression $R^2$ increases slightly with increasing number of parameters in the model. This however is not a problem here as all the models set forth in Chapter 3 have the same number of parameters. It is also possible to use other methods to compare models and optimally the result will be the same, i.e. both methods show the same model as the best one.

To calculate the $R^2$ for a model from one of the regressions a few formulas and definitions are required. The observed order book data, i.e. the actual order book prices, are denoted by $y_k$ where $k = 1, \dots, n$ and $n$ is the number of observed points on the supply curve at that hour. For each $y_k$ there is an associated modelled value $f_k$ derived from the estimated model. The mean of the observed data is denoted by $\bar{y} = \frac{1}{n}\sum_{i=1}^{n} y_i$ and the total sum of squares and the residual sum of squares by $TSS = \sum_i (y_i - \bar{y})^2$ and $RSS = \sum_i (y_i - f_i)^2$ respectively. The coefficient of determination is then given by $R^2 = 1 - \frac{RSS}{TSS}$.

The value of *TSS* only depends on the actual observed data so for each regression the value of *TSS* is the same for all the models. The model with the lowest residual sum of squares, *RSS*, is therefore the one with the highest $R^2$. Comparing the values of *RSS* and finding the model with the lowest one will therefore give the same results as finding the model with the highest $R^2$. Later on in this section results from the comparison of the *RSS* values will be shown.

Another method to compare models is to use the sum of absolute residuals and finding the model with the lowest value. The sum of absolute residuals is given by $SAR = \sum_i |y_i - f_i|$. Optimally the results from the *SAR* comparison will be the same as from the *RSS*, i.e. both give the same model as the best model. Later on in this section results from the comparison of the *SAR* values will be shown and also an analysis on whether the two different methods return the same results or not.

Table 9 shows the main results from the comparison of the value of the residual sum of squares, *RSS*, between the models. The results are represented in per cents where the per cent value shows how often each model had the lowest value of *RSS*. From the first line of Table 9, showing results for HFF24 before the crisis, it can be seen that the *linear* model had the lowest *RSS* value in 2.0% of the regressions, the *exponential* model also had the lowest *RSS* value in 2.0% of the regressions and the *root* model had the lowest value in 22.3% of the regressions. The *logarithm* model however had the lowest *RSS* value in 73.7% of the regressions made for HFF24 before the crisis. Therefore the *logarithm* model is selected as the best model to describe the supply curve for HFF24 before the crisis based on the *RSS* values.

The data in Table 9 clearly shows that based on the *RSS* values the *logarithm* model is the best to describe the supply curves for these three bonds issued by The Icelandic Housing Financing Fund. It turns out that in only one case, for HFF24 during the crisis, the *logarithm* model is the best model less than 50% of the time. This in fact is the only case where the *linear* and *exponential* model come close to be as good fits at the *logarithm* model, scoring 20.4% and 24.5% respectively while the *logarithm* model had 45.9%. For HFF34 and HFF44 the *root* model gets closest to scoring as well as the *logarithm* model during the crisis, hitting 33.7% and 30.3% while the *logarithm* model has 52.6% and 55.0%. In other cases the *logarithm* model scores from 67% up to 81%. For convenience the highest value in each line is written in bold.

Table 10 and Table 11 show even more clearly, how much better the *logarithm* model is than the other models based on the *RSS*. In these two tables the values from Table 9 have been added up to show the aggregated numbers for each time period and bond, respectively. It turns out that the *logarithm* model is the best fit about 75% of the time and only dropping down to 51.3% in one case. The 51.3% value is taken during the crisis when the market was illiquid and few orders where in the market. This low value for the *logarithm* model can in fact be more related to the fact that orders were few and far between during the crisis rather than the model not being good. The *linear* and *root* models do a much better job during the crisis than they do during other time periods. From Table 11 it is also clear that the *root* model is the second best to the *logarithm* model to describe the supply curves based on *RSS* values.

*Table 9: Modes performances (showing number of lowest RSS values)*

| Time Period | Series | *N* | Linear | Exponential | Root | Logarithm |
|---|---|---|---|---|---|---|
| Before the Crisis | HFF24 | 3399 | 2.0% | 2.0% | 22.3% | **73.7%** |
| Before the Crisis | HFF34 | 3415 | 2.2% | 1.5% | 21.1% | **75.2%** |
| Before the Crisis | HFF44 | 3365 | 3.4% | 2.4% | 26.8% | **67.5%** |
| During the Crisis | HFF24 | 98 | 9.2% | 20.4% | 24.5% | **45.9%** |
| During the Crisis | HFF34 | 95 | 10.5% | 3.2% | 33.7% | **52.6%** |
| During the Crisis | HFF44 | 109 | 9.2% | 5.5% | 30.3% | **55.0%** |
| Weak Market Making | HFF24 | 420 | 4.0% | 1.0% | 21.4% | **73.6%** |
| Weak Market Making | HFF34 | 428 | 1.6% | 2.3% | 15.9% | **80.1%** |
| Weak Market Making | HFF44 | 422 | 3.6% | 2.8% | 19.2% | **74.4%** |
| Normal Market Making | HFF24 | 516 | 4.8% | 1.0% | 26.0% | **68.2%** |
| Normal Market Making | HFF34 | 522 | 1.0% | 1.0% | 16.3% | **81.8%** |
| Normal Market Making | HFF44 | 520 | 1.7% | 0.8% | 22.5% | **75.0%** |

*Table 10: Model performances in different time periods (lowest RSS values)*

| **Time Period** | **Series** | ***N*** | **Linear** | **Exponential** | **Root** | **Logarithm** |
|---|---|---|---|---|---|---|
| Before the Crisis | All | 10179 | 2.5% | 2.0% | 23.4% | **72.2%** |
| During the Crisis | All | 302 | 9.6% | 9.6% | 29.5% | **51.3%** |
| Weak Market Making | All | 1270 | 3.1% | 2.0% | 18.8% | **76.1%** |
| Normal Market Making | All | 1558 | 2.5% | 0.9% | 21.6% | **75.0%** |

*Table 11: Model performances for different bonds (lowest RSS values)*

| **Time Period** | **Series** | ***N*** | **Linear** | **Exponential** | **Root** | **Logarithm** |
|---|---|---|---|---|---|---|
| All | HFF24 | 4433 | 2.7% | 2.2% | 22.7% | **72.4%** |
| All | HFF34 | 4460 | 2.2% | 1.6% | 20.3% | **76.0%** |
| All | HFF44 | 4416 | 3.4% | 2.3% | 25.7% | **68.7%** |

Table 12 shows the main results from comparison of the value of the sum of absolute residuals, *SAR*, between the models. As in Table 9 to Table 11 the results are represented in per cents where the per cent value shows how often each model had the lowest value of *SAR*. From the first line of Table 12, showing results for HFF24 before the crisis, it can be seen that the *linear* model had the lowest *SAR* value in 1.8% of the regressions, the *exponential* model had the lowest *SAR* value in 2.0% of the regressions and the *root* model had the lowest value in 22.2% of the regressions. The *logarithm* model however had the lowest *SAR* value in 74.0% of the regressions made for HFF24 before the crisis. Therefore the *logarithm* model is selected as the best model to describe the supply curve for HFF24 before the crisis based on the *SAR* values.

The data in Table 12 clearly shows that based on the *SAR* values the *logarithm* model is the best to describe the supply curves for these three bonds issued by The Icelandic Housing Financing Fund. It turns out that in only one case, for HFF24 during the crisis, the *logarithm* model is the best model less than 60% of the time. This in fact is the only case where the other models all score more than 10% and this is also the case where the *root* model is closest to scoring as high as the *logarithm* model. In other cases the *logarithm* model scores from 63% up to more than 80%.

Table 13 and Table 14 show even more clearly how much better the *logarithm* model is than the other models based on the *SAR* values. In these two tables the values from Table 12 have been added up to show the aggregated numbers for each time period and bond, respectively. It turns out that the *logarithm* model is the best fit about 75% of the time and only dropping down to 57.3% in one case. The 57.3% value is during the crisis when the market was illiquid and few orders where in the market. This low value for the *logarithm* model can in fact be more related to the fact that orders were few and far between during the crisis rather than the model not being good. The other models turn out to be better performing during the crisis than they were in other periods. This fact is most likely resulting from the fact that during the crisis the market was moving very fast and volatility was high so market participants where more cautious to place their orders alongside other orders in the order book. It is, however, clear from Table 14 that the *root* model is the second best to the *logarithm* model to describe the supply curves based on *SAR* values.

*Table 12: Model performances (showing number of lowest SAR values)*

| Time Period | Series | *N* | Linear | Exponential | Root | Logarithm |
|---|---|---|---|---|---|---|
| Before the Crisis | HFF24 | 3399 | 1,8% | 2,0% | 22,2% | **74,0%** |
| Before the Crisis | HFF34 | 3415 | 1,8% | 1,7% | 20,1% | **76,3%** |
| Before the Crisis | HFF44 | 3365 | 3,1% | 2,4% | 26,0% | **68,5%** |
| During the Crisis | HFF24 | 98 | 10,2% | 12,2% | 33,7% | **43,9%** |
| During the Crisis | HFF34 | 95 | 8,4% | 4,2% | 24,2% | **63,2%** |
| During the Crisis | HFF44 | 109 | 5,5% | 4,6% | 25,7% | **64,2%** |
| Weak Market Making | HFF24 | 420 | 4,0% | 1,4% | 19,8% | **74,8%** |
| Weak Market Making | HFF34 | 428 | 1,2% | 1,9% | 17,3% | **79,7%** |
| Weak Market Making | HFF44 | 422 | 2,6% | 3,8% | 20,4% | **73,2%** |
| Normal Market Making | HFF24 | 516 | 4,5% | 2,1% | 25,0% | **68,4%** |
| Normal Market Making | HFF34 | 522 | 0,6% | 1,5% | 15,1% | **82,8%** |
| Normal Market Making | HFF44 | 520 | 1,9% | 0,8% | 22,3% | **75,0%** |

*Table 13: Model performances in different time periods (lowest SAR values)*

| Time Period | Series | *N* | Linear | Exponential | Root | Logarithm |
|---|---|---|---|---|---|---|
| Before the Crisis | All | 10179 | 2,2% | 2,0% | 22,8% | **73,0%** |
| During the Crisis | All | 302 | 7,9% | 6,9% | 27,8% | **57,3%** |
| Weak Market Making | All | 1270 | 2,6% | 2,4% | 19,2% | **75,9%** |
| Normal Market Making | All | 1558 | 2,3% | 1,5% | 20,8% | **75,4%** |

*Table 14: Model performances for different bonds (lowest SAR values)*

| Time Period | Series | *N* | Linear | Exponential | Root | Logarithm |
|---|---|---|---|---|---|---|
| All | HFF24 | 4433 | 2,5% | 2,2% | 22,6% | **72,8%** |
| All | HFF34 | 4460 | 1,7% | 1,7% | 19,3% | **77,1%** |
| All | HFF44 | 4416 | 3,0% | 2,4% | 25,0% | **69,6%** |

The results from the model comparison above shows that the *logarithm* model is the best to fit the supply curve for these three bonds issued by The Icelandic Housing Financing Fund both in terms of *RSS* and *SAR* values. Table 15 shows that based on the lowest *RSS* values the *logarithm* model is the best fit 72.4% of the time, while based on *SAR* values it is the best fit 73.2% of the time. From the same table it is also clear that the *root* model is the second best fit. This result is in line with the description put forth in Chapters 3 and 4 where the rather high bid-ask spread was mentioned and also the market participants tend to place their orders alongside other orders. These two factors are better described by the *S*-shape of the *root* and *logarithm* models than they are by the shape of the *linear* and *exponential* models.

*Table 15: Model performances comparison aggregated by comparison method*

| ***Comparison*** | **Linear** | **Exponential** | **Root** | **Logarithm** |
|---|---|---|---|---|
| RSS | 2,7% | 2,0% | 22,9% | **72,4%** |
| SAR | 2,4% | 2,1% | 22,3% | **73,2%** |

# 6 Conclusion

The liquidity analysis of Cetin et al [6], which depends on a supply curve, was taken as given. Using historical limit order data provided by the Icelandic Stock Exchange, it was possible to reconstruct historical order books for three bonds. First this order book data was used to show that the supply curve is non-trivial and that liquidity was considerably less during the crisis in 2008 than it was both before and after the crisis. Then four different models for the supply curve were fitted and then finally the residual values were used to select the model that best fitted the data. It turned out that the logarithm model had the best fit and therefore is the best model to describe the supply curve. This result is in line with what was expected for the Icelandic Housing Financing Fund bonds as high bid-ask spread and low volume are well known characteristics of the Icelandic Housing Financing Fund bond market.

It would be possible to use the methods from this thesis to model a supply curve for other bonds and stocks trading in the Icelandic Stock Exchange. However, the data period would not be long as the market is only just now becoming active again after the financial crash in 2008. Bonds that would be possible to examine now are for example the newly issued bonds by the Icelandic Municipality Credit Fund and also the newly issued bonds by the municipality of Reykjavik. All these bonds have good market making agreements and therefore information from the order books can be used. Covered Bonds issued by the newly established banks in Iceland would not be suitable as they usually only have one market maker showing orders in the order book. It would be possible to use some of the stocks trading in the Exchange today too but volume there is still rather low and the market making agreements differ a lot between stocks so it would be necessary to account for that difference while analysing the supply curve for the stocks.

The results from this thesis should be welcomed by market participants in Iceland, the Stock Exchange, bond and stock issuers and last, but not least, the risk management departments within the banks. It has long been known among market participants in Iceland that there is a tendency in the market to place orders alongside other orders in the order book, or at least not too far from them. This has now been shown to be true with the good performances of the *S*-shaped models and therefore the existence of this tendency is very likely. The Icelandic Stock Exchange should realise from the results of this thesis that the existence of market makers and market making agreements is vital for liquidity in the market. This was shown by the very low liquidity during the crisis period when market makers abandoned their posts. The Stock Exchange should therefore focus on making a business friendly environment for market makers and thus ensure that liquidity in the market stays good. Bond and stock issuers should be able to grasp from the results of this thesis that a good market making agreement is vital to the liquidity of their product. Good liquidity attracts investors so a bond issuer should get lower rates and a stock issuer should get a higher price with good market making agreements. Finally, risk management departments within the banks should welcome the results from this thesis as they finally have a way of modelling liquidity risk in the market. The model presented in this thesis could be used as a standalone model to manage the banks' liquidity or be used as inputs into other risk models used. The models can be used to price derivatives with respect to liquidity risk and also as an input to Value-at-Risk (VaR) models. One of the setbacks of using standard VaR models is that the method only uses the size of the banks' positions, the price volatility for each position and the correlation between each price process to model in some way how risky the banks' total position is. The result from this thesis shows that the price obtained from selling a given exposure in the market is highly related to the size of the exposure and therefore all VaR models should use supply curves as an input when estimating the Value-at-Risk for the bank.

# References

1. **Duffie, D.** *Dynamic asset pricing theory.* 2nd edn. New Jersey : Prinston University Press, 1996.

2. **Bielecki, T. and Rutkowski, M.** *Credit risk: Modeling, valuation and hedging.* New York : Springer-Verlag, 2002.

3. **Jorion, P.** *Value at Risk.* 2nd edn. New York : McGraw Hill, 2001.

4. **Jarrow, R.** Default parameter estimation using market prices. *Financial Anal.* 2001, September/October, pp. 75-92.

5. **Jarrow, R. and Turnbull, S.** An integrated approach to the hedging and pricing of eurodollar derivatives. 1997, Vol. 62, 2, pp. 271-299.

6. **Cetin, U., Jarrow, R. and Protter, P.** Liquidity risk and arbitrage pricing theory. *Finance Stochasts.* 2004, Vol. 8, pp. 311-674.

7. **Heath, D., Jarrow, R. and Morton, A.** Bond pricing and the term structure of interst rates: A new methodology for contingent claims caluation. *Econometrica.* 1992, pp. 77-105.

8. **Blais, M. and P.Protter.** An analysis of the supply curve for liquidty risk through book data. *International journal of theoretical and applied finance.* 6, 2010, Vol. 13, pp. 821-838.

9. **Kyle, A.** Continuous auctions and insider trading. *Econometrica.* 1985, Vol. 53, pp. 1315-1335.

10. **Glosten, L. and Milgrom, P.** Bid, ask and transaction prices in a specialist market with heterogeneously informed traders. *Journar of Financial Economics.* 1985, Vol. 14, March, pp. 71-100.

11. **Grossman, S. and Miller, M.** Liquidity and market structure. *The Journal of Finance.* 3, 1988, Vol. 43, pp. 617-637.

12. **Jarrow, R. and Protter, P.** Liquidity risk and option pricing theory. *Handbook in operation research and management science: Financial engineering.* s.l. : J. Brige and V. Linetsky, eds. (North Holland), 2007, pp. 727-762.

13. **Björk, T.** *Arbitrage theory in contionious time.* 2nd edn. New York : Oxford University Press, 2004.

14. **Lee, C. and Ready, M.** Ingerring trade direction from intraday data. *The Journal of Finance.* 1991, Vol. 49, pp. 733-746.

15. **Protter, M. Blais and P.** *Signing trades and an evalutation of the Lee-Ready algorithm.* s.l. : Springer-Verlag, 2011.

# Appendices

## A. Time Series of α-values

The following figures show the same information as Figure 4, Figure 5 and Figure 6 from Section 5.2 but for the *exponential*, *root* and *logarithmic* models. All these figures show the same behaviour of the time series of α-values as described in Section 5.2 for the *linear* model. By visual examination it is clear, and this was shown statistically in Section 5.3, that liquidity was considerably less during the *weak market making* period than it was *before the crisis* and *after normal market making*.

From all the figures below, and also the ones in Section 5.2 for that matter, it is clear that the α-values *during the crisis* are on average much higher than during other periods. However, as mentioned in Section 5.3, it is not possible to state statistically that there is a difference between the *crisis* and other periods as the data points *during the crisis* are too few and their variance is too high.

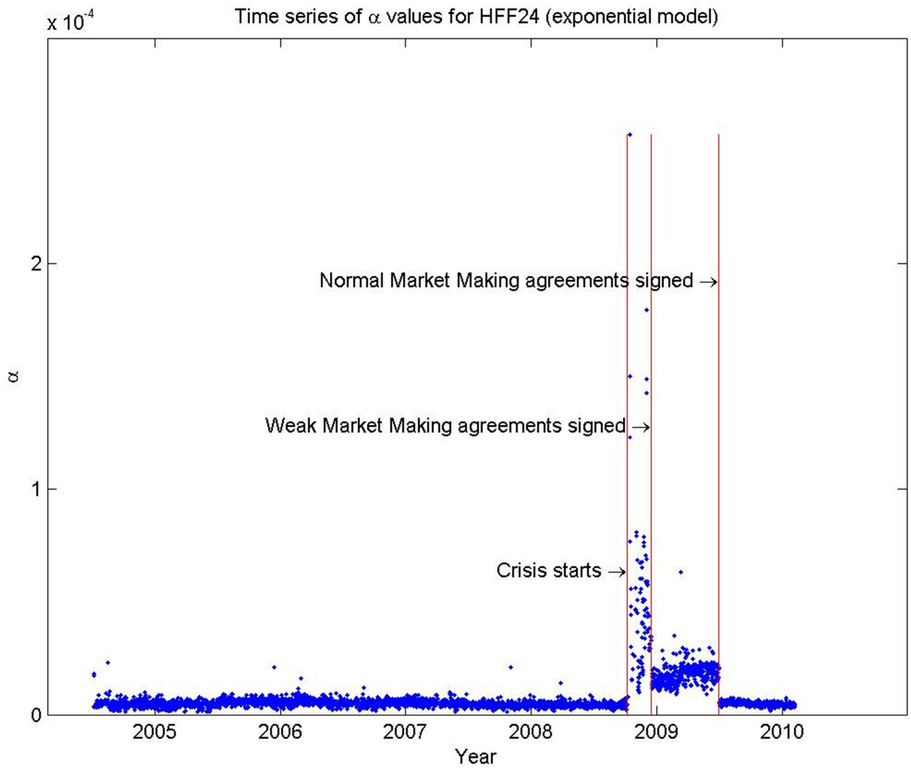


*Figure 7: Time series of α-values for HFF24 (exponential model)*

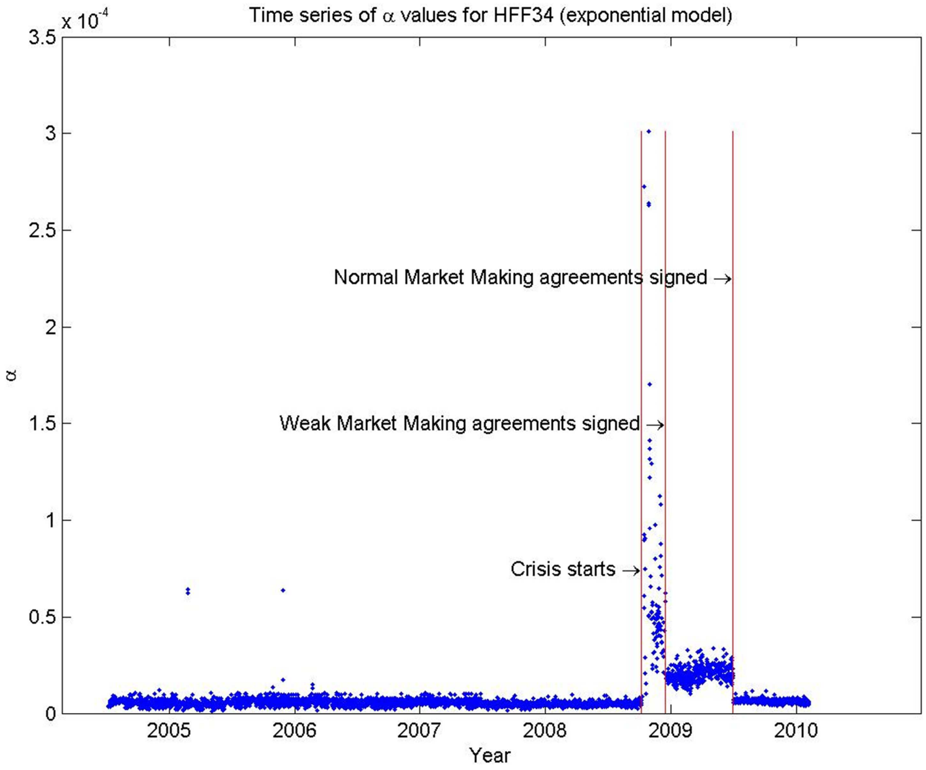


*Figure 8: Time series of α-values for HFF34 (exponential model)*

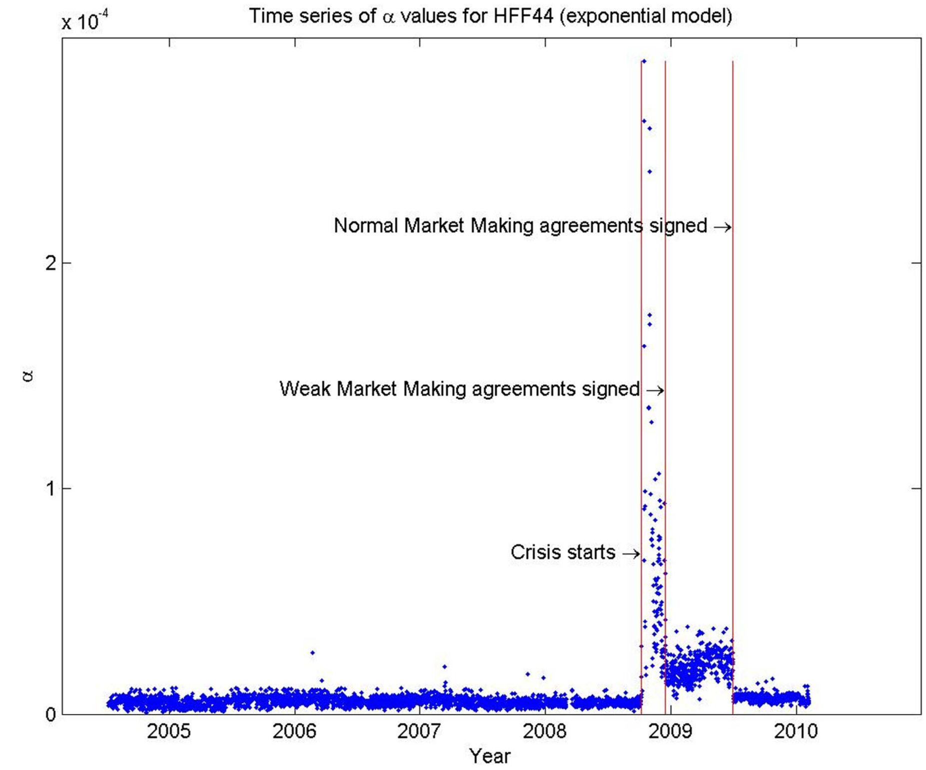


*Figure 9: Time series of α-values for HFF44 (exponential model)*

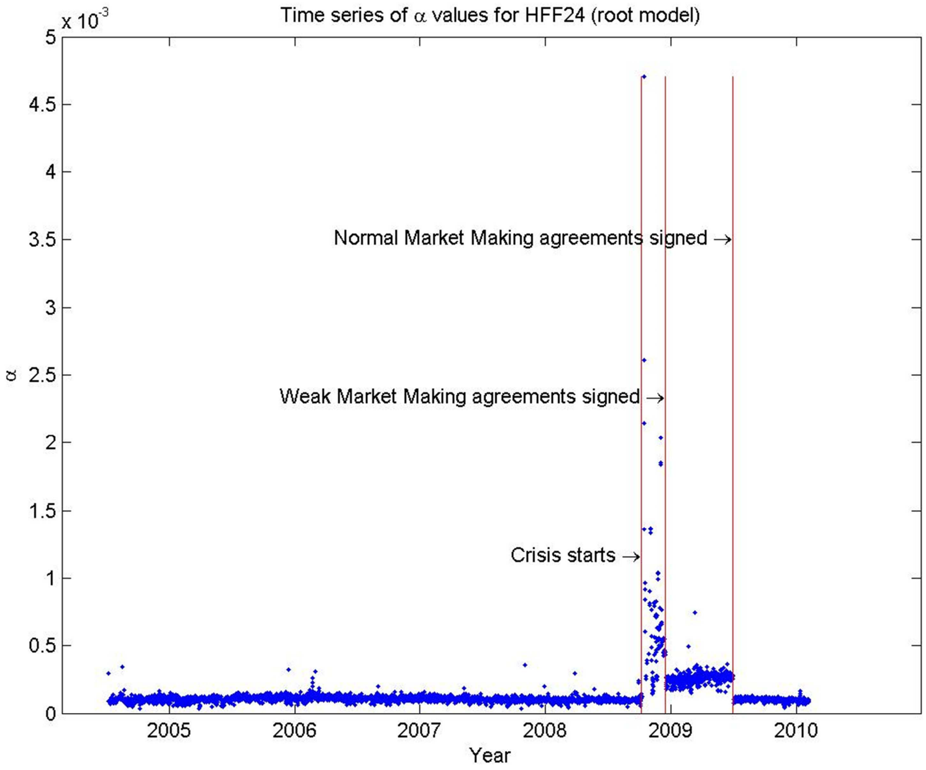


*Figure 10: Time series of α-values for HFF24 (root model)*

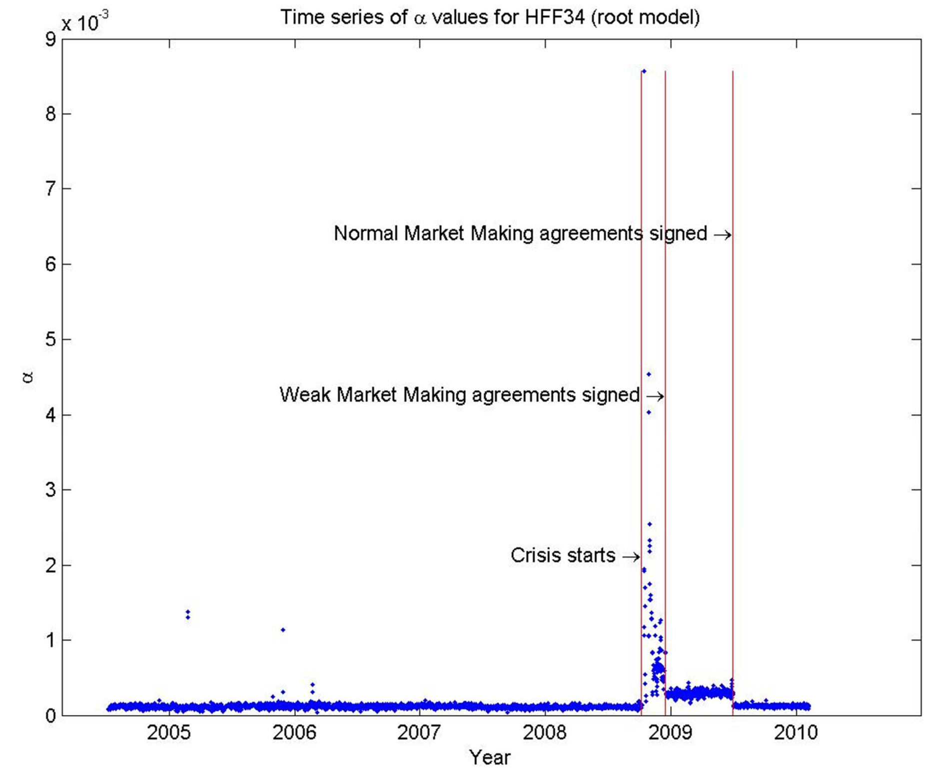


*Figure 11: Time series of α-values for HFF34 (root model)*

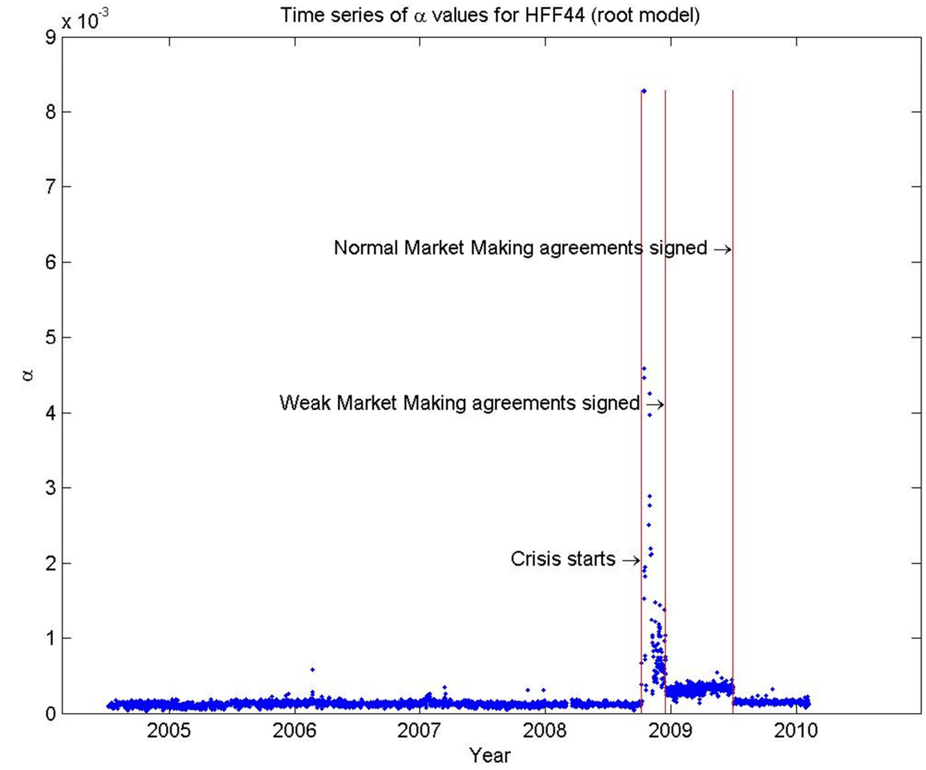


*Figure 12: Time series of α-values for HFF44 (root model)*

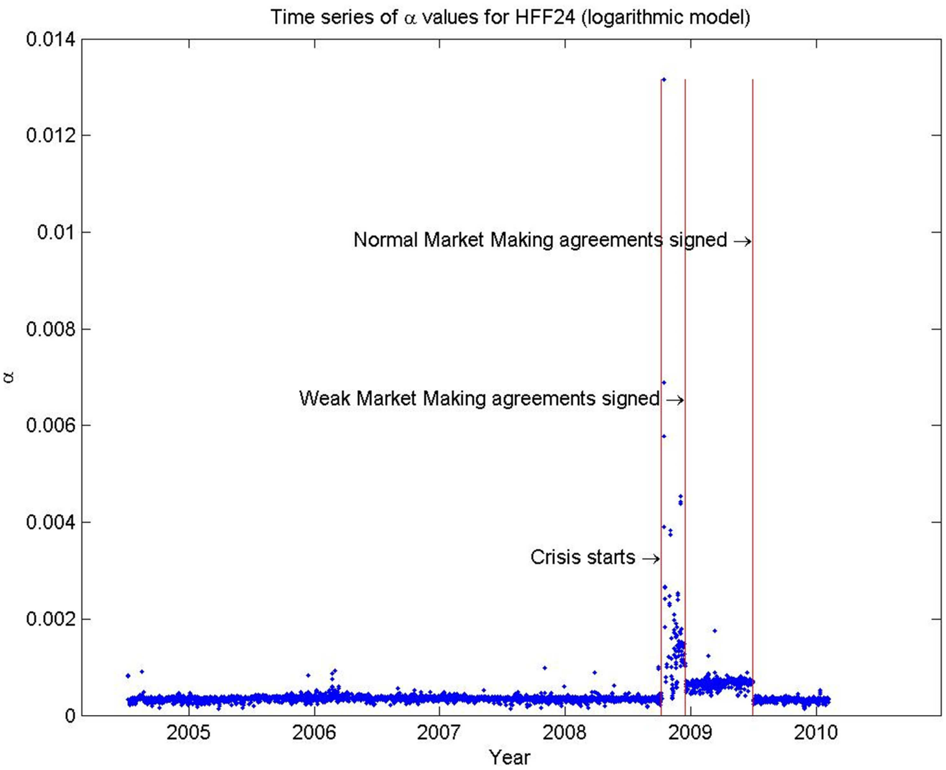


*Figure 13: Time series of α-values for HFF24 (logarithmic model)*

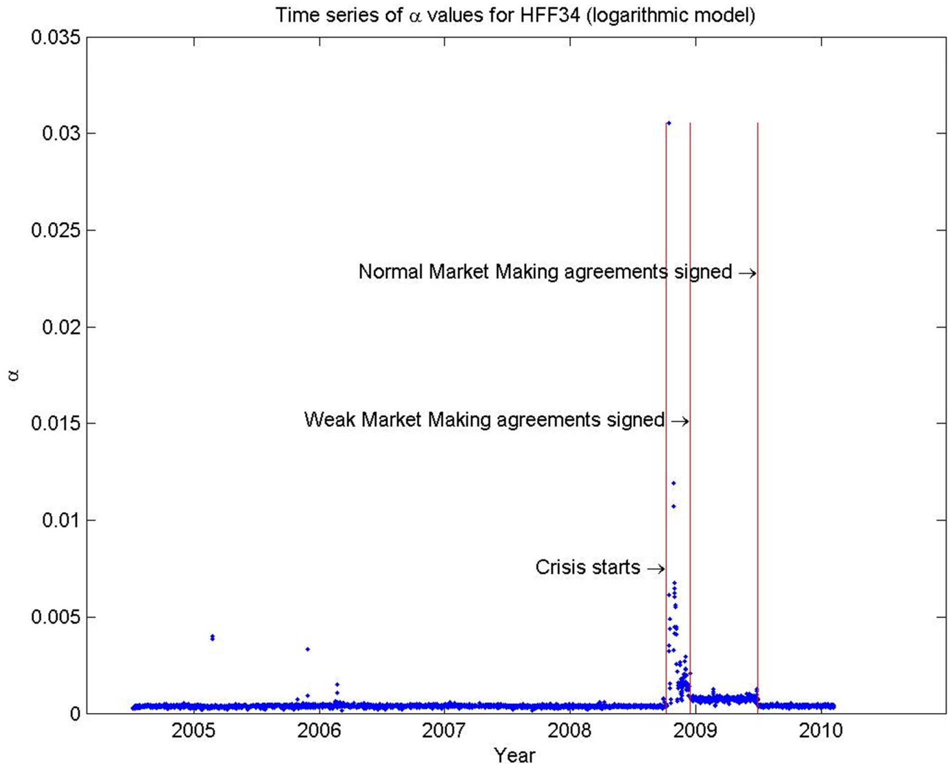


*Figure 14: Time series of α-values for HFF34 (logarithmic model)*

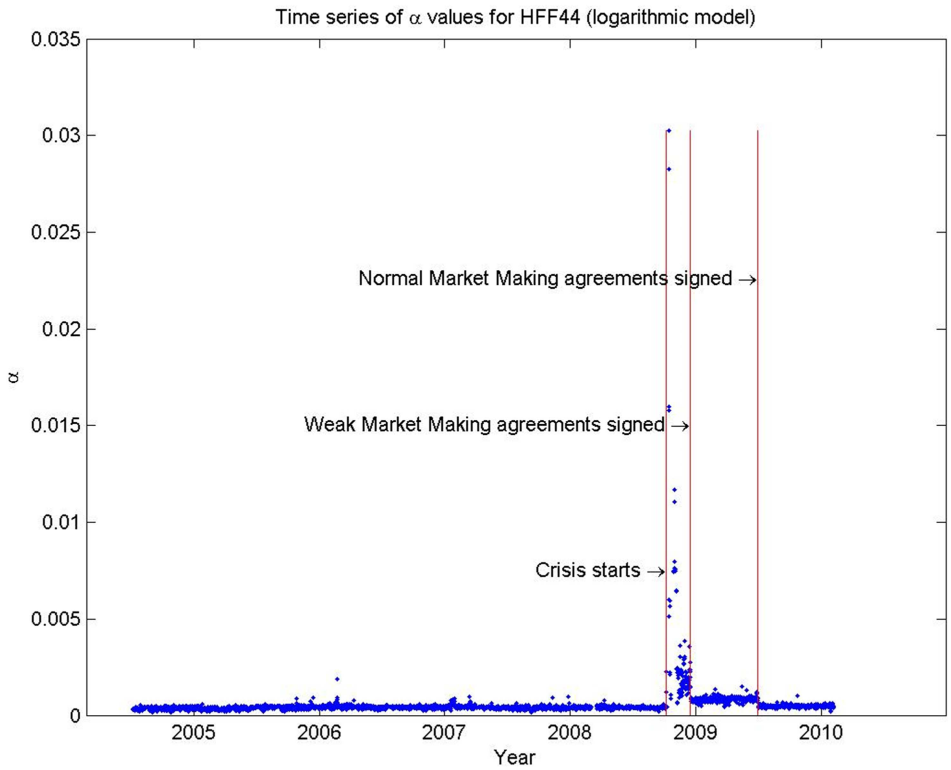


*Figure 15: Time series of α-values for HFF44 (logarithmic model)*

## B. Normality of α Residuals

To be able to use the *t*-test to show that the value of α in the models is significantly different from zero, that is to reject the null hypothesis $H_0: \alpha = 0$ against the alternative $H_1: \alpha \neq 0$, it is necessary to know that the estimated residuals are normally distributed. To argue that this is the case probability plots are used. The next three figures show normal probability plots. The first one, Figure 16, shows the residuals from the *linear* model for HFF24 before the crisis, the second one, Figure 17, shows the residuals from the *root* model for HFF34 during the weak market making period, and the last one, Figure 18, shows the residuals from the *logarithmic* model for HFF44 after the normal market making agreement was signed. It is worth noting that although the plots do not show a perfect line and this might need more investigation this is quite enough for our test to perform well due to the central limit theorem.

For simplicity reasons only these three plots are shown although this argument is needed in all cases, i.e. for all bonds and models during all the data periods.

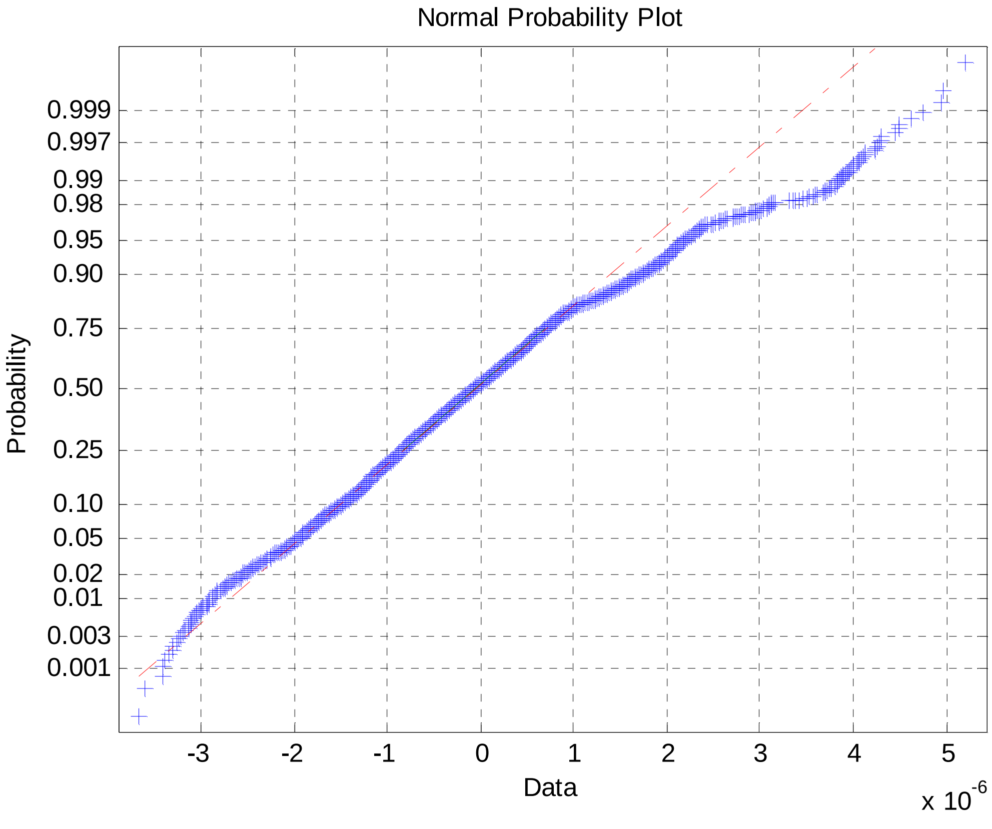


*Figure 16: Residuals probability plot from the linear model for HFF24 before the crisis*

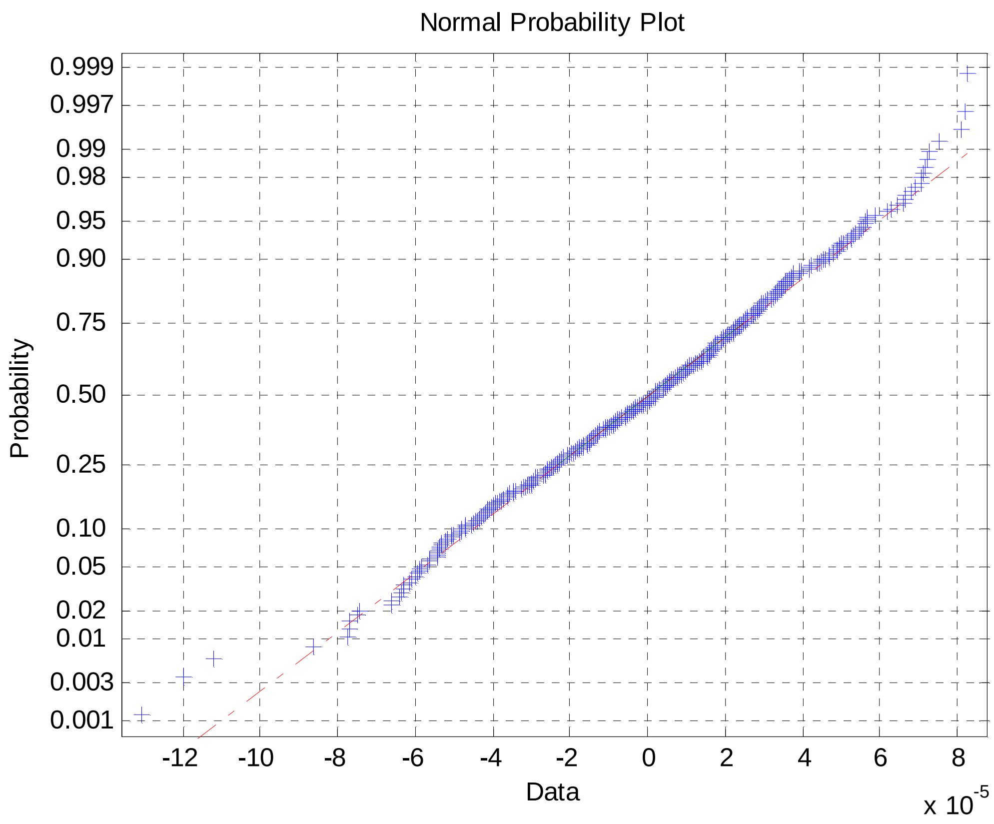


*Figure 17: Probability plot from the root model for HFF34 in weak market marking*

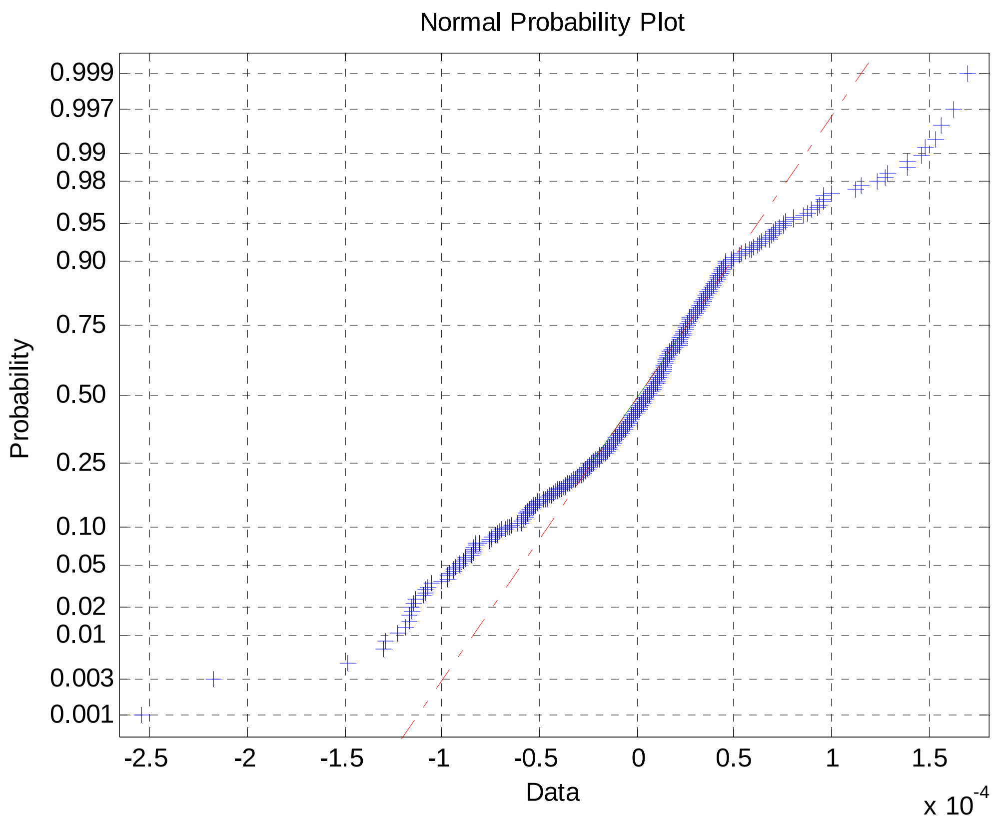


*Figure 18: Probability plot from the log model for HFF44 in normal market marking*