Risk-return optimization in the absence and presence of liabilities: comparative solution analysis and related methods

Guðmundur Magnússon

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Abstract

In practical asset management, traditional mean-variance portfolio optimization is frequently applied. Surplus optimization has a wide range of applications including benchmarking of portfolio returns with liabilities. In this thesis, mean-variance optimization models in absence and presence of liabilities are derived and expressed using simplifying notation that also explains certain properties of the optimal sets. Additional methods closely connected to surplus optimization are proposed. Asset allocation strategies in absence and presence of liabilities are compared in terms of generated surplus and performance.

By defining a 3×3 symmetric matrix, whose elements are composed from the inputs to the surplus optimization model, the optimal portfolios and respective return variances can be expressed in a simple form, as a function of return requirements. Also the determinant of this matrix and it‘s sub-determinants provide an efficient way of understanding the difference between the traditional mean-variance frontier and surplus frontier(s) in risk return space. The surplus optimization approach allows for multiple index benchmarking as the hedge component can be decomposed into several components resulting from the different benchmark indices. The theoretical absolute minimum surplus return variance for all feasible portfolios can be found and the classic market portfolio in presence of liabilities can be hedged by including a separate liability hedging component. The log-returns assumption allows for a simple probabilistic measure on gaining positive surplus and also allows for the application of shortfall constraints on the funding ratio. Using historical data from July 2008 to January 2012, a comparison of optimal allocation strategies in absence and presence of liabilities indicates that strategies considering liabilities are superior over their asset-only counterparts in terms of generating surplus and in recovering from the market downturn in 2008.

Keywords: Asset-Liability management, Mean-Variance optimization, Surplus optimization, Surplus risk-return frontier.

Útdráttur

Titill ritgerðar á íslensku: Bestun eignasafna út frá áhættu og væntri ávöxtun með og án tillits til skuldbindinga: samanburðargreining lausna og tengdar aðferðir.

Í hefðbundinni eignastýringu er oftar en ekki stuðst við bestun eignasafna út frá áhættu og væntri ávöxtun. Eignasöfn má besta með tilliti til ýmissra viðmiða, til að mynda skuldbindinga. Í þessari ritgerð eru líkön til bestunar eignasafna út frá áhættu og væntri ávöxtun með og án tillits til skuldbindinga leidd út og lausnirnar ritaðar með einföldum rithætti sem útskýrir einnig ákveðna eiginleika lausnamengjanna. Aðferðir sem nota má samhliða bestun eignasafna með skuldbindingaviðmiði eru lagðar fram. Fjárfestingastefnur með og án skuldbindingaviðmiðs eru bornar saman á grundvelli vaxtar eigna umfram skuldbindinga og frammistöðu.

Með því að skilgreina 3×3 fylki, með stökum samsettum úr stærðum sem teknar eru inn í bestunarlíkanið með skuldbindingaviðmiði, er hægt að rita lausnir fyrir eignasöfn á framfalli og dreifni ávöxtunar þeirra á einfaldan hátt sem föll af ávöxtunarkröfu. Ákveða og undirákveður fylkisins segja ennfremur til um mismuninn á framföllum eignasafna með og án tillits til skuldbindinga á einfaldan hátt. Bestunarlíkanið með skuldbindingaviðmiði getur tekið tillit til margra viðmiða í einu þar sem að hægt er að kljúfa varnarsafn skuldbindinga niður í þætti tengda einstökum viðmiðum. Finna má fræðilega lágmarksdreifni milli ávöxtunar eigna og skuldbindinga á möguleg eignasöfn og hið hefðbundna markaðssafn í návist skuldbindinga má verja með sérstöku varnarsafni. Ef gert er ráð fyrir normaldreifingu lógaritmískrar ávöxtunar má setja fram einfaldan líkindamælikvarða á eignir umfram skuldbindingar og einnig má beita skammfallsskorðum á hlutfall eigna og skuldbindinga. Samanburður eignasafna með og án skuldbindingaviðmiðs með gögnum frá júlí 2008 til janúar 2012 leiðir í ljós að eignasöfn með skuldbindingaviðmiði stóðu sig betur í að ávaxta eignir umfram skuldbindingar ásamt því að ná sér hraðar upp úr niðursveiflunni frá árinu 2008.

Lykilorð: Eigna- og skuldbindingastýring, Bestun eignasafna út frá áhættu og væntri ávöxtun, Bestun umframeigna (eignasafna með skuldbindingaviðmiði), Framfall umframeigna.

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List of abbreviations

ALM	Asset - liability management.
CDF	Cumulative distribution function.
GBM	Geometric Brownian motion.
i.i.d.	independent and identically distributed.
MSV	Absolute minimum surplus return variance portfolio.
MSVP	Minimum surplus return variance portfolio.
MVF	Mean – variance framework.
MVP	Minimum return variance portfolio.
s.t.	subject to.
w.r.t.	with respect to.
PDF	Probability density function.

List of propositions

List of theorems

List of notations

$\tau = \{0,..,t,..,T\}$	A time set from 0 to T, where $0 < t < T$.
$I = \{1,...,i,...,j,...,n\}$	A set of *n* risky assets.
$\Sigma_A = \left[\sigma_{ij}\right]_{i,j=1,...,n}$	A n by n symmetric, positive definite covariance matrix of risky asset returns.
$\mu_A = \left[\mu_i\right]_{i=1,...,n}$	A *n* by 1 vector for estimates of risky asset mean returns.
$E\left[R_L\right]$, μ_L	A scalar estimate for return on liabilities.
$VAR\left[R_L\right]$, σ_L^2	A scalar estimate for return variance on liabilities.
$\Sigma_{AL} = \left[\sigma_{iL}\right]_{i=1,...,n}$	A *n* by 1 vector of covariances between assets and liabilities.
$\upsilon = \left[1_i\right]_{i=1,...,n}$	A *n* by 1 vector of ones.
$A(t)$	Value of assets at time t.
$L(t)$	Value of liabilities at time t.
$S(t)$	Value of surplus, i.e. assets net of liabilities at time t.
θ	An importance parameter for surplus optimization.
λ_h	A Lagrangian multiplier in a constrained Lagrangian optimization problem.
$w_P{}'$	A *n* by 1 unspecified asset allocation vector in absence of liabilities.
$w_{P,S}{}'$	A *n* by 1 unspecified asset allocation vector in presence of liabilities.
w_P^*, $w_{P,S}^*$	A *n* by 1 unspecified optimal asset allocation vector.
w_{MVP}	The minimum return variance portfolio asset allocation vector.
ϕ_{MSVP}	The minimum surplus variance correction component asset allocation vector.
w_{MSVP}	The minimum surplus return variance portfolio asset allocation vector.
w_η	The return generating component asset allocation vector.
$w_{\eta,S}$	The return generating correction component asset allocation vector.
w_{rp}	The optimal return variance portfolio asset allocation.
$w_{rp,S}$	The optimal surplus return variance portfolio asset allocation vector.

w_{cov} The unit normalized asset-liability covariance portfolio asset allocation vector.

w_{MKT} The allocation vector for the market portfolio in absence of liabilities.

ϕ_{MKT} The allocation vector for the market portfolio liability correction component.

$w_{MKT,S}$ The allocation vector for the market portfolio in presence of liabilities.

w_{MSV} The absolute minimum surplus return variance portfolio allocation vector.

W The set of all optimal return variance portfolios, $W = \left\{ w_{MVP}, w_{rp} \right\}$.

W_S The set of all optimal surplus return variance portfolios, $W_S = \left\{ w_{MSVP}, w_{rp,S} \right\}$.

W_E The set of all efficient optimal return variance portfolios, $W_E \subseteq W$

$W_{E,S}$ The set of all efficient surplus return variance portfolios, $W_{E,S} \subseteq W_S$

F A funding ratio, defined as the ratio of assets to liabilities.

F_{cov} The unique funding ratio where $w_{MSVP} = w_{MKT,S} = w_{COV}$.

F_{MSV} The unique funding ratio where $w_{MSVP} = w_{MSV}$.

r_P A return requirement on a portfolio.

r_f The risk-free rate of return.

$E\left[R_P\left(w_P \right) \right]$ Expected return on a portfolio w_P.

$\sigma\left[R_P\left(w_P \right) \right] = \sigma_P$ Standard deviation of returns on a portfolio w_P.

$VAR\left[R_P\left(w_P \right) \right] = \sigma_P^2$ Variance of returns on a portfolio w_P.

$\rho_{P,L}$ A correlation coefficient between portfolio w_P and liabilities returns.

$COV\left[R_P\left(w_P \right), R_L \right]$ Covariance between returns on a portfolio w_P and liabilities returns.

$VAR\left[R_P\left(w_P \right) - \frac{\theta}{F} R_L \right]$, $\sigma_{S,n}^2\left(w_P \right)$, $\sigma^2\left[R_S\left(w_P \right) \right]$ Normalized surplus return variance.

$E\left[R_P\left(w_P \right) - \frac{\theta}{F} R_L \right]$, $E\left[R_S\left(w_P \right) \right]$ Expected normalized surplus returns.

$\mathrm{VAR}\left[R_P\left(w_P \right) - R_L \right]$, $\sigma_S^2\left(w_P \right)$ Surplus return variance.

1. Introduction

Pension fund systems play various roles in developed communities. Initially, they should provide the basis for the accumulation of savings in order to cover the spending of the majority of the population after retirement. Second, they should provide some sort of a financial insurance due to longevity. Third, they should include some diversification of individual risks in society and provide a form of common public insurance. Fourth, it is somewhat desirable that pension systems include a form of income levelling amongst retirees (Guðmundsson, 2000).

It is important to distinguish between the roles and objectives of pension funds. The role of pension funds is defined by law whereas the objectives are defined by the management of the funds. The objectives can be quite different unlike the predefined roles by law. Essentially, the role of pension funds is to secure pensions to their members or their surviving spouse or children according to provisions of laws and funds agreements. The laws imply that the objectives of pension funds are to provide highest possible pension at each time, but pension funds have certain flexibility in reaching that goal along with other objectives that can vary somewhat between funds. Their ability to enhance the rights in excess of minimum rights is mainly dependent on their members' demographic composition and historical assets returns (Kaupþing, 2006).

In modern economies, the increasing importance of pension plans has been recognized as a consequence of demographic trends in developed communities. The establishment of the pension systems in the last century introduced a great step forward, where the burden of the subsistence of older generation was systematically organized so that the coming generations would be better capable of financing their pensions themselves, without needing to rely on the subsistence of their children. In many societies, pension assets have been invested in capital markets in order to decrease present contributions and let the asset growth finance a greater portion of future pension payments. Long term growth has been intended to be stable and despite of individual market shocks, the long term growth has been presumed to be capable of growing pension assets well enough to cover pension liabilities.

Many share the opinion that markets have become more volatile than the pension plans could afford. Two equity market shocks have occurred since the high-tech bubble burst in late 2000 and a number of economies have not recovered from the 2008 crisis. In many cases, the liability covering target has been missed and as a result, pension funds have been criticized for being too heavily invested in equities, irrespective of their liability structure and the ability to bear the equity-risk (Ryan, Fabozzi, 2002; Fabozzi, Focardi, Jonas, 2004; Martellini, 2006). World‘s pension systems have suffered repeatedly from losses and adverse market conditions have affected the funding status of pension plans, often to a severe degree. Declining values of equities have resulted in falling pension plan assets while at the same time, decrease in interest rates has pushed up the pension liabilities along with lowering yields and returns on the low-risk fixed income asset classes. The low interest rate environment combined with the poor performance of global stock markets has lowered the funding status of many pension plans even further (Martellini, 2006).

The adverse market events occurring in the last two decades have shed light on the weakness of risk management and asset allocation practices in the pension fund industry. It is increasingly being recognized among academics and practitioners that pension fund risk management needs some rethinking and an extensive analysis of what went wrong and how the pension fund industry should adopt to a more volatile and cross-connected environment (Ryan, Fabozzi, 2002).

A frequently asked question in the discussion of pension funds still remains unanswered: “Could the pension fund crisis have been avoided or could the consequences of the recent financial crisis on the pension funds have been smoothened with proper risk management practices?” No one can really tell, but academics argue that better risk management and broader base of knowledge would improve the current situation and help to guide the industry into better future prospects (Fabozzi, Focardi, Jonas, 2004; Martellini, 2006).

Risk factors in pension fund management

Pension fund management is exposed to many sources of uncertainty and a simple overview of risk factors in pension fund management can be found in table 1.1. The core role of asset allocation modelling is to get a better grip on uncertainty and to act as a helping tool in making important decisions. Modelling can help in managing uncertainty and suggest preventive or corrective actions against risk factors, using statistical analysis and a blend of various theoretical methods. In the pension fund industry, the growing importance of modelling is being recognized although the capabilities of implementing modelling techniques vary between and within countries (Fabozzi, Focardi, Jonas, 2004).

Table 1.1: Risk factors in pension fund management

Liability Risk	Solvency Risk	Counterparty risk	Market Risk	Operational Risk
The risk that pension fund cannot meet its liabilities.	The risk of insufficient liquidity to cover pension payouts.	The risk of a counterparty will not meet its contractual obligations.	The risk of financial loss due to on or off balance sheet terms; changes in market value due to changes in interests rates, currency rates or equity values etc.	The risk of less due to insufficient or defective operational procedures, personell, systems or due to external events in the environment.
- Contribution Risk - Risk of benefits reduction - Political Risk - Demographic Risks - Recession Risk - Interest rate Risk	- Cash outflow Risk - Liquidity Risk	- Delivery Risk - Country Risk - Credit Risk	- Interest rate Risk - Currency Risk - Equity Risk - Inflation Risk - Repayment Risk - Reinvestment risk - Off-balance sheet asset risk - Discrepancy risk - Basis Risk - Commodity Risk	Human Factor Risk Fraud Risk Employee Risk Technology Risk Information risk Legal Risk Reputation Risk Pension verdict Risk

Table 1.1: Risk factors in pension fund and asset management (Crouhy, Galai, Mark, 2001; Icelandic pension fund association [1], 2010; Kaupthing 2006; Sigurðsson et. al 2010)

As a consequence of the increasing frequency and magnitude of pension plan deficits, the regulatory environment of pension funds has been altered in many countries. Increased emphasis on the importance of Asset-Liability Management (ALM) in managing pensions and greater requirements in risk management, have increased the need for proper modelling techniques (Fabozzi, Focardi, Jonas, 2004; Martellini, 2006). But nice things cost money; the capabilities of implementing modelling often depend on the size of pension funds. Larger pension funds are more likely to use advanced models and are quicker to adopt new investment strategies than smaller funds, which are more likely to suffer from a lack of in-house knowledge on proper risk management practices (Fabozzi, Focardi, Jonas, 2004).

The net performance of pension funds and therefore the rights of its beneficiaries are mainly controlled by two factors – assets and liabilities. One of the largest risk factors in pension fund operation is the risk of whether the assets can cover the liabilities. The development of liabilities can be more of a concern than the return on assets as the observation of demographic trends in developed communities implies. The changing demographic factors have presented increasing problems for pension schemes and are one of many reasons for the current difficulties of pension systems in developed countries. In some countries, increased disability trends have raised concerns regarding public and private insurance and pension systems. The reasons for trends in disability rates are controversial but this development increases the burden of pension systems, regardless of the reasons for the trends.

Estimation of demographic trends for prediction purposes is a challenging task. Various models for the purposes of predicting these trends have been developed and applied; their prediction power will not be observed until decades have passed. These models include uncertainty factors and the analysis of trend behaviour of first or second degree, e.g. the Lee-Carter mortality model (McCarthy and Miles, 2011; Ísleifsson 2012). One of many problems regarding the hedging of demographic risk factors is the autocorrelation in demographic trends and also, the correlation between demography and financial markets.

A large risk factor in the interaction between assets and liabilities is the development of interest rates. Despite varying conventions between countries in discounting liabilities, movements in interest rates affect liabilities regardless of discounting conventions used. During the last decade, changes in regulatory environment in many countries have resulted in increasing number of pension funds discounting liabilities using market rates instead of fixed or semi-fixed discount rates (Fabozzi, Focardi, Jonas, 2004). This has resulted in more changes in liabilities due to changes in market rates and hence more volatility of liabilities. For the last years, market rates have been very low on historical basis in many economies, resulting in somewhat lower return on fixed income assets and increased liabilities which has increased pension fund deficits. In countries using fixed or averaging discount rates, the interest rate risk is somewhat hidden and does not emerge although it basically exists.

Inflation risk is a large risk factor for pension funds providing indexed pensions. Obligatory full indexation of benefits creates a substantial risk for pension funds, especially in countries where inflation jumps are not uncommon, whereas their assets are only partially indexed. Accordingly, the funding status of a pension plan can deteriorate considerably if inflation is high. Risk management practices should account for how to respond to inflation jumps, when and especially before inflation is expected to increase.

The contribution of country risk can be relatively large and countries facing high country risk can benefit from international risk diversification (Driessen and Laeven, 2007). International asset allocation includes risks due to underlying foreign assets as well as currency risk. Currency movements affect the return on foreign assets in domestic currency. As a result, hedging against currency fluctuations is considered as a natural response to risk caused by foreign investments. The estimation of foreign asset risk can be challenging as it can be difficult to predict short term currency fluctuations whereas long term fluctuations are generally more predictable. The long term mean reversion tendency of real currency rates results in less long term currency risk. From that perspective, currency risk is not necessarily a major risk factor for properly diversified long term overseas investors. Nevertheless, large portions of assets invested internationally can induce considerable fluctuations in short term returns due to currency rate fluctuations (Magnússon, 2006).

Asset-liability modelling issues

The deficits of many pension plans in recent years are mainly the result of development in demographic trends, risk management and accounting practices along with asset performance. The sharp drop of stock markets around 2000 resulted in lower asset values along with a period of low interest rates which raised the value of liabilities. Another and greater market shock followed in 2008 in which equity prices fell dramatically and pension funds suffered further. In a comparative study by Fabozzi, Focardi and Jonas (2004), bad modelling or even absence of modelling was considered as a substantial explanation of the current pension fund crisis. Nevertheless, their results implied that there was a growing use of computer-based modelling in the pension funds industry in western societies. They concluded that modelling allows for better decision making by reducing uncertainty and those plans that have implemented a computer assisted decision making process are generally safer and more efficient with respect to cost, risk and performance than their less sophisticated peers. It would serve the interests of all parties concerned to accurately model a pension fund's cash flow. Their report was based on conversations with over forty persons mainly from pension funds but also regulators, consultants and academics in Europe and the U.S.A. Funds with assets under EUR 1.5 billion were not included in this study as they typically depend on external consultants and have little in-house knowledge on modelling issues. Unfortunately in some cases examined by the study, modelling was used to justify investment decisions taken or even used as some kind of cosmetics to make the decisions look more sophisticated.

According to article 3.4 in OECD guidelines on pension fund asset management, pension funds investment policies should include a sound risk management process that measures and seeks to appropriately control portfolio risk. Furthermore, assets and liabilities should be managed in a coherent and integrated manner (OECD, 2006). The guidelines suggest that asset management practices in pension funds should minimize any imbalance between assets and liabilities and that the risk structure of assets and liabilities is as similar as possible at each moment in time (Icelandic pension fund association [2], 2010; OECD, 2006; Sigurðsson et. al., 2010). Accordingly, the regulatory environment of pension funds has been adopting this methodology, posing challenges to the pension funds regarding duration matching of assets and liabilities and to model assets and liabilities in more accurate manner. In the near future, regulatory requirements of asset-liability modelling are likely to force pension funds to respond and adopt some form of liability driven investment strategies.

Questions have been raised on whether the practices and procedures used in the pension fund industry are sufficient following the market difficulties since the millennium. The market difficulties have motivated the development of risk management solutions, including asset-liability management (ALM) methodology. The ALM literature suggests directly or indirectly that a fraction of portfolio wealth should be invested in a liability-hedging portfolio that is supposed to minimize the risk due to liabilities, i.e. reduce the likelihood and magnitude of shortfall caused by a mismatch between assets and liabilities.

Various optimization methods in the field of ALM have been developed for allocating assets in the presence of liability constraints and some are used by institutional investors and pension funds. According to previously mentioned study by Fabozzi, Focardi and Jonas (2004), frequently used methods in pension fund applications are simulation and optimization under uncertainty. Flexible methods are provided by optimization under uncertainty, allowing for dynamic decision making where the portfolio policy is the output. The optimization methods can be distinguishing between as the modelling methods for uncertainty vary. The main methods for numerical optimization under uncertainty can be classified into stochastic, dynamic and fuzzy programming methods (Reynisson, 2012). According to Fabozzi, Focardi and Jonas (2004), the literature in ALM applications is highly concentrated on stochastic programming. In stochastic programming, various scenario generation techniques are used along with methods that use algorithms for solving problems numerically, since the main assumption in these methods is that analytic solutions do not exist. From the perspective of non-specialists, simulation methods are more easily understood as the portfolio policy is an input.

Portfolio optimization

The foundation of modern portfolio theory is based on Markowitz‘s (1952) famous article, Portfolio Theory. In his article, he introduced an idea of risk-return representation of portfolios, where the set of efficient portfolios could be represented via expected return and volatility. The common practice of focusing on maximizing expected returns only is rejected by Markowitz‘s portfolio theory. He showed in his article that volatility of returns should be minimized, given some level of expected returns, or equivalently, expected returns should be maximized for given levels of volatility. In his approach, the efficient set results in well diversified portfolios where the minimization of volatility is achieved with diversification. The total asset risk is reduced by increasing the number of assets in a portfolio and by choosing the assets with respect to cross-correlation of asset returns. Although the systematic risk of the markets cannot be eliminated by diversification, specific asset risk can be reduced and theoretically diversified by selecting all assets from the market. As the inputs to risk-return models are easily estimated, these models are nevertheless quite sensitive to the inputs and the optimal allocation can change substantially due to quite small changes in inputs.

Markowitz‘s approach led to many innovative ideas in asset management, e.g. the "mutual fund separation theorem" where the set of optimal portfolios can be represented as a linear combination of other optimal portfolios. These ideas and the work that followed provided the foundation for modern asset management modelling techniques. Asset allocation models have improved since the early days of asset management and due to the growing importance of appropriate modelling techniques for large institutional investors, various classes of portfolio models taking miscellaneous factors and benchmarks into account have been developed.

The mean-variance frontier can be interpreted as the set of all optimal portfolios, i.e. portfolios with minimum variance for a given value for expected return. The portfolio that has the absolute minimum variance is referred to as the minimum variance portfolio and represents the minimum return variance point on the efficient frontier. The risk-return space is a simple, yet an efficient way of reducing the investment problem‘s dimensionality from n dimensions ($n + 1$ if a risk-free asset is included) into two dimensions.

The traditional mean-variance efficient portfolios can be decomposed into two separate portfolios; the minimum risk or minimum variance component and a return generating component. The minimum risk component is not dependent upon preferences whereas the return generating component is chosen by the investor upon return preferences and resulting risk.

In general, modern portfolio theory assumes that returns on assets are normally distributed but models with other distributional assumptions have been developed. In financial modelling, logarithmic returns or continuously compounded returns are often assumed to be normally distributed, based on the roughly symmetric character of stock return distributions. The assumption on normality has strong empirical support (Stewart, Piros, Heisler, 2011). In general, normality tests of daily portfolio return do not reject the normality assumption, suggesting that the assumption of normality for stock portfolio returns may be acceptable even if individual stock returns are not normally distributed. By lengthening the observation horizon, the assumption on normality appears to become more acceptable for individual stock returns. Another reason for the normality assumption is the central limit theorem, which states that, under mild conditions, the mean of a large number of random variables independently drawn from the same distribution is distributed approximately normally, irrespective of the form of the original distribution. Accordingly, the normality assumption of returns is merely a reasonable and a simplifying assumption for modelling purposes.

Surplus management and surplus optimization

By extending the traditional mean variance framework to take account of liabilities, one focuses on growing the pension asset portfolio in excess of the liability portfolio, i.e. achieving pension plan surplus. The pure surplus is generally defined as the total asset value being held by an investor, net of liabilities. As the investment problem is now affected by the presence of liabilities, the investor‘s strategy is not a traditional asset allocation problem, since the investment decisions must enable the investor to cover his liabilities without excessive risk of falling short of this goal. The objective of the problem has transferred into maintaining the value of assets above the value of liabilities, irrespective of actuarial or accounting rules used to calculate the present value of the liabilities. The investment problem has now been transferred from portfolio risk-return space into surplus risk-return space, where fluctuations in the funding ratio, i.e. the ratio of present value of assets to present value of liabilities, have emerged as units of financial risk.

According to the previously mentioned study by Fabozzi, Focardi and Jonas (2004), many practitioners in the pension fund industry have been somewhat reluctant to take full account of liabilities into the mean-variance framework and still favour the traditional Markowitz approach in absence of liabilities. However, practitioners in the pension fund industry and regulatory bodies are slowly accepting the importance of considering liabilities in the investment process. Investors are frequently focused on the return on assets rather than the return on assets net of liabilities and as a consequence, the surplus risk might be undetected. On many occasions, it’s likely that the different dynamics of assets and liability returns are involved in generating an imbalance between assets and liabilities for many pension plans. As an initial step towards considering liabilities in the investment process, models with strong link to standard theory might be helpful since they provide simple and easily understood extensions to the widely applied risk-return analysis. In fact, the traditional mean-variance optimization model is a special case of the surplus optimization model, where the liabilities are ignored.

Surplus management can be thought of as liability benchmarking where investments are made by considering the dynamics of the liabilities. Benchmarking is commonly applied practice in asset management where the returns of portfolios are supposed to be closely related to or exceed some pre-specified benchmark. Liability benchmarking is only one type of benchmarking and as a result, surplus optimization can be applied in a much broader context. The goal of exceeding a benchmark return involves optimizing surplus returns on that particular benchmark. An investor with the goal of optimizing surplus might have a certain view on the typical asset return benchmark, but he might as well focus on the importance of benchmarking his asset return with liabilities returns.

Along with gaining positive surplus, the surplus optimizer‘s strategy aims at investing in a portfolio which smoothes the fluctuation of the surplus returns caused by different volatility factors, i.e. in a liability hedge portfolio that has high return correlation with the most relevant risk factors. Surplus optimization presents a strategic asset allocation using asset classes that are not perfectly correlated with the liabilities. The investment in the liability hedge portfolio depends on the funding ratio and the preferences of the investor have not to be specified.

From the regulators point of view, this allows for a simple technique to monitor pension funds asset management strategies as the liability hedge portfolio depends only on current funding ratio, not on preferences. In order to maximize pension fund‘s expected surplus return with respect to volatility, the fund should hedge the liabilities according to the financial status as the hedge portfolio for the liabilities is dependent on the funding ratio and the properties of liabilities. A pension fund should consider investing in a portfolio that provides the best hedge against fluctuations in wages and salaries, inflation, demographic trends etc. Pension funds, are legally obliged to cover their liabilities with their assets at each point in time during their lifetime. As a natural response, pension management should consider some sort of ALM practices in asset and risk management.

Literature Review

The foundation of modern portfolio theory was laid by Markowitz (1952), with his idea of risk-return representation of portfolios. In his approach, the efficient set results in well diversified portfolios where the minimization of volatility is achieved via diversification. Markowitz‘s approach led to many innovative ideas in asset management, e.g. the “mutual fund separation theorem" where the set of optimal portfolios can be represented as a linear combination of any optimal portfolios. Solnik (1978) extended the traditional mean-variance approach with portfolio choice under inflation where it was assumed that assets have nominal payoffs where investors care about real returns. Keel and Muller (1995) refined Solnik‘s approach for portfolio choice under inflation and applied it to the asset-liability problem where they derived the surplus optimal set in terms of Lagrangian parameters. They showed that the presence of liabilities leads to a shift of the efficient set where the shift vector can be decomposed in different components. Furthermore, they showed how shortfall constraints for a pension fund can be reconciled with efficiency. Sharpe and Tint (1990) presented an approach that avoids either asset only or full surplus optimization by introducing an importance parameter that permits full or partial emphasis on liabilities They introduced a procedure that can be used to measure the exact relationships among expected returns, risks, and hedging characteristics via a hedging credit and the hedging ability of assets.

Among the first steps forward in optimal portfolio selection theory in continuous-time was taken by Robert Merton in a series of papers on intertemporal selection analysis, e.g Merton (1969), (1971), (1973). Merton (1969) derived the optimality equations for a multi-asset problem when the rates of returns are generated by Wiener processes. His methods can be used to examine wide class of intertemporal economic problems under uncertainty. Several authors have extended Merton‘s intertemporal selection analysis which has resulted in a spectrum of continuous-time models considering various aspects of the asset allocation problem, including those who account for the presence of liability constraints in the asset allocation policy. Rudolf and Ziemba (2004) presented an intertemporal portfolio selection model for pension funds that maximizes the intertemporal expected utility of the surplus, where state variables are interpreted as currency rates that affect the value of the asset portfolio. The optimum occurs for investors holding four funds: the market portfolio, the hedge portfolio for the state variables, the hedge portfolio for the liabilities, and the risk-free asset. The model provides an intertemporal portfolio selection approach for surplus optimizers. Martellini 2006 considered an intertemporal portfolio problem in the presence of liability constraints. Using the value of the liability portfolio as a natural numéraire, he finds that the solution involves a three-fund separation theorem; the risk-free asset, the standard market portfolio and a liability hedging portfolio.

The core of the continuous-time literature is that the presence of risk factors induces a specific hedging component in the optimal allocation strategy. These risk factors can be financial risks such as liabilities, inflation, interest rates, currencies etc. and the optimal strategy aims at hedging one or more risk factors simultaneously. Stochastic models can be particularly useful in relation to risk-return space, where they allow for the representation of the uncertainty relating to a set of risk factors that impact the objectives of the portfolio strategy.

Motivation and objectives

The previously mentioned articles by Keel and Muller (1995) and Rudolf and Ziemba (2004) motivated the author of this thesis to understand and analyse the surplus optimization model in details and search for further explanations of the properties of the optimal solutions. The contents of Keel's and Muller's article provide the foundation in this thesis but different and perhaps more practical expressions of the optimal solutions are proposed in this thesis, using constrained Lagrangian optimization to find the optimal solutions. Keel and Muller expressed the optimal solutions in terms of Lagrangian parameters but since inputs into practical asset allocation models in terms of raw Lagrangian multipliers are somewhat inconvenient, the optimal solutions were modified so that return requirement could be used directly as an input. Also, a convenient notation for expressing the optimal solutions is introduced where elements of a matrix play the role of simplifying the expressions. Interestingly, this matrix can also be used to explain certain properties of the optimal solutions and the structure of the frontiers in risk-return space and surplus risk-return space.

As the motivation behind the work has been stated, the goal of this thesis is:

1. To derive simple and convenient expressions for the optimal portfolios and return variances of the surplus optimization problem so that the problem can easily be expressed and programmed in software.
2. Provide relatively simple explanations of certain properties of optimal solutions of the surplus optimization problem and the difference between the risk-return frontiers in absence and presence of liabilities.
3. Explore probabilistic measures and constraints that can be applied in parallel with surplus optimization.
4. Apply the derived methodology by comparing asset allocation strategies in absence and presence of liabilities in terms of generating surplus, using recent historical asset and liabilities data.

To address these topics and thereby the goals of this thesis, the optimal mean-variance portfolios in absence and presence of liabilities are derived in chapter 2 and additional methods in close relation to mean-variance analysis are proposed. By introducing a 3×3 symmetric matrix with elements consisting of the inputs to the surplus optimization model, the optimal portfolios and respective return variances can be expressed in a simple form as a function of return requirement. Also the determinant of this matrix and it‘s sub-determinants provide a simple method for understanding the difference between the traditional mean-variance frontiers and surplus frontiers in risk-return space. The surplus optimization approach allows for multiple benchmarking, since the hedging component can be decomposed into different components resulting from each benchmark. The theoretical absolute minimum surplus return variance on all feasible portfolios can be found and a market portfolio in the presence of liabilities can be considered by including a separate liability hedging component. The log-returns assumption allows for a simple probability measure on gaining positive surplus and also allows for the application of shortfall constraints on the funding ratio. The derived methods are applied to a hypothetical pension fund where the value of its liabilities is assumed to follow the pension obligation index (POI) for employees in the Icelandic public sector (Statistics Iceland, 2013). Using historical data from July 2008 to January 2012, a comparison of optimal allocation strategies in absence and presence of liabilities indicates that strategies considering liabilities are superior to their asset-only counterparts in terms of generating surplus and recovering from the market downturn in 2008.

Although the surplus optimization model is well studied, the main contribution of this thesis is to provide simple, yet detailed explanations of certain properties of the model. Such explanations can be helpful in academic studies since many published papers on the topic do not provide detailed derivations. The optimal solutions associated with risk-return optimization in absence and presence of liabilities are composed out of separate components. This thesis aims to analyze these components, to explain some of their characteristics and to give a geometrical interpretations of the optimal solutions in the associated risk-return space. This is done by proposing a simple notation that can be used in explaining the difference between the risk-return frontiers in absence and presence of liabilities and in expressing the optimal solutions. This thesis provides a numerical example of risk-return optimization in absence and presence of liabilities, using assets and liabilities return data with ISK returns. Also, the performance and the surplus generating ability of such asset allocation strategies is compared during and after the recent 2008 market downturn.

The remaining text in this thesis is organized as follows: Section 2.1 states necessary definitions for later sections and a notation that is convenient for expressions and for explaining certain properties of the optimal solutions. The standard mean-variance optimization problem is derived in section 2.2 and the results are explained and discussed. The risky asset market portfolio is derived in section 2.3. In section 2.4, the surplus optimization model is introduced by altering the model from section 2.3 to take account of liabilities. For a better understanding of the objective of the surplus optimization model, surplus return variance is analysed in section 2.5. The surplus optimization model and associated optimal solutions are derived and expressed in section 2.6, where the text aims to give a detailed explanation of the components of the optimal solutions. Section 2.7 covers additional assumptions on liabilities where the liability component is decomposed according to the assumption of linear dependence between the growth rate of the liabilities and other factors. The quadratic relationships between expected returns and return variances for the models from section 2.3 and 2.6 are shown in section 2.8. Section 2.9 is the surplus risk-return counterpart of section 2.8 where the relationship between expected surplus returns and surplus return variance are shown. In section 2.10, certain properties of the optimal solutions are explained, using the notation from section 2.2. The funding ratio that minimizes surplus return variance is expressed and discussed in section 2.11. The market portfolio in presence of liabilities is expressed in section 2.12 and briefed on the connections with the ALM literature. Section 2.13 presents a probability measure for gaining positive surplus at the end of a specific horizon. In section 2.14, traditional shortfall constraints and shortfall constraints on funding ratio are introduced. In a numerical example in chapter 3, the methods derived in chapter 2 are applied to a hypothetical pension fund to analyse the characteristics of the optimal portfolios in absence and presence of liabilities. In the final section of the numerical example, asset allocation strategies in absence and presence of liabilities are compared in terms of performance and surplus generating ability, using recent historical asset class and liabilities data. Section 4 concludes the thesis. Appendices 1 - 26, provide proofs and derivations to pertinent expressions in chapter 2. Finally, appendices 27 - 28 cover fundamental topics of non-linear optimization methods.

2. Methods

The mean-variance framework (MVF) attempts to maximize expected portfolio return for a given amount of portfolio risk, or equivalently, minimize risk for a given level of expected return by choosing the proportions of various assets. The MVF aims to select a collection of investment assets that have collectively lower risk than any individual asset due to the diversification of risk. The MVF models an asset's return as a normally distributed function, defines risk as the standard deviation of return, and models a portfolio as a weighted combination of assets, so that the return of a portfolio is the weighted combination of the assets returns. The fundamental concept behind the MVF is that the assets in an investment portfolio should not be selected individually but rather by considering how individual asset prices change relative to other asset prices. As a result, the best possible diversification strategy can be found in accordance with the given assumptions.

With this brief foreword on the material under consideration in this chapter, the contents of the chapter are as following:

Initially, a symmetric matrix is defined whose determinants are used in expressing and explaining certain properties of the optimal sets. In section 2, the optimal sets and their properties for the traditional mean-variance model are derived and explained. An optimal solution in presence of a risk-free asset is added in section 3. Surplus optimization is introduced in section 4 and a brief analysis of the objective function follows in section 5. The surplus optimization model is expressed, solved an explained in section 6. Section 7 covers additional assumptions on liabilities. In sections 8 and 9, the risk-return and surplus risk-return relationships are analyzed. In section 10, the properties of the symmetric matrix determinants from the first section are explained further. Section 11 covers an analysis on minimum surplus variance from theoretical point of view. In section 12, optimal allocations in presence of liabilities is expressed and discussed where a risk-free asset is assumed to be available for investing. In section 13, a simple probability measure of surplus is derived and explained. Finally, section 14 covers certain type of shortfall constraints in absence and presence of liabilities.

2.1 Definitions and convenient notation

The definitions of scalars, vectors and matrices given in this section are used throughout the thesis. Vector notation is dominant for the derivations but component notation is used occasionally for clarity. In the following derivations, it is assumed that n risky assets are available for investing and where it is appropriate, there exists a risk free asset yielding the return r_f per time unit. It is assumed that the risky assets are valued according to market prices at each point in time and those risky assets logarithmic returns are normally distributed with values for the first two moments, i.e. means and variances, estimated by an appropriate estimation method. For modelling the risky assets returns in presence of liabilities, it is assumed that liabilities are not readily marketable and do therefore not have a market value; specific accounting or actuarial rules are used to calculate the present value of liabilities. It is assumed that the first two moments for the growth rate of liabilities can be estimated by the same estimation methods as for assets at each point in time.

For the clarity of the derivations in following sections, the notation behind the set of observations of a stochastic asset price process will be stated in the following lines. Initially, let the time period under examination be a time set $\{0 \le t \le T\}$, the set of risky assets is defined as $I = \{1, ..., i, ..., j, ..., n\}$, a n by 1 vector of ones as $\upsilon = [1_i]_{i=1,...,n}$, a n by 1 vector of portfolio risky asset allocation as $w_P = [w_i]_{i=1,...,n}$, the n by 1 vector of risky asset mean returns as

$$\mu_A = [\mu_i]_{i=1,...,n}$$

σ_{ij} is the covariance between assets *i* and *j* and

$$\Sigma_A = [\sigma_{ij}]_{i,j=1,...,n}$$

is the n by n covariance matrix of the risky asset returns, a non-singular, symmetric and positive definite matrix. Also, it is assumed that $\forall\ i, j \in I,\ \mu_i \ne \mu_j$. Finally, Σ_{AL} is a n by 1 vector containing the covariance elements between individual asset returns and returns on liabilities;

$$\Sigma_{AL} = [\sigma_{iL}]_{i=1,...,n}$$

The assumptions and definitions that have been listed already provide conditions for existence of unique solutions to the optimization problems under consideration in this thesis. The investment strategy of an investor is given by the choice of securities w_i, $i \in I$ in accordance with the definition above.

Before the optimal solutions will be derived, a convenient and useful notation in expressing the optimal solutions is introduced. The following definition is essential for the formulation of the sets of optimal solutions.

Definition 2.1.1: All optimal portfolios in the absence of liabilities, w_P^*, are elements in the optimal set W, $w_P^* \in W$, and all optimal portfolios in the presence of liabilities, $w_{P,S}^*$, are elements in the optimal set W_S, $w_{P,S}^* \in W_S$ (Keel & Muller, 1995).

As mentioned in the introduction, the traditional mean-variance optimization model (in absence of liabilities) is a special case of the surplus optimization model (in presence of liabilities) when no consideration is given to liabilities. In that case, $W = W_S$.

For convenience and clarity of notation in expressing the optimal solutions and as well as for understanding certain properties of the frontier sets, the matrix Q is introduced as:

$$Q = \begin{bmatrix} Q_{11} & Q_{12} & Q_{13} \\ Q_{21} & Q_{22} & Q_{23} \\ Q_{31} & Q_{32} & Q_{33} \end{bmatrix} = \begin{bmatrix} \mu_A^T \Sigma_A^{-1} \mu_A & \mu_A^T \Sigma_A^{-1} \upsilon & \mu_A^T \Sigma_A^{-1} \Sigma_{AL} \\ \upsilon^T \Sigma_A^{-1} \mu_A & \upsilon^T \Sigma_A^{-1} \upsilon & \upsilon^T \Sigma_A^{-1} \Sigma_{AL} \\ \Sigma_{AL}^T \Sigma_A^{-1} \mu_A & \Sigma_{AL}^T \Sigma_A^{-1} \upsilon & \Sigma_{AL}^T \Sigma_A^{-1} \Sigma_{AL} \end{bmatrix} \tag{2.1.1}$$

The elements of this matrix are composed of the inputs to the optimization models d, defined before in this section. Since the elements of the matrix are scalar, it can be seen that $Q_{12} = Q_{21}$, $Q_{13} = Q_{31}$ and $Q_{23} = Q_{32}$, and this fact is used throughout the thesis in the formulation of optimal solutions and associated sizes. Accordingly, the matrix Q is symmetric and can be rewritten for further convenience as:

$$Q = \begin{bmatrix} Q_{11} & Q_{12} & Q_{13} \\ Q_{12} & Q_{22} & Q_{23} \\ Q_{13} & Q_{23} & Q_{33} \end{bmatrix} \tag{2.1.2}$$

The determinant of the matrix Q can be written as

$$|Q| = Q_{11}Q_{22}Q_{33} + 2Q_{12}Q_{13}Q_{23} - Q_{11}Q_{23}^2 - Q_{12}^2 Q_{33} - Q_{13}^2 Q_{22} \tag{2.1.3}$$

Five sub-matrices of Q are defined here as:

$$Q_{\#1} = \begin{bmatrix} Q_{11} & Q_{12} \\ Q_{12} & Q_{22} \end{bmatrix}, \quad Q_{\#2} = \begin{bmatrix} Q_{12} & Q_{22} \\ Q_{13} & Q_{23} \end{bmatrix} \tag{2.1.4) - (2.1.5}$$

$$Q_{\#3} = Q_{\#2}^T = \begin{bmatrix} Q_{12} & Q_{13} \\ Q_{22} & Q_{23} \end{bmatrix}, \quad Q_{\#4} = \begin{bmatrix} Q_{22} & Q_{23} \\ Q_{23} & Q_{33} \end{bmatrix} \tag{2.1.6) - (2.1.7}$$

and

$$Q_{\#5} = \begin{bmatrix} Q_{11} & Q_{13} \\ Q_{12} & Q_{23} \end{bmatrix} \tag{2.1.8}$$

Their determinants are, respectively:

$$|Q_{\#1}| = Q_{11}Q_{22} - Q_{12}^2, \quad |Q_{\#2}| = |Q_{\#3}| = Q_{12}Q_{23} - Q_{13}Q_{22} \tag{2.1.9) - (2.1.10}$$

$$|Q_{\#4}| = Q_{22}Q_{33} - Q_{23}^2 \quad \text{and} \quad |Q_{\#5}| = Q_{11}Q_{23} - Q_{12}Q_{13} \tag{2.1.11) - (2.1.12}$$

The matrix Q is convenient for simplification in expressing optimal solutions through the derivations. Furthermore, the determinants of Q and its sub-matrices are prove to be useful in determining the existence of the optimal solutions of the frontier sets, W and W_S, and determining certain properties of the surplus frontiers. These properties will be derived further in section 2.10 after the optimal solutions and sets have been derived.

2.2 The mean-variance optimization model in absence of liabilities

This section covers the fundamental modelling issues related to the well-known mean-variance optimization model where liabilities are not taken into account. Complete proofs of the derivations in sections 2.2 – 2.3 are given in appendices 1 - 7.

The assumptions that have been listed already in section 2.1 provide conditions for the existence of unique solutions to the optimization problem derived in this section. As noted in last section, the investment strategy is given by the choice of securities w_i, $i \in I$. For the investor‘s portfolio P which is characterized by the asset allocation vector w_P, the expected value of returns on P, composed out of the n risky assets, can be expressed as

$$E\left[R_P\left(w_P\right)\right] = \mu_A^T w_P$$

where μ_A is the vector of risky asset mean returns as introduced in section 2.1.

The return variance on the portfolio P is expressed as

$$VAR\left[R_P\left(w_P\right)\right]=\sigma_P^2\left(w_P\right)$$

The objective of the optimization model is to minimize the return variance resulting from the risky asset portfolio, σ_P^2, for a given value of expected return on the portfolio, r_P. Accordingly, the optimization model can be expressed as

$$\underset{w}{Min}\quad VAR\left[R_P\left(w_P\right)\right]$$

s.t.

$$E\left[R_P\left(w_P\right)\right]=r_P$$

$$\sum_{i=1}^{n} w_i = 1$$

In order to obtain a solution to the optimization model, the model has to be expressed in more concrete mathematical way. The return variance on a portfolio P, containing *n* risky assets, is commonly expressed as

$$\sigma_P^2\left(w_P\right)=\sum_{i,j=1}^{n} w_i\sigma_{i,j}w_j \quad \text{or} \quad \sigma_P^2\left(w_P\right)=w_P^T\Sigma_A w_P$$

Using the latter variance expression and multiplying the objective with ½ for convenience, the optimization problem from above can formally be rewritten as

$$\underset{w}{Min}\quad \frac{1}{2}w_P^T\Sigma_A w_P \tag{2.2.1}$$

s.t.

$$w_P^T\mu_A = r_P \tag{2.2.2}$$

$$w_P^T\upsilon = 1 \tag{2.2.3}$$

To solve this equality constrained optimization problem, a Lagrangian function is defined so that

$$L(w,\lambda_1,\lambda_2)=\frac{1}{2}w_P^T\Sigma_A w_P+\lambda_1\left(r_P-w_P^T\mu_A\right)+\lambda_2\left(1-w_P^T\upsilon\right)$$

The partial derivatives of L w.r.t. w_P yield necessary first order conditions for stationarity and primal feasibility as:

$$\Sigma_A w_P^*-\lambda_1\mu_A-\lambda_2\upsilon=0,\quad w_P^T\mu_A=r_P \quad \text{and} \quad w_P^T\upsilon=1$$

A second order sufficient condition for a minimum is satisfied as the covariance matrix is positive definite. Isolating the optimal allocation vector w_P^* from the first order stationarity condition gives

$$w_P^* = \Sigma_A^{-1} \left[\mu_A \quad \upsilon \right] \left[\lambda_1 \quad \lambda_2 \right]^T \tag{2.2.4}$$

Equation (2.2.4) is the general solution to the portfolio risk-return optimization problem (2.2.1) - (2.2.3) in terms of Lagrangian parameters and is used for deriving the solutions specified in theorems 1 – 3 and propositions 1 - 3 that follow.

As a starting point in solving the model (2.2.1) - (2.2.3) for specific optimal solutions, theorem 1 gives the minimum return variance portfolio, with the shorthand notation of MVP or w_{MVP}. The MVP is obtained by omitting the constraint on portfolio return, requiring $\lambda_1 = 0$ and (2.2.4) simplifies accordingly to

$$w_P^* = \lambda_2 \Sigma_A^{-1} \upsilon \tag{2.2.5}$$

Theorem 1: The minimum return variance portfolio.

The minimum return variance portfolio allocation vector is obtained by solving (2.2.5) *explicitly for* w_P^* *which results in*

$$w_{MVP} = \frac{\Sigma_A^{-1} \upsilon}{Q_{22}} \tag{2.2.6}$$

The expected return on the minimum return variance portfolio is

$$E\left[R_P \left(w_{MVP} \right) \right] = r_{MVP} = \frac{Q_{12}}{Q_{22}} \tag{2.2.7}$$

and the minimum return variance is

$$VAR\left[R_P \left(w_{MVP} \right) \right] = \sigma_{MVP}^2 = \frac{1}{Q_{22}} \tag{2.2.8}$$

Proof: See Appendix 1.

Equation (2.2.8) expresses the absolute minimum return variance that can be achieved when a portfolio consists of *n* risky assets and (2.2.6) is the portfolio in the set of optimal portfolios with the minimum return variance. All other portfolios in the set of optimal portfolios can be achieved using the general solution of the optimization problem that can be obtained from equation (2.2.4) The portfolios are distinguished between by different preferences on return requirement, r_P, Using the general solution (2.2.4) where $\lambda_1 \neq 0$, the optimal portfolio with a return requirement r_P is given by theorem 2.

Theorem 2: The optimal return variance portfolio.

Given a preferred return requirement r_P*, the optimal return variance portfolio asset allocation vector is obtained by solving* (2.2.4) *for* w_P^* *which results in*

$$w_{rp} = \frac{\Sigma_A^{-1}\upsilon}{Q_{22}} + \frac{r_P Q_{22} - Q_{12}}{|Q_{\#1}|}\left[\Sigma_A^{-1}\mu_A - \frac{Q_{12}}{Q_{22}}\Sigma_A^{-1}\upsilon\right] \tag{2.2.9}$$

The optimal allocation vector (2.2.9) *is composed out of two separate portfolios*

$$w_{rp} = w_{MVP} + w_\eta \tag{2.2.10}$$

where

$$w_\eta = \frac{r_P Q_{22} - Q_{12}}{|Q_{\#1}|}\left[\Sigma_A^{-1}\mu_A - \frac{Q_{12}}{Q_{22}}\Sigma_A^{-1}\upsilon\right] \tag{2.2.11}$$

and $\upsilon^T w_\eta = 0$.

Proof: See Appendix 2.

Theorem 2 states that optimal portfolios given by (2.2.9) can be distinguished between by different values of the preferred return requirement r_P. It should be noted in addition to definition 2.1.1, the optimal set W is composed out of all optimal solutions of the constrained Lagrangian optimization problem (2.2.1) - (2.2.3), i.e. both w_{MVP} (2.2.6) and w_{rp} (2.2.9). Theorem 2 shows that the optimal portfolios are composed out of the minimum variance portfolio, w_{MVP} given by (2.2.6), and a separate return generating portfolio, w_η (2.2.11). Equation (2.2.9) states that an optimal portfolio can be achieved for any return requirement r_P. For further analysis of the optimal solution, it is convenient to decompose the return generating component (2.2.11) as

$$w_\eta = \alpha_\eta w_\alpha \tag{2.2.12}$$

where the return requirement multiplier

$$\alpha_\eta = \frac{r_P Q_{22} - Q_{12}}{|Q_{\#1}|} \tag{2.2.13}$$

is a function of required rate of return, r_P, on a portfolio w_{rp}, and

$$w_\alpha = \left[\Sigma_A^{-1} \mu_A - \frac{Q_{12}}{Q_{22}} \Sigma_A^{-1} \upsilon \right] \tag{2.2.14}$$

is the portfolio component that shifts the optimal portfolio within the optimal set, W. The component w_α is linear in μ_A and involves an investment in a optimal growth portfolio $\Sigma_A^{-1}\mu_A$ and selling the MVP proportionally to Q_{12}. In fact, $\Sigma_A^{-1}\mu_A$ is a component that maximizes expected return for a given return variance. Interestingly, w_α (2.2.14) involves no net additional investment as it serves as a rebalancing term in order to tune the asset portfolio w_{rp} so as to achieve the expected return of r_P. In other words, the sum of all asset weights in (2.2.11) is zero as theorem 2 implies.

As the solutions of the optimal set have been stated in theorems 1 – 2, two definitions follow.

Definition 2.2.1: A portfolio w_P^* is called efficient if it solves the optimality problem for some $r_P \geq r_{MVP}$ (Keel & Muller, 1995).

Definition 2.2.2: The set of all efficient portfolios, W_E, is called the efficient set for the optimization problem (2.2.1) - (2.2.3); $W_E \subseteq W$ (Keel & Muller, 1995)..

The optimal portfolios (2.2.9) form the efficient set if $\alpha_\eta \geq 0$ in (2.2.12) which is equivalent to letting $r_P \geq r_{MVP}$ in (2.2.9). This can easily be seen by inserting r_{MVP} (2.2.7) into (2.2.9). The return generating component, w_α, is linear in μ_A and shifts the optimal portfolio w_{rp} within the optimal set, where the shift is proportional to α_η and therefore also proportional to r_P. This shift within the optimal set due to return requirement preferences is illustrated by figure 2.1 as different points on the line that represents the efficient set, W_E. The figure is a geometrical interpretation of the optimization problem and illustrates the special case when the number of risky assets is 3. Accordingly, the triangle on the figure represents the Euclidian 2-simplex where allocations in each of the risky assets are between 0 and 1 as the triangle represents only a small part of the surface defining feasible allocations. If optimal allocations for some assets are negative, the point on the line representing the efficient set is outside the triangle.

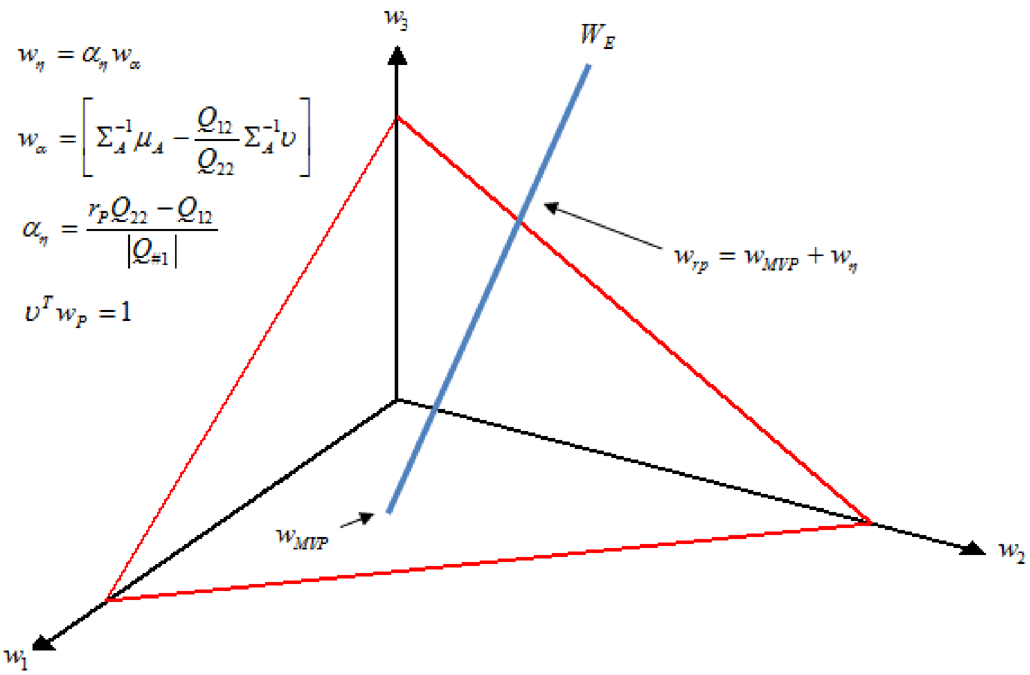


Figure 2.1: Geometrical interpretation of the optimization problem for the special case when n = 3. By varying the return requirement, the optimal portfolio shifts within the optimal set, W_E, due to that w_α is linear in μ_A.

As the optimal allocations of the traditional mean-variance model have been expressed, the associated expected returns and return variances are yet to be expressed. The expected return on the optimal portfolio is the sum of the expected return of two portfolio components as suggested by (2.2.10). Proposition 1 that follows, gives the expected return on the return generating component (2.2.11).

Proposition 1: The expected return on the return generating component.

The expected return on the return generating component (2.2.11) *is*

$$E\left[R_P\left(w_\eta\right)\right] = r_P - \frac{Q_{12}}{Q_{22}} \tag{2.2.15}$$

Proof: See Appendix 3.

As a result of proposition 1, the expected return on the optimal portfolio (2.2.9) equals the return requirement on that portfolio. This can easily be seen as the sum of (2.2.7) and (2.2.15) equals the return requirement parameter r_P;

$$E\left[R_P\left(w_{rp}\right)\right] = E\left[R_P\left(w_{MVP}\right)\right] + E\left[R_P\left(w_\eta\right)\right] = \frac{Q_{12}}{Q_{22}} + r_P - \frac{Q_{12}}{Q_{22}} = r_P$$

Since the optimal portfolio is composed out of two portfolio components, the return variance is composed out of the return variances of the two components and a return covariance term, i.e.

$$VAR\left[R_P\left(w_{rp}\right)\right]=VAR\left[R_P\left(w_{MVP}\right)\right]+VAR\left[R_P\left(w_\eta\right)\right]+2COV\left[R_P\left(w_{MVP}\right),R_P\left(w_\eta\right)\right]$$

Interestingly, the return covariance between the two portfolio components is zero as given by proposition 2 that follows.

Proposition 2: The zero return covariance between the minimum return variance and the return generating components.

Any optimal return variance portfolio can be written as $w_{rp}=w_{MVP}+w_\eta$. *The return covariance between the minimum return variance component,* w_{MVP}, *and the return generating component,* w_η *is*

$$COV\left[R_P\left(w_{MVP}\right)\ ,\ R_P\left(w_\eta\right)\right]=0 \tag{2.2.16}$$

Proof: See Appendix 4.

Since the return covariance between the two portfolio components is zero, the return variance on the optimal portfolio is simply the sum of the variances of the portfolio components. Proposition 3 gives the return variance on the return generating component (2.2.11).

Proposition 3: The return variance on the return generating component.

The return variance associated with the return generating component (2.2.11) *is*

$$VAR\left[R_P\left(w_\eta\right)\right]=\sigma_\eta^2=\frac{\left(r_PQ_{22}-Q_{12}\right)^2}{Q_{22}\left|Q_{\#1}\right|} \tag{2.2.17}$$

Proof: See Appendix 5.

As the return variances on both constituents of the optimal return portfolio have been given by theorem 1 and proposition 3, theorem 3 gives the expression for the variance on the optimal return portfolio (2.2.9).

Theorem 3: The return variance on the optimal return variance portfolio.

Any optimal return variance portfolio can be written as $w_{rp} = w_{MVP} + w_{\eta}$. *As a result of theorem 1 and propositions 2 and 3, the return variance associated with the optimal return variance portfolio* (2.2.9) *is*

$$\sigma_{rp}^2 = \sigma_{MVP}^2 + \sigma_{\eta}^2 = \frac{1}{Q_{22}}\left(1 + \frac{\left(r_P Q_{22} - Q_{12}\right)^2}{\left|Q_{\#1}\right|}\right)$$

Simplification of this expression results in

$$VAR\left[R_P\left(w_{rp}\right)\right] = \sigma_{rp}^2 = \frac{r_P^2 Q_{22} - 2r_P Q_{12} + Q_{11}}{\left|Q_{\#1}\right|} \tag{2.2.18}$$

Proof: See Appendix 6.

For a preferred return requirement parameter r_P, equation (2.2.18) expresses the return variance on the optimal portfolio w_{rp} associated with r_P.

2.3 The optimal portfolio in presence of a risk-free asset – The market portfolio

Now assume that in addition to the *n* risky assets available, there exists a risk-free asset yielding the risk-free rate r_f, and that this risk-free asset is available for investing in addition to the *n* risky assets. Let the allocation in the risk-free asset be w_0. Then the asset allocation vector becomes

$$\tilde{w}_P = \left[w_0, w_P\right]^T,\ \tilde{w}_P \in \mathbb{R}^{n+1} \tag{2.3.1}$$

where w_P denotes the portfolio of risky assets as before, $\tilde{\upsilon}^T \tilde{w}_P = 1$ where $\tilde{\upsilon} = \left[1_i\right]_{i=1,\ldots,n+1}$

As a starting point, lets consider (2.3.1) fully invested in risky assets, i.e $w_0 = 0$ and $\upsilon^T w_P = 1$. The expected return on the risky asset portfolio w_P can be decomposed into the risk-free rate, r_f, and the portfolio‘s risk premium on the risk-free rate, $w_P^T\left(\mu_A - r_f\upsilon\right)$. Formally, the decomposed expected return on the portfolio of risky assets w_P, can be expressed as

$$E\left[R_P\left(w_P\right)\right] = w_P^T\mu_A = w_P^T\left(\mu_A - r_f\upsilon\right) + r_f \tag{2.3.2}$$

Using (2.3.2) instead of (2.2.2) as a constraint in the optimization problem (2.2.1) - (2.2.3), the well-known market portfolio can be derived using the same method as before. The market portfolio is an optimal portfolio composed out of the *n* risky assets available and not including the risk free asset. Solving the optimization problem (2.2.1) s.t. (2.3.2) yields the market portfolio expressed by theorem 4 that follows.

Theorem 4: The market portfolio.

Given the risk-free rate r_f *, the risky assets market portfolio asset allocation vector is*

$$w_{MKT} = \frac{\Sigma_A^{-1}\left(\mu_A - r_f\upsilon\right)}{\upsilon^T\Sigma_A^{-1}\left(\mu_A - r_f\upsilon\right)} \tag{2.3.3}$$

The expected return on the market portfolio is

$$E\left[R_P\left(w_{MKT}\right)\right] = r_{MKT} = \frac{\mu_A^T\Sigma_A^{-1}\left(\mu_A - r_f\upsilon\right)}{\upsilon^T\Sigma_A^{-1}\left(\mu_A - r_f\upsilon\right)} \tag{2.3.4}$$

and the market portfolio return variance is

$$VAR\left[R_P\left(w_{MKT}\right)\right] = \sigma_{MKT}^2 = \frac{\left(\mu_A - r_f\upsilon\right)^T\Sigma_A^{-1}\left(\mu_A - r_f\upsilon\right)}{\left(\upsilon^T\Sigma_A^{-1}\left(\mu_A - r_f\upsilon\right)\right)^2} \tag{2.3.5}$$

Proof: See Appendix 7.

According to (2.3.3), the market portfolio can be considered as a risk premium portfolio that is linear in the risk premium vector $\left(\mu_A - r_f\upsilon\right)$. The market portfolio is a unique portfolio for any given value of the risk-free rate and is expressed independently from the minimum variance portfolio, only dependent on μ_A, Σ_A and r_f as implied by expression (2.3.3). The market portfolio is also referred to as the CAPM tangency portfolio as it is the optimal portfolio of risky assets where the capital market line (CML) is tangent to the set of optimal portfolios in risk-return space. At the CML, the investor invests in the portfolio given by (2.3.1); w_0 in the risk-free asset and $\left(1 - w_0\right)w_{MKT}$ in the risky asset market portfolio. The risk-return space will be given further attention in following sections, especially in section 2.8. Furthermore, the market portfolio will be analyzed further in section 2.12, including the consideration of liabilities.

2.4 Introduction to surplus optimization

In the introduction, a feasible method for incorporating liabilities into the traditional mean-variance framework was discussed; surplus return optimization. In following sections, a surplus risk-return optimization model is derived including a simple extra term in addition to the traditional Markowitz model derived in section 2.2. As mentioned before, the traditional model is a special case of the surplus optimization model where the value of the liabilities is zero, the growth rate of the liabilities is zero or the covariance between assets and liabilities is zero.

The surplus return optimization model depends upon a fixed initial funding ratio, i.e. the ratio of assets to liabilities and represents a static solution as the solution is not derived from a continuous-time framework. It should be noted that initial funding ratio is not a variable although the funding ratio varies with time as a result of changes in values of assets and liabilities. For solving the optimization problem derived in the following sections, the necessary conditions, assumptions and notation are given in section 2.1 and also added appropriately through the text. Complete proofs of the derivations in section 2.6 are given in appendices 8 - 20.

The assumptions that have been listed already in section 2.1 provide conditions for the existence of unique solutions to the optimization problem derived in this and next sections. As in previous sections, the investment strategy of an investor is given by the choice of securities w_i, $i \in I$.

For the investor‘s risky asset portfolio P, which is characterized by the asset allocation vector w_P, the expected value of returns on P, composed out of the n risky assets available is the same as given in previous section, i.e.

$$E\left[R_P\left(w_P\right)\right] = \mu_A^T w_P$$

where μ_A is vector of risky asset mean returns as defined in section 2.1.

The return on total assets, *A*, and on liabilities, *L*, can be written as, respectively,

$$R_A = \frac{A(t_1) - A(t_0)}{A(t_0)} \quad \text{and} \quad R_L = \frac{L(t_1) - L(t_0)}{L(t_0)}$$

The surplus of a pension scheme at each time t with the importance parameter θ included (Sharpe & Tint, 1990), is defined as

$$S(t) = A(t) - \theta L(t)$$

The importance parameter $\theta \geq 0$ allows for surplus optimization approach that avoids either asset only ($\theta = 0$) or full surplus optimization ($\theta = 1$). By incorporating θ, an investor can choose the consideration level given to liabilities in his allocation process. This allows for flexibility in asset allocation and partial ALM consideration. This increases the potentials for the surplus optimization model applicability in asset management and allows for gradually increasing the application of ALM practices in pension schemes where ALM practices are being adopted, by increasing θ with time.

The funding ratio of a pension scheme at time t w.r.t. θ, is written as the ratio of assets to liabilities;

$$F(t) = \frac{A(t)}{\theta L(t)}$$

The value of the assets and liabilities at time t_1, can be written as

$$A(t_1) = A(t_0)[1 + R_A] \quad \text{and} \quad L(t_1) = L(t_0)[1 + R_L]$$

According to the above, the pension plan surplus can be expressed as

$$S(t_1) = A(t_1) - \theta L(t_1) = A(t_0)(1 + R_A) - \theta L(t_0)(1 + R_L)$$

The change in surplus during the time interval is therefore

$$\begin{aligned} S(t_1) - S(t_0) &= A(t_0)(1 + R_A) - \theta L(t_0)(1 + R_L) - \left(A(t_0) - \theta L(t_0)\right) \\ &= A(t_0) R_A - \theta L(t_0) R_L \end{aligned}$$

Using the same method as in Sharpe & Tint (1990), Keel & Müller (1995) and Rudolf & Ziemba (2004), the normalized return on surplus, where the change in surplus is written in terms of assets, can be written as

$$R_{S,n} = \frac{S(t_1) - S(t_0)}{A(t_0)} = R_A - \theta \frac{L(t_0)}{A(t_0)} R_L = R_A - \frac{\theta}{F(t_0)} R_L \qquad (2.4.1)$$

The normalized return on surplus is used to avoid division with zero which could happen if the numerator was $S(t_0)$.

It now follows that the expected normalized return on surplus is written as

$$E\left[R_{S,n}(w_P)\right] = E\left[R_P(w_P) - \frac{\theta}{F} R_L\right] = \mu_A^T w_P - \frac{\theta}{F} \mu_L \qquad (2.4.2)$$

where the growth rate of the liabilities has been replaced with its estimator; $R_L = \mu_L$.

Also, a clear distinction is made between return on surplus, i.e. without the normalization of $1/F$ and the normalized return on surplus. Accordingly, the expected return on surplus is

$$E\left[R_S\left(w_P\right)\right]=E\left[R_P\left(w_P\right)-\theta R_L\right]=\mu_A^T w_P-\theta\mu_L \tag{2.4.3}$$

In this thesis, the normalized surplus returns are mainly under consideration and the adjective "normalized" is not used except when both concepts are under consideration in parallel. Therefore, the expected normalized surplus returns and normalized surplus return variance are referred to as surplus returns and surplus return variance, except when a clear distinction is made between those two concepts.

Since a feasible expression for the expected return on normalized surplus has now been written (2.4.2), the goal is now to find a set of portfolios where the assets are selected in such a way that the volatility of the surplus return is minimized. Before the derivation of the optimization model is continued, some properties of the surplus return variance are analyzed.

2.5 Analysis of the surplus return variance.

In order to prepare the ground for the surplus return optimization model, it is useful to analyze the normalized surplus return variance. Given the asset portfolio w_P, the variance of surplus returns is

$$\mathrm{VAR}\left[R_P\left(w_P\right)-R_L\right]\equiv\sigma_S^2\left(w_P\right)$$

and the variance of normalized surplus returns, mainly under consideration in this thesis, is

$$\mathrm{VAR}\left[R_P\left(w_P\right)-\frac{\theta}{F}R_L\right]\equiv\sigma_{S,n}^2\left(w_P\right)$$

Using general methodology, the variance of the normalized surplus return can be decomposed as

$$\sigma_{S,n}^2\left(w_P\right)=\sigma_P^2\left(w_P\right)+\frac{\theta^2}{F^2}\sigma_L^2-\frac{2\theta}{F}\sigma_P\sigma_L\rho_{P,L}$$

and similarly, the decomposition of surplus return variance can be written as

$$\sigma_S^2\left(w_P\right)=\sigma_P^2\left(w_P\right)+\theta^2\sigma_L^2-2\theta\sigma_P\sigma_L\rho_{P,L}$$

where $\sigma_P^2\left(w_P\right)$ is the variance of asset portfolio returns, σ_L^2 is the variance of liability returns and $\rho_{P,L}$ is the correlation coefficient between portfolio returns and returns on liabilities.

As the general expression for the return variance of a portfolio w_P is $\sigma_P^2\left(w_P\right) = w_P^T \Sigma_A w_P$, the variance of normalized surplus returns can equivalently be expressed as

$$\sigma_{S.n}^2\left(w_P\right) = w_P^T \Sigma_A w_P + \frac{\theta^2}{F^2}\sigma_L^2 - \frac{2\theta}{F}\Sigma_{AL}^T w_P \tag{2.5.1}$$

where Σ_{AL} is a vector containing the covariance elements between individual asset returns and returns on liabilities; $\Sigma_{AL} = \left[\sigma_{iL}\right]_{i=1,\dots,n}$. Similarly, the variance of surplus returns can be expressed as

$$\sigma_S^2\left(w_P\right) = w_P^T \Sigma_A w_P + \theta^2\sigma_L^2 - 2\theta\Sigma_{AL}^T w_P \tag{2.5.2}$$

According to Sharpe and Tint (1990), the third term in (2.5.1) is referred to as liability hedging credit (LHC) for the asset portfolio, i.e.

$$LHC_P = \frac{2\theta}{F}\Sigma_{AL}^T w_P \tag{2.5.3}$$

Surplus optimization involves a willingness to accept lower expected portfolio return and/or greater asset portfolio risk in order to increase the ability of an asset mix to hedge against changes in liabilities. The liability hedging credit relates to the ability of an asset portfolio to reduce the risk associated with surplus as can be directly seen from (2.5.1). As the name implies, the liability hedging credit provides certain credit in terms of reducing surplus return variance.

From (2.5.1), it is readily observable that the variance of normalized surplus returns can be decomposed into three components; the variance of asset portfolio returns, the variance of liabilities returns and the covariance term between returns on assets and liabilities. This observation has important implications for asset allocation purposes in presence of liabilities. As the first two terms are positive and add together, the surplus return variance can be reduced by selecting the assets in such a way that the covariance between the assets and liabilities is positive and preferably high, i.e. select assets where their returns are highly correlated with the returns on liabilities. Theoretically, this implies that the investor should invest in assets that are perfectly correlated with the liabilities. Such a complete market situation does not exist in practice, as the growth of liabilities is driven by more uncertainty factors than market variables alone. The uncertainty of liabilities growth is frequently decomposed in academic studies into two separate factors that can be decomposed further; variability due to market movements and variability due to factors that are not directly connected to market behaviour. The second factor includes variability due to factors like demographic trends, actuarial factors etc. The division of the surplus return risk makes sense as systematic market risk and specific liability risk are only subject to the same risk factors up to a certain point. The co-movement in the worth of liabilities and marketable securities differs between economies; liabilities in one economy may be more correlated with market variables than in other for a number of reasons.

One question that rises naturally in relation to the surplus return variance (2.5.1), irrespective of asset allocation, is what value of importance given to liabilities, θ, minimizes the variance of surplus returns. A necessary first order condition for a minimum is found via the partial derivative of $\sigma_{S,n}^2$ w.r.t. θ which yields

$$\frac{\theta}{F} = \frac{w_P^T \Sigma_{AL}}{\sigma_L^2} = \frac{COV\left[R_P\left(w_P\right), R_L\right]}{VAR\left[R_L\right]} \tag{2.5.4}$$

By observing (2.5.4) and thinking in terms of CAPM, the liabilities can be considered as a "market index" where the performance of the assets is measured against liabilities in a similar way as stock β in the CAPM model; θ/F represents the β of the liabilities. The ratio θ/F plays an important role in the asset allocation decision in presence of liabilities as it directly affects the optimal investment decision by determining the covariance between the liabilities and the optimal investment policy. If the ratio θ/F increases, the covariance between assets and liabilities must increase through the asset portfolio in order to maintain equilibrium in(2.5.4). This can be interpreted in the following way: If a fund manager wants to maintain a constant surplus return risk in his asset portfolio when the ratio θ/F increases, he must increase the covariance between assets returns and returns on liabilities via rebalancing the portfolio. According to this, the appropriate action against decreasing funding ratio or increasing importance is to select the assets so as to increase the covariance between asset and liabilities returns. One problem with this CAPM-like interpretation is that the liabilities risk can only be partially hedged for by the securities available in the market, i.e. the market is not "complete" with respect to liabilities as a random variable.

2.6 The surplus optimization model

Some properties of the objective have now been explained and discussed in previous section. Since the purpose of the surplus optimization model should be clear, the model can be expressed as

$$\underset{w}{Min} \;\; \mathrm{VAR}\left[R_P\left(w_P\right) - \frac{\theta}{F} R_L \right]$$

s.t.

$$E\left[R_P\left(w_P\right)\right] = r_P$$

$$\sum_{i=1}^{n} w_i = 1$$

where r_P is a return requirement value for expected return on the portfolio as before. Using (2.5.1) from last section and dropping the irrelevant constant term for the liabilities return variance, the surplus return optimization model is expressed as

$$\underset{w}{Min} \;\; \frac{1}{2} w_{P,S}^T \Sigma_A w_{P,S} - \frac{\theta}{F} w_{P,S}^T \Sigma_{AL} \tag{2.6.1}$$

s.t.

$$w_{P,S}^T \mu_A = r_P \tag{2.6.2}$$

$$w_{P,S}^T \upsilon = 1 \tag{2.6.3}$$

For the problem (2.6.1) - (2.6.3), the Lagrangian is defined as

$$L\left(w_{P,S}, \lambda_1, \lambda_2\right) = \frac{1}{2} w_{P,S}^T \Sigma_A w_{P,S} - \frac{\theta}{F} w_{P,S}^T \Sigma_{AL} + \lambda_1\left(r_P - w_{P,S}^T \mu_A\right) + \lambda_2\left(1 - w_{P,S}^T \upsilon\right)$$

The partial derivatives of L w.r.t. $w_{P,S}$ yield necessary first order conditions for stationarity and primal feasibility

$$\Sigma_A w_{P,S}^* - \frac{\theta}{F}\Sigma_{AL} - \lambda_1 \mu_A - \lambda_2 \upsilon = 0, \quad w_{P,S}^T \mu_A = r_P \quad \text{and} \quad w_{P,S}^T \upsilon = 1$$

A second order sufficient condition for a minimum is satisfied as the covariance matrix is positive definite. Isolating the optimal allocation vector $w_{P,S}^*$ from the first order stationarity condition gives

$$w_{P,S}^* = \Sigma_A^{-1}\left[\mu_A \quad \upsilon \quad \Sigma_{AL}\right]\left[\lambda_1 \quad \lambda_2 \quad \theta F^{-1}\right]^T \tag{2.6.4}$$

Equation (2.6.4) is the general solution to the surplus risk-return optimization problem (2.6.1) - (2.6.3) in terms of Lagrangian parameters and is used for deriving the solutions specified in theorems 5 – 8 and propositions 4 - 12 that follow.

As a starting point in solving the model (2.6.1) – (2.6.3) for specific optimal solutions, theorem 5 gives the minimum surplus return variance portfolio, with the shorthand notation of MSVP or w_{MSVP}. The MSVP is obtained by omitting the constraint on portfolio return (2.6.2), requiring $\lambda_1 = 0$ and (2.6.4) simplifies accordingly to

$$w^*_{P,S} = \Sigma_A^{-1}\left[\upsilon \quad \Sigma_{AL}\right]\left[\lambda_2 \quad \theta F^{-1}\right]^T \tag{2.6.5}$$

Theorem 5: The minimum surplus return variance portfolio.

The minimum surplus return variance portfolio (MSVP) allocation vector is obtained by solving (2.6.5) *explicitly for* $w^*_{P,S}$ *which results in*

$$w_{MSVP} = \frac{\Sigma_A^{-1}\upsilon}{Q_{22}} + \frac{\theta}{F}\left[\Sigma_A^{-1}\Sigma_{AL} - \frac{Q_{23}}{Q_{22}}\Sigma_A^{-1}\upsilon\right] \tag{2.6.6}$$

The optimal allocation vector (2.6.6) *is composed out of two separate portfolios*

$$w_{MSVP} = w_{MVP} + \theta\phi_{MSVP} \tag{2.6.7}$$

where w_{MVP} (2.2.6) *is in accordance with theorem 1, the minimum surplus variance correction component is*

$$\phi_{MSVP} = \frac{1}{F}\left[\Sigma_A^{-1}\Sigma_{AL} - \frac{Q_{23}}{Q_{22}}\Sigma_A^{-1}\upsilon\right] \tag{2.6.8}$$

and $\upsilon^T\phi_{MSVP} = 0$.

Proof: See Appendix 8.

As (2.6.6) implies, the MSVP is composed out of two separate components; the traditional minimum variance portfolio, w_{MVP} (2.2.6), and an additional surplus return variance correction component, ϕ_{MSVP}, that is linear in Σ_{AL} and leads to a shift of the optimal set by $\theta\phi_{MSVP}$. Equation (2.6.8) represents a separate liability hedging component added to w_{MVP}, that plays the role of tuning the returns on the MVP with liabilities returns by minimizing surplus return variance. This correction is achieved by rebalancing the MVP allocation in proportion to θ/F

where θ/F acts as a rebalancing multiplier or leveraging coefficient. The rebalancing resulting from the minimization of surplus return variance leads to a shift in the optimal set in proportion to θ/F and an increase in correlation between the MVP and the liabilities through $\Sigma_A^{-1}\Sigma_{AL}$. Minimizing the variance of surplus returns is strongly related to maximizing the return covariance between assets and liabilities. From a certain point of view, minimizing surplus return variance involves maximizing the return covariance between assets and liabilities as the maximum asset-liability return covariance component, $\Sigma_A^{-1}\Sigma_{AL}$, is included in the minimum surplus return solution.

The problem of maximizing the return covariance between assets and liabilities can be expressed in the following way:

$$\underset{w}{Max} \quad w_P^T\Sigma_{AL}$$

s.t.

$$w_P^T\Sigma_A w_P = \sigma_P^2$$

Necessary first order conditions for stationarity and primal feasibility are

$$\Sigma_{AL} - \lambda\Sigma_A w_P^* = 0 \quad \text{and} \quad w_P^T\Sigma_A w_P = \sigma_P^2$$

and the second order sufficient condition for a maximum is satisfied as Σ_A is positive definite. Isolating w_P^* from the first order stationarity condition gives

$$w_{COV} = \frac{1}{\lambda}\Sigma_A^{-1}\Sigma_{AL} \tag{2.6.9}$$

Equation (2.6.9) expresses the portfolio component that maximizes the return covariance between asset portfolio returns and returns on liabilities on the normalization basis of λ. This component appears in the expression for the minimum surplus return variance correction component ϕ_{MSVP} (2.6.8) and plays the role of maximizing the return covariance between assets and liabilities. If the MSVP (2.6.6) is recalled,

$$w_{MSVP} = w_{MVP} + \theta\phi_{MSVP} = \frac{\Sigma_A^{-1}\upsilon}{Q_{22}} + \frac{\theta}{F}\left[\Sigma_A^{-1}\Sigma_{AL} - \frac{Q_{23}}{Q_{22}}\Sigma_A^{-1}\upsilon\right]$$

the minimum surplus return variance portfolio involves partial investment $\left(1-\theta Q_{23}/F\right)$ in the MVP and partial investment in the covariance maximizing component $\left(\theta\, Q_{23}/F\right)$.

This is consistent with isolating λ from (2.6.9) and under the normalization of $\upsilon^T w_P = 1$, λ becomes

$$\lambda = Q_{23}$$

Accordingly, inserting this expression for λ back into (2.6.9) gives the unit normalized maximum return covariance component

$$w_{\text{cov}} = \frac{\Sigma_A^{-1}\Sigma_{AL}}{Q_{23}} \tag{2.6.10}$$

This component appears again in section 2.12 where its association with other optimal portfolios in presence of liabilities is discussed.

Since the minimum surplus return variance allocation vector has been expressed, the associated expected returns and return variances are yet to be expressed. The expected return on the MSVP is the sum of the expected returns of the two portfolio components as suggested by (2.6.7) and proposition 4 confirms.

Proposition 4: The expected return on the minimum surplus return variance portfolio.
The expected return on the minimum surplus return variance correction component (2.6.8) *is*

$$E\left[R_P\left(\phi_{MSVP}\right)\right] = -\frac{1}{F}\frac{|Q_{\#2}|}{Q_{22}} \tag{2.6.11}$$

Accordingly, the expected return on the minimum surplus return variance portfolio (2.6.6) *is*

$$\begin{aligned} E\left[R_P\left(w_{MSVP}\right)\right] &= E\left[R_P\left(w_{MVP}\right)\right] + \theta E\left[R_P\left(\phi_{MSVP}\right)\right] \\ &= \frac{1}{Q_{22}}\left(Q_{12} - \frac{\theta}{F}|Q_{\#2}|\right) \end{aligned} \tag{2.6.12}$$

Proof: See Appendix 9.

Since the MSVP is composed out of two portfolio components, the return variance is composed out of the return variances of the two portfolio components and a return covariance term, i.e.

$$\begin{aligned} VAR\left[R_P\left(w_{MSVP}\right)\right] = VAR\left[R_P\left(w_{MVP}\right)\right] &+ \theta^2 VAR\left[R_P\left(\phi_{MSVP}\right)\right] \\ &+ 2\theta COV\left[R_P\left(w_{MVP}\right)\ ,\ R_P\left(\phi_{MSVP}\right)\right] \end{aligned}$$

In order to find an expression for the return variance on the MSVP (2.6.6), the expression for the return variance on the minimum surplus return variance correction component is given by proposition 5 that follows.

Proposition 5: The return variance on the minimum surplus return variance correction component.

The return variance associated with the minimum surplus return variance correction component (2.6.8) *is;*

$$VAR\left[R_P\left(\phi_{MSVP}\right)\right]=\sigma^2_{\phi MSVP}=\frac{1}{F^2}\frac{\left|Q_{\#4}\right|}{Q_{22}} \tag{2.6.13}$$

Proof: See Appendix 10.

As the return variance associated with both components of the MSVP have been found, proposition 6 that follows states that the return covariance between the two portfolio components is zero .

Proposition 6: The zero return covariance between the minimum return variance component and the minimum surplus return variance correction component.

The minimum surplus return variance portfolio is written as $w_{MSVP}=w_{MVP}+\theta\phi_{MSVP}$. *The return covariance between the minimum return variance component,* w_{MVP}, *and the minimum surplus return variance correction component,* ϕ_{MSVP}, *is*

$$COV\left[R_P\left(w_{MVP}\right)\ ,\ R_P\left(\phi_{MSVP}\right)\right]=0 \tag{2.6.14}$$

Proof: See Appendix 11.

Since all components of the return variance on the MSVP have been stated in theorem 1, and propositions 5 - 6, theorem 6 gives the expression for the return variance on the MSVP.

Theorem 6: The return variance on the minimum surplus return variance portfolio.

The minimum surplus return variance portfolio is written as $w_{MSVP}=w_{MVP}+\theta\phi_{MSVP}$. *The return variance associated with the minimum surplus return variance portfolio is*

$$\begin{aligned}VAR\left[R_P\left(w_{MSVP}\right)\right]&=\sigma^2_{MSVP}\\&=\sigma^2_{MVP}+\theta^2\sigma^2_{\phi MSVP}\\&=\frac{1}{Q_{22}}\left(1+\frac{\theta^2}{F^2}\left|Q_{\#4}\right|\right)\end{aligned} \tag{2.6.15}$$

This result is in accordance with the results of theorem 1, and propositions 5 - 6.

Proof: See Appendix 12.

For given values of θ and F, equation (2.6.15) expresses the minimum surplus return variance that can be achieved when a portfolio consists of n risky assets and (2.6.6) gives the portfolio in the set of optimal surplus return variance portfolios with the minimum surplus return variance.

All other portfolios in the set of optimal surplus return portfolios, W_S, can be achieved using the general solution of the optimization problem that can be obtained from equation (2.6.4). The optimal portfolios are distinguished between by different preferences on return requirement, r_P, Using the general solution (2.6.4) where $\lambda_1 \neq 0$, the optimal portfolio with return requirement r_P is given by theorem 7.

Theorem 7: The optimal surplus return variance portfolio.

Given the preferred return requirement r_P, *the optimal surplus return portfolio asset allocation vector is obtained by solving* (2.6.4) *for* $w^*_{P,S}$ *which results in*

$$\begin{aligned} w_{rp,S} &= \frac{\Sigma_A^{-1}\upsilon}{Q_{22}} + \frac{\theta}{F}\left[\Sigma_A^{-1}\Sigma_{AL} - \frac{Q_{23}}{Q_{22}}\Sigma_A^{-1}\upsilon\right] \\ &\quad + \frac{r_P Q_{22} - Q_{12} + \theta F^{-1}\left|Q_{\#2}\right|}{\left|Q_{\#1}\right|}\left[\Sigma_A^{-1}\mu_A - \frac{Q_{12}}{Q_{22}}\Sigma_A^{-1}\upsilon\right] \end{aligned} \tag{2.6.16}$$

The optimal allocation vector is composed out of four separate portfolios

$$w_{rp,S} = w_{MVP} + \theta\phi_{MSVP} + w_\eta + \theta w_{\eta,S} \tag{2.6.17}$$

where w_{MVP}, w_η *and* ϕ_{MSVP} *are in accordance with theorems 1, 2 and 5, respectively. The return generating correction component is*

$$w_{\eta,S} = \frac{1}{F}\frac{\left|Q_{\#2}\right|}{\left|Q_{\#1}\right|}\left[\Sigma_A^{-1}\mu_A - \frac{Q_{12}}{Q_{22}}\Sigma_A^{-1}\upsilon\right] \tag{2.6.18}$$

and $\upsilon^T w_{\eta,S} = 0$.

Proof: See Appendix 13.

Theorem 7 states that the optimal portfolios given by (2.6.16) can be distinguished between by different values of the preferred return requirement r_P. It should be noted in addition to definition 2.1.1, the optimal set W_S is composed of all optimal solutions of the constrained

Lagrangian optimization problem (2.6.1) – (2.6.3), i.e. both w_{MSVP} (2.6.6) and $w_{rp,S}$ (2.6.16). Theorem 7 shows that the optimal portfolios are composed out of the minimum variance portfolio, w_{MVP} given by equation (2.2.6), the minimum surplus return variance correction component, ϕ_{MSVP} (2.6.8), which is linear in Σ_{AL} and leads to a shift of the optimal set by $\theta\phi_{MSVP}$. Recall that these two are the components of the MSVP, i.e. $w_{MSVP} = w_{MVP} + \theta\phi_{MSVP}$. Additionally, two separate components are included; the return generating components w_η and $w_{\eta,S}$ which both shift the optimal portfolio within the optimal set. The shift resulting from w_η is proportional to r_P whereas the shift resulting from $w_{\eta,S}$ is proportional to θ/F as $w_{\eta,S}$ results from considering liabilities. The former return generating component, w_η (2.2.11), is the component that shifts the optimal portfolio within the optimal set so as to achieve return requirement r_P on the portfolio. The latter component, $w_{\eta,S}$ (2.6.18), corrects the return on the optimal portfolio $w_{rp,S}$ as to neutralize the return shift that ϕ_{MSVP} adds. The expected return on the liability hedging component, ϕ_{MSVP}, is included in the return on the optimal portfolio and must be corrected for if a specified return requirement input is used for the return generating component, w_η. The return generating correction component, $w_{\eta,S}$, corrects the additional return stemming from ϕ_{MSVP} so as to match the expected return on the portfolio and the return requirement input, r_P. This is confirmed by the results of proposition 7, after a little more discussion on the optimal set, where the sum of the expected returns on ϕ_{MSVP} and $w_{\eta,S}$ is zero.

By closer examination of equation (2.6.16)/(2.6.17), an optimal surplus management policy involves investing in at least two "funds"; a liability hedging portfolio ϕ_{MSVP} for risk management purposes in proportion to the importance attached to it by θ, and the standard optimal growth portfolio, $w_{MVP} + w_\eta$ (2.2.10), for conventional asset management purposes. The latter portfolio (2.2.10) is the standard mean-variance efficient portfolio; a conventional investment portfolio or an "alpha-boosting strategy" in addition to the liability hedge. In a simple way, the two/three portfolios (2.6.16)/(2.6.17) combine the classic performance-seeking portfolio and a liability hedging portfolio.

For further analysis of the optimal solution, it is convenient to decompose the net return generating portfolio $w_\eta + \theta w_{\eta,S}$ form (2.6.16)/(2.6.17) as

$$w_\eta + \theta w_{\eta,S} = \alpha w_\alpha \tag{2.6.19}$$

where

$$\alpha = \alpha_\eta + \theta \alpha_{\eta,S} \tag{2.6.20}$$

$$\alpha_{\eta,S} = \frac{1}{F}\frac{|Q_{\#2}|}{|Q_{\#1}|} \tag{2.6.21}$$

and α_η and w_α are according to (2.2.13) and (2.2.14), respectively;

$$\alpha_\eta = \frac{r_P Q_{22} - Q_{12}}{|Q_{\#1}|} \quad \text{and} \quad w_\alpha = \left[\Sigma_A^{-1}\mu_A - \frac{Q_{12}}{Q_{22}}\Sigma_A^{-1}\upsilon\right]$$

Accordingly, equation (2.6.20) becomes

$$\alpha = \alpha_\eta + \theta \alpha_{\eta,S} = \frac{r_P Q_{22} - Q_{12} + \theta F^{-1}|Q_{\#2}|}{|Q_{\#1}|} \tag{2.6.22}$$

As w_α is the portfolio component that shifts the optimal portfolio within the optimal set, W or W_S depending on the framework, the multiplier (2.6.22) ensures that required rate of return, r_P, on a portfolio is acquired.

Since the solutions of the surplus optimal set have been stated in theorems 5 and 7, two definitions follow.

Definition 2.6.1: A portfolio $w_{P,S}$ is called efficient if it solves the optimality problem for some $r_P \geq r_{MSVP}$ (Keel and Muller, 1995).

Definition 2.6.2: The set of all efficient portfolios, $W_{E,S}$, is called the efficient set for the surplus optimization problem; $W_{E,S} \subseteq W_S$ (Keel and Muller, 1995).

The optimal portfolios form the efficient set if $\alpha \geq 0$ which is equivalent to letting $r_P \geq r_{MSVP}$ in (2.6.16). This can easily be seen by inserting r_{MSVP} (2.6.12) into (2.6.16). The return generating component, w_α, is linear in μ_A and shifts the optimal portfolio within the optimal set. The shift is proportional to α and therefore also proportional to r_P. The shift of the efficient set due to the liability hedge and the shift due to return requirement preferences is illustrated by figure 2.2 where the occurrence of liabilities leads to a shift of the efficient set by $\theta\phi_{MSVP}$. The figure illustrates the special case when the number of risky assets is 3 and is the same figure as figure 2.1, except it shows the shift of the efficient set due to the liability hedge. Just as for figure 2.1, the triangle on the figure represents the Euclidian 2-simplex where allocations in each of the risky assets are between 0 and 1 as the triangle represents only a small part of the surface defining feasible allocations. If optimal allocations for some assets are negative, the point on the line representing the efficient set is outside the triangle.

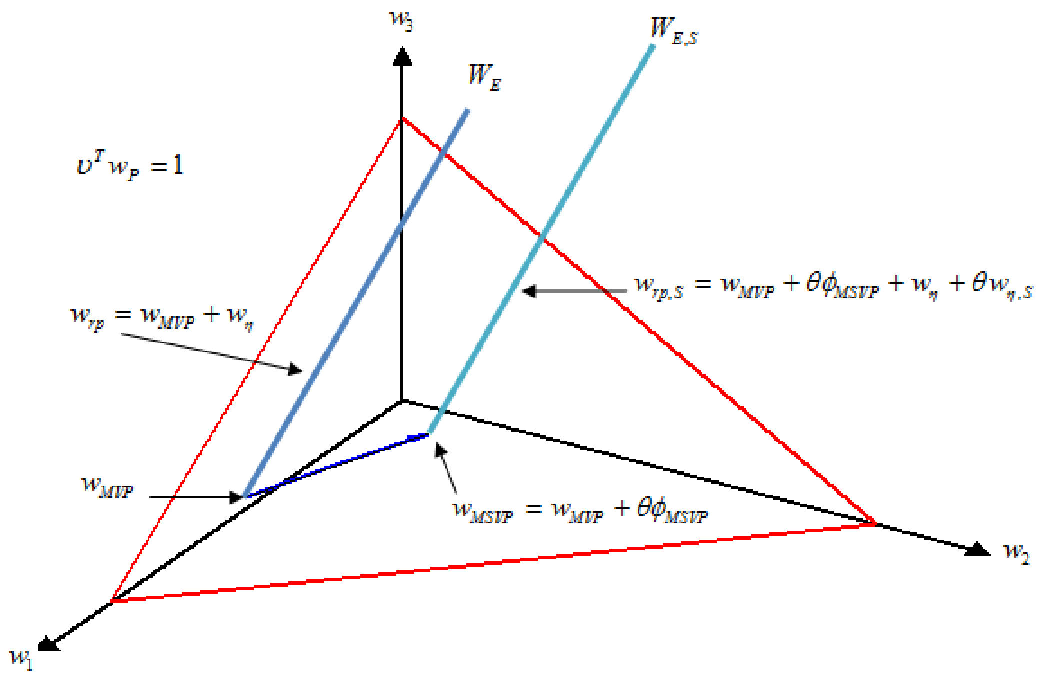


Figure 2.2: Geometrical interpretation of the optimization problem in the presence of liabilities for the special case when n = 3. The shift of the frontier set W_E, due to , results in a shifted set $W_{E,S}$.

As discussed before, the return shift resulting from the liability hedging component, ϕ_{MSVP}, is neutralized by the expected return on the return generating correction component $w_{\eta,S}$ as shown by proposition 7 that follows,

Proposition 7: The expected return on the return generating correction component.

The expected return on the return generating correction component $w_{\eta,S}$ (2.6.18) *is;*

$$E\left[R_P\left(w_{\eta,S}\right)\right]=\frac{1}{F}\frac{\left|Q_{\#2}\right|}{Q_{22}} \tag{2.6.23}$$

From (2.6.11) *and* (2.6.23), *it can be seen that*

$$E\left[R_P\left(\phi_{MSVP}\right)\right]+E\left[R_P\left(w_{\eta,S}\right)\right]=0 \tag{2.6.24}$$

Proof: See Appendix 14.

As a result of proposition 7, the expected portfolio return on the optimal portfolio (2.6.16) equals the return requirement on the optimal portfolio. This can easily be seen from (2.6.24) and that the sum of (2.2.7) and (2.2.15) equals the return requirement parameter r_P ;

$$\begin{aligned}E\left[R_P\left(w_{rp,S}\right)\right]&=E\left[R_P\left(w_{MVP}\right)\right]+\theta E\left[R_P\left(\phi_{MSVP}\right)\right]+E\left[R_P\left(w_{\eta}\right)\right]+\theta E\left[R_P\left(w_{\eta,S}\right)\right]\\&=\frac{Q_{12}}{Q_{22}}-\frac{\theta}{F}\frac{\left|Q_{\#2}\right|}{Q_{22}}+r_P-\frac{Q_{12}}{Q_{22}}+\frac{\theta}{F}\frac{\left|Q_{\#2}\right|}{Q_{22}}=r_P\end{aligned}$$

As the name implies, the purpose of the return generating correction component, $w_{\eta,S}$ (2.6.18), is to correct the expected return stemming from the liability correction component ϕ_{MSVP} (2.6.8). Including the $w_{\eta,S}$ in the optimal solution changes the return variance of the optimal portfolio $w_{rp,S}$ by the return variance and covariance terms associated with $w_{\eta,S}$. Proposition 8 that follows shows the return variance on $w_{\eta,S}$.

Proposition 8: The return variance on the return generating correction component.

The return variance associated with the return generating correction component (2.6.18) *is;*

$$VAR\left[R_P\left(w_{\eta,S}\right)\right]=\sigma^2_{\eta,S}=\frac{1}{F^2}\frac{\left|Q_{\#2}\right|^2}{Q_{22}\left|Q_{\#1}\right|} \tag{2.6.25}$$

Proof: See Appendix 15.

As the allocation and expected returns on the optimal surplus return portfolio components has been shown in the preceding text, the return variance has yet to be expressed. The return variances on the portfolio components have been stated in theorem 1 and propositions 3, 5 and 8. Two return covariance terms between individual portfolio components have been stated in propositions 2 and 6. For deriving the expression for the return variance on the optimal surplus return portfolio (2.6.16), propositions 9 - 12 provide the remaining covariance terms and the return variance is finally expressed in theorem 8.

Proposition 9: The zero return covariance between the minimum return variance and the return generating correction component.

The return covariance between the minimum return variance component, w_{MVP}*, and the return generating correction component,* $w_{\eta,S}$*, is*

$$COV\left[R_P\left(w_{MVP}\right)\ ,\ R_P\left(w_{\eta,S}\right)\right] = 0 \tag{2.6.26}$$

Proof: See Appendix 16.

Proposition 10: The return covariance between the minimum surplus return variance correction and the return generating components.

The return covariance between the minimum surplus return variance correction component, ϕ_{MSVP}*, and the return generating component,* w_{η}*, is*

$$COV\left[R_P\left(\phi_{MSVP}\right)\ ,\ R_P\left(w_{\eta}\right)\right] = -\frac{1}{F}\frac{\left|Q_{\#2}\right|}{\left|Q_{\#1}\right|}\left(r_P - \frac{Q_{12}}{Q_{22}}\right) \tag{2.6.27}$$

Proof: See Appendix 17.

Proposition 11: The return covariance between the minimum surplus return variance correction and the return generating correction components.

The return covariance between the minimum surplus return variance correction component, ϕ_{MSVP}*, and the return generating correction component,* $w_{\eta,S}$*, is*

$$COV\left[R_P\left(\phi_{MSVP}\right)\ ,\ R_P\left(w_{\eta,S}\right)\right] = -\frac{1}{F^2}\frac{\left|Q_{\#2}\right|^2}{Q_{22}\left|Q_{\#1}\right|} \tag{2.6.28}$$

Proof: See Appendix 18.

Proposition 12: The return covariance between the return generating and the return generating correction components.

The return covariance between the return generating component, w_η *, and the return generating correction component,* $w_{\eta,S}$ *, is*

$$COV\left[R_P\left(w_\eta\right)\ ,\ R_P\left(w_{\eta,S}\right)\right]=\frac{1}{F}\frac{\left|Q_{\#2}\right|}{\left|Q_{\#1}\right|}\left(r_P-\frac{Q_{12}}{Q_{22}}\right) \qquad (2.6.29)$$

Proof: See Appendix 19.

As all necessary terms for expressing the return variance on the optimal surplus return portfolio have been derived, the return variance can be expressed in a clear and simple form as a function of the return requirement on the optimal portfolio, r_P. Furthermore, the return variance expression is the same as for the traditional model (2.2.18) except a simple term adds stemming from the presence of liabilities. Theorem 8 that follows provides the expression for the return variance on the optimal surplus return portfolio.

Theorem 8: The return variance on the optimal surplus return variance portfolio.

The optimal surplus risk-return portfolio is expressed as $w_{rp,S}=w_{MVP}+\theta\phi_{MSVP}+w_\eta+\theta w_{\eta,S}$.
The return variance associated with the optimal surplus risk-return portfolio is

$$VAR\left[R_P\left(w_{rp.S}\right)\right]=\sigma^2_{rp,S}=\frac{r_P^2Q_{22}-2r_PQ_{12}+Q_{11}+\theta^2F^{-2}\left|Q\right|}{\left|Q_{\#1}\right|} \qquad (2.6.30)$$

This result is in accordance with the results of theorems 1, 3, and 6, and propositions 2, 3, 5, 6 and 9 - 12.
Proof: See Appendix 20.

For a preferred return requirement on the optimal portfolio, r_P, equation (2.6.30) expresses the return variance on the optimal surplus return variance portfolio $w_{rp,S}$ associated with r_P. Equation (2.6.30) is basically the same as the return variance on the optimal portfolio in the absence of liabilities (2.2.18) but includes an additional term as a result of the liabilities hedge and the return correction stemming from the hedge. Recall that the optimal portfolio is composed out of four components, i.e. $w_{rp,S}=w_{MVP}+\theta\phi_{MSVP}+w_\eta+\theta w_{\eta,S}$, and the return variance (2.6.30) is lower than the sum of the return variances of individual components. The return covariance terms between ϕ_{MSVP} and w_η on the one hand and w_η and $w_{\eta,S}$ on the other hand cancel each other. The negative covariance terms between ϕ_{MSVP} and $w_{\eta,S}$ reduce the return variance of the optimal surplus return portfolio and as a result, the total return variance is lower than if the components returns had zero covariance between them in all cases.

The surplus return optimization model has been derived along with the optimal solutions in terms of asset allocations, expected returns and associated return variances. Before next steps are taken in further analysis of the model properties, further discussion of the model is necessary to complete certain interpretations of the equations that have already been derived.

From the MSVP expression (2.6.6), it can also be seen that the funding ratio of a pension fund does not only determine the capability to bear risk but should also reflect the willingness to take risk. The higher the funding ratio is, the lower the necessity for liability hedging. This is also confirmed by (2.5.4); the higher the funding ratio is, the lower needs the return covariance between asset portfolio and liabilities to be if the volatility on liabilities is fixed. This makes intuitive sense; the better the funding status of a pension fund is, the lower is the need to hedge against liabilities and the pension fund can afford to take more risk by investing in a traditional growth portfolio. As a practical consequence of the relationship of the funding ratio with liabilities hedge, the hedging policy of pension funds could easily be monitored by authorities. This is due to the fact that for optimal portfolios with a fixed θ, only the funding ratio is decisive for the liabilities hedge. From this discussion, it can be seen that the funding ratio provides an objective measure to quantify attitudes towards risk and that the funding ratio is directly related to the ability to bear risk. For a specific pension fund, the investment in the liability hedging portfolio thus depends on the funding status and the funding ratio value incorporates the dynamical properties that the asset portfolio should have as shown by (2.5.4). The lower the funding ratio, the higher should the portion invested in the liability hedge portfolio be as to establish proper dynamics between assets and liabilities.

Furthermore, the widely used traditional risk-return optimization model derived in section 2.2 is a special case of the surplus return optimization model derived in this section. This can easily be seen by setting zero importance to the liabilities ($\theta = 0$), letting the funding ratio approach infinity, setting the growth rate of the liabilities to zero or if the covariance between assets and liabilities is zero. It can also be seen from the expressions for w_{MSVP} (2.6.6) and $w_{rp,S}$ (2.6.16), that the surplus optimization set converges with the traditional mean-variance set w_{MVP} (2.2.6) and w_{rp} (2.2.9), if $\theta = 0$ or when the funding ratio approaches infinity. This makes intuitive sense; as the liabilities become very small in value compared to the assets, the liabilities play less role in the management of the fund and as the funding ratio approaches infinity, the liabilities can be ignored.

2.7 Optimal portfolios under additional assumptions on liabilities.

This section is based on Keel and Muller (1995) with small changes for adapting their approach to the solutions expressed in this thesis.

In theorem 5, the liability correction component ϕ_{MSVP} was derived, which depends linearly on the vector containing the covariance elements between the returns on assets and liabilities, Σ_{AL}. If returns on liabilities, R_L, are assumed to depend on several factors $\ell = \{1,...,M\}$, we can assume that

$$R_L = a + \sum_{m=1}^{M} b_m R_{L(m)} \tag{2.7.1}$$

where $R_{L(m)}$ is the return component resulting from return factor m. One way of applying (2.7.1), is to consider it as a multiple linear regression model by adding an error term ε. According to (2.7.1), the return on liabilities is assumed to be linearly dependent upon the m factors.

A direct consequence of (2.7.1) is that the covariance element between individual asset returns and liabilities is a linear combination of the covariance elements between individual asset return and each return factor:

$$\Sigma_{AL} = \sum_{m=1}^{M} b_m \Sigma_{AL(m)} \tag{2.7.2}$$

with

$$\Sigma_{AL(i,m)} = COV\left[R_{L(m)}, R_i\right] \quad \forall i \in I,\ m \in \ell \quad \text{and} \quad \Sigma_{AL(m)} = \left[\Sigma_{AL(1,m)},...,\Sigma_{AL(n,m)}\right]^T$$

As a result of the above, (2.7.1) makes the liability portfolio component ϕ_{MSVP} a linear combination of return covariance vectors resulting from the return factorization (2.7.1), i.e.

$$\phi_{MSVP} = \sum_{m=1}^{M} b_m \phi_{MSVP(m)} \tag{2.7.3}$$

with

$$\phi_{MSVP(m)} = \frac{1}{F}\left[\Sigma_A^{-1}\Sigma_{AL(m)} - \frac{\upsilon^T\Sigma_A^{-1}\Sigma_{AL(m)}}{Q_{22}}\Sigma_A^{-1}\upsilon\right] \tag{2.7.4}$$

From (2.7.1) – (2.7.4), it can be seen that each of the m factors $R_{L(m)}$ leads to a liability correction $\phi_{MSVP(m)}$ on the optimal surplus return portfolio and the total correction is a linear combination of $\phi_{MSVP(1)},...,\phi_{MSVP(M)}$. Hence as a consequence of (2.7.1), all optimal portfolios (2.6.16) can be written as

$$w_{rp,S} = w_{MVP} + \theta\sum_{m=1}^{M} b_m\phi_{MSVP(m)} + w_\eta + \theta w_{\eta,S} \tag{2.7.5}$$

All factors $R_{L(m)}$ can be neglected, having $\left\|\phi_{MSVP(m)}\right\|$ sufficiently small or if coefficients of linear regression are not significant w.r.t. chosen confidence level. Equation (2.7.1) allows for using any statistically significant components as factors in generating returns on liabilities, e.g. inflation, wage growth, economic growth and demographic factors. The point that can be taken out of this section is that taking liabilities into account leads to a shift of the efficient set where the shift vector depends linearly on the investor's sensitivity to different factors (inflation, economic growth, etc.). This approach allows for risk-return optimization with respect to more than one benchmark instead of optimizing only with respect to one benchmark like the model in section 2.6 assumes.

The shift due to the component wise liability hedge (2.7.3) is illustrated by figure 2.3 where the occurrence of different liability return factors leads to a shift of the efficient set in total of $\theta\phi_{MSVP}$. The figure illustrates the special case when the number of risky assets is three and the number of factors affecting liability returns are two. Just as for figures 2.1 and 2.2, the triangle on the figure represents the Euclidian 2-simplex where allocations in each of the risky assets are between 0 and 1 as the triangle represents only a small part of the surface defining feasible allocations. If optimal allocations for some assets are negative, the point on the line representing the efficient set is outside the triangle.

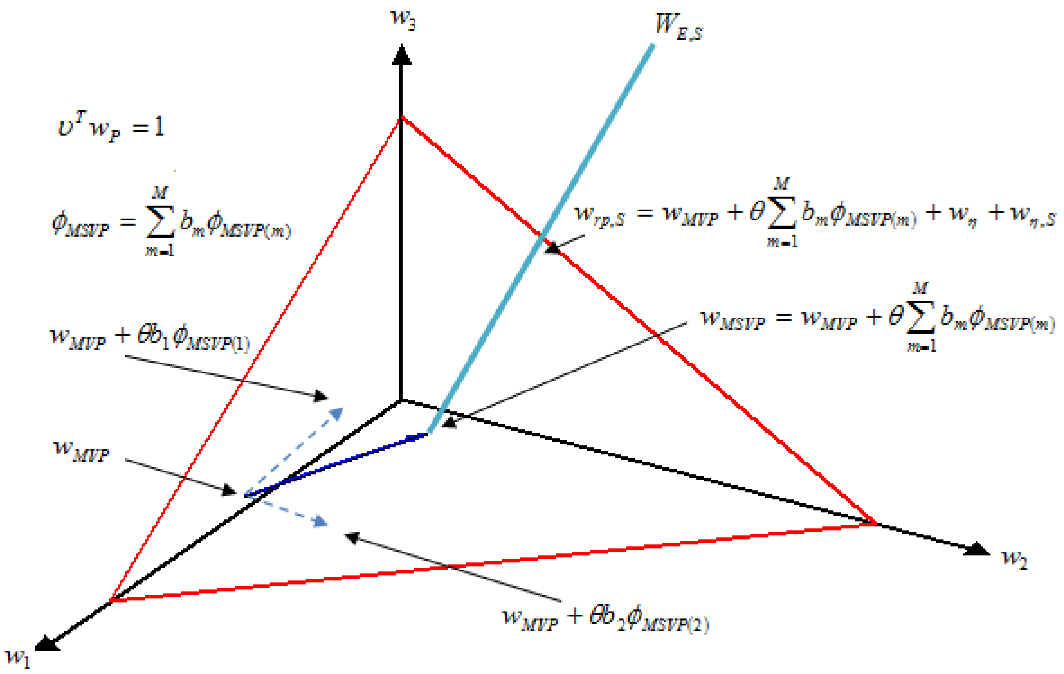


Figure 2.3: Geometrical interpretation for the special case where it is assumed that return on liabilities depends on two factors (n = 3 and M = 2.) The minimum surplus return variance portfolio (MSVP) vector is a linear combination of the traditional minimum variance portfolio (MVP) vector and the two shift vectors.

2.8 The mean-variance frontiers in absence and presence of liabilities in risk-return space.

In this section, the relationship between expected returns and return variance are analysed, commonly known as the risk-return relationship. This relationship is analysed both in the absence and presence of liabilities. The optimal sets, W in the absence and W_S in the presence of liabilities, can be rewritten as, respectively,

$$w_{rp} = w_{MVP} + \alpha_\eta w_\alpha \tag{2.8.1}$$

in accordance with (2.2.10) - (2.2.14) and

$$\begin{aligned} w_{rp,S} &= w_{MSVP} + \alpha w_\alpha \\ &= w_{MSVP} + \left(\alpha_\eta + \theta \alpha_{\eta,S}\right) w_\alpha \end{aligned} \tag{2.8.2}$$

in accordance with (2.6.17) - (2.6.18), and (2.6.19) - (2.6.22).

The two coefficients, α_η and α, are multipliers for yielding excess return on the MVP and the MSVP, respectively, along with w_α (2.2.14). Both optimal portfolios, w_{rp} and $w_{rp,S}$, consist of two separate portfolio components according to (2.8.1) and (2.8.2); w_{rp} consists of the minimum return variance portfolio, w_{MVP}, and the return generating portfolio, $\alpha_\eta w_\alpha$ and

$w_{rp,S}$ consists of the minimum surplus return variance portfolio, w_{MSVP}, and the net return generating portfolio, αw_α. Since the MVP and MSVP can be considered as unique portfolios in terms of the risk-return relationship of the optimal portfolios and that they are components in other optimal portfolios, the relationship between expected return and return variance can built on that fact. Accordingly, for the optimal set in the absence of liabilities, the result of theorems 1 – 3, propositions 1 – 3 and equations (2.2.12) – (2.2.14) and (2.8.1) show that all portfolios w_{rp} satisfy

$$E\left[R_P\left(w_{rp}\right)\right]=E\left[R_P\left(w_{MVP}\right)\right]+\alpha_\eta E\left[R_P\left(w_\alpha\right)\right] \tag{2.8.3}$$

and

$$VAR\left[R_P\left(w_{rp}\right)\right]=VAR\left[R_P\left(w_{MVP}\right)\right]+\alpha_\eta^2 VAR\left[R_P\left(w_\alpha\right)\right] \tag{2.8.4}$$

where α_η was defined according to (2.2.13) as

$$\alpha_\eta = \frac{r_P Q_{22}-Q_{12}}{\left|Q_{\#1}\right|}$$

Equation (2.8.4) is equivalent to the expression (2.2.18) for optimal portfolio return variance in absence of liabilities. For a complete proof of (2.8.4), see Appendix 21.

Hence it can be seen that there is a quadratic relationship between

$$E\left[R_P\left(w_{rp}\right)\right] \quad \text{and} \quad VAR\left[R_P\left(w_{rp}\right)\right]$$

Furthermore, for the optimal set in presence of liabilities, the results of theorems 5 – 8, propositions 4 - 12 and equations (2.6.19) – (2.6.22) and (2.8.2) show that all portfolios $w_{rp,S}$ satisfy

$$E\left[R_P\left(w_{rp,S}\right)\right]=E\left[R_P\left(w_{MSVP}\right)\right]+\alpha E\left[R_P\left(w_\alpha\right)\right] \tag{2.8.5}$$

and

$$\begin{aligned} VAR\left[R_P\left(w_{rp,S}\right)\right]=&VAR\left[R_P\left(w_{MSVP}\right)\right]+\alpha^2 VAR\left[R_P\left(w_\alpha\right)\right] \\ &+2\alpha COV\left[R_P\left(w_{MSVP}\right)\ ,\ R_P\left(w_\alpha\right)\right] \end{aligned} \tag{2.8.6}$$

where α was defined according to (2.6.22) as

$$\alpha = \frac{r_P Q_{22}-Q_{12}+\theta F^{-1}\left|Q_{\#2}\right|}{\left|Q_{\#1}\right|}$$

Equation (2.8.6) is equivalent to the much simpler expression (2.6.30) for optimal portfolio return variance in presence of liabilities. For a complete proof of (2.8.6), see Appendix 22.

Hence it can also be seen that there is a quadratic relationship between

$$E\left[R_P\left(w_{rp,S}\right)\right] \quad \text{and} \quad VAR\left[R_P\left(w_{rp,S}\right)\right]$$

The covariance term in (2.8.6) is only an extra term that survives according to section 2.6 and theorem 1; it does not influence the quadratic relationship between expected returns and the return variance of the surplus optimal set.

Figure 2.4 shows the quadratic relationship between standard deviation of returns and expected returns, commonly known as the risk-return or mean-variance frontier(s). The blue/cyan (left/right) frontiers show the optimal sets in absence/presence of liabilites. The frontier for the surplus optimal portfolios is shifted to the right on the volatility axis due to the variance shift from the minimum surplus variance correction component $\theta\phi_{MSVP}$, the return generating correction component $\theta w_{\eta,S}$ and return covariance terms between components. The blue frontier is in accordance with theorems 1 – 3 and propositions 1 – 3. For a specified return requirement of $r_{P\psi}$, the optimal portfolios contains both w_{MVP} and w_η where $r_P = r_{P\psi}$. For the cyan colored (right) surplus optimal frontier, the component-wise buildup of the optimal portfolio, $w_{rp,S}$, can clearly be seen in figure 2.4. The MSVP is formed by adding $\theta\phi_{MSVP}$ to the w_{MVP} and to construct a suplus optimal portfolio with return requirement $r_{P\psi}$, w_η has to be added which results in $w_{MVP} + \theta\phi_{MSVP} + w_\eta$. This portfolio overshoots in terms of expected return and thus $w_{\eta,S}$ is added to correct the expected return to $r_P = r_{P\psi}$, resulting in a surplus optimal portfolio $w_{rp\psi,S}$ with $E\left[R_P\left(w_{rp\psi,S}\right)\right] = E\left[R_P\left(w_{rp\psi}\right)\right] = r_{P\psi}$.

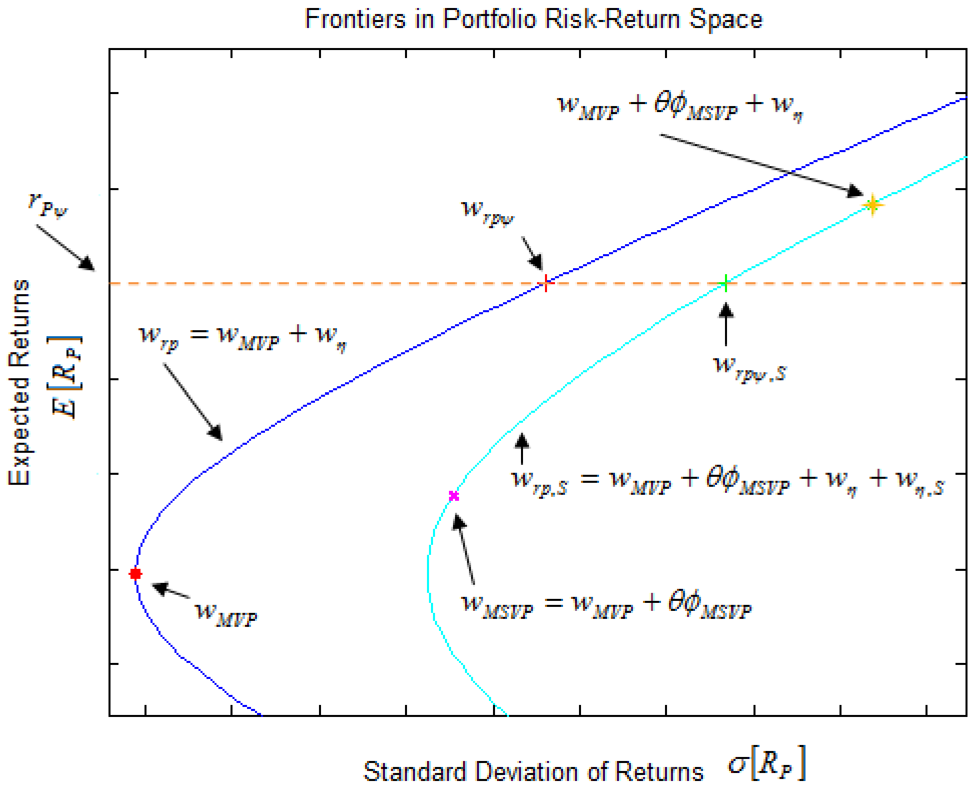


Figure 2.4: The risk-return frontiers in the absence (blue) and presence (cyan) of liabilities. The shift due to adding individual portfolio components are shown for a return requirement value of $r_{P\psi}$.

2.9 The surplus frontier(s) in surplus risk-return space.

In this section, the relationship between expected surplus returns and surplus return variance are analyzed. A portfolio in the surplus optimal set, W_S, was defined in accordance with (2.8.2) as

$$w_{rp,S} = w_{MSVP} + \alpha w_\alpha$$

with

$$\alpha = \frac{r_P Q_{22} - Q_{12} + \theta F^{-1}\left|Q_{\#2}\right|}{\left|Q_{\#1}\right|} \quad \text{and} \quad w_\alpha = \left[\Sigma_A^{-1}\mu_A - \frac{Q_{12}}{Q_{22}}\Sigma_A^{-1}\upsilon\right]$$

as given by (2.6.22) and (2.2.14), respectively. The parameter α can be considered as a return requirement parameter for yielding excess return on the MSVP along with w_α, as the only controllable parameter in α is r_P. Here, $w_{rp,S}$ consists of two separate portfolio components; the minimum surplus return variance portfolio, w_{MSVP}, and the net return generating portfolio, αw_α. The surplus risk-return frontier can therefore be constructed in a similar way as the traditional risk-return frontier, i.e. composed out of the MSVP, w_{MSVP}, and the net return generating portfolio, αw_α. Proposition 13 gives a useful result for the analysis of the surplus risk-return structure of the optimal set, W_S.

Proposition 13: The zero surplus return covariance between the minimum surplus return variance and the return generating components.

All portfolios in the frontier set, W_S*, can be written as* $w_{rp,S} = w_{MSVP} + \alpha w_\alpha$*. The surplus return covariance between the minimum variance component,* w_{MSVP}*, and the return generating component,* w_α*, is*

$$COV\left[R_P\left(w_\alpha\right)\ ,\ R_P\left(w_{MSVP}\right) - \frac{\theta}{F}R_L\right] = 0 \qquad (2.9.1)$$

Proof: See Appendix 23.

From proposition 13, the results of theorems 5 – 8, propositions 4 - 12 and equations (2.6.19) – (2.6.22) and (2.8.2) show that all portfolios $w_{rp,S}$ satisfy

$$E\left[R_P\left(w_{rp,S}\right) - \frac{\theta}{F}R_L\right] = E\left[R_P\left(w_{MSVP}\right) - \frac{\theta}{F}R_L\right] + \alpha E\left[R_P\left(w_\alpha\right)\right] \qquad (2.9.2)$$

and

$$VAR\left[R_P\left(w_{rp,S}\right) - \frac{\theta}{F}R_L\right] = VAR\left[R_P\left(w_{MSVP}\right) - \frac{\theta}{F}R_L\right] + \alpha^2 VAR\left[R_P\left(w_\alpha\right)\right] \qquad (2.9.3)$$

where α was defined according to (2.6.22) as

$$\alpha = \frac{r_P Q_{22} - Q_{12} + \theta F^{-1}\left|Q_{\#2}\right|}{\left|Q_{\#1}\right|}$$

For a complete proof of (2.9.3), see Appendix 24.

Hence it can be seen that there is a quadratic relationship between

$$E\left[R_P\left(w_{rp,S}\right) - \frac{\theta}{F}R_L\right] \quad \text{and} \quad VAR\left[R_P\left(w_{rp,S}\right) - \frac{\theta}{F}R_L\right]$$

Just as the optimization problem in absence of liabilities in section 2.2 can be projected into risk-return space, the surplus optimization problem can be projected into surplus risk-return space. In both cases, the relationship between expected returns and variance are observed as quadratic functions. Figure 2.5 shows the relationship between expected surplus return and surplus return variance, i.e. the surplus risk-return frontier in surplus risk-return space.

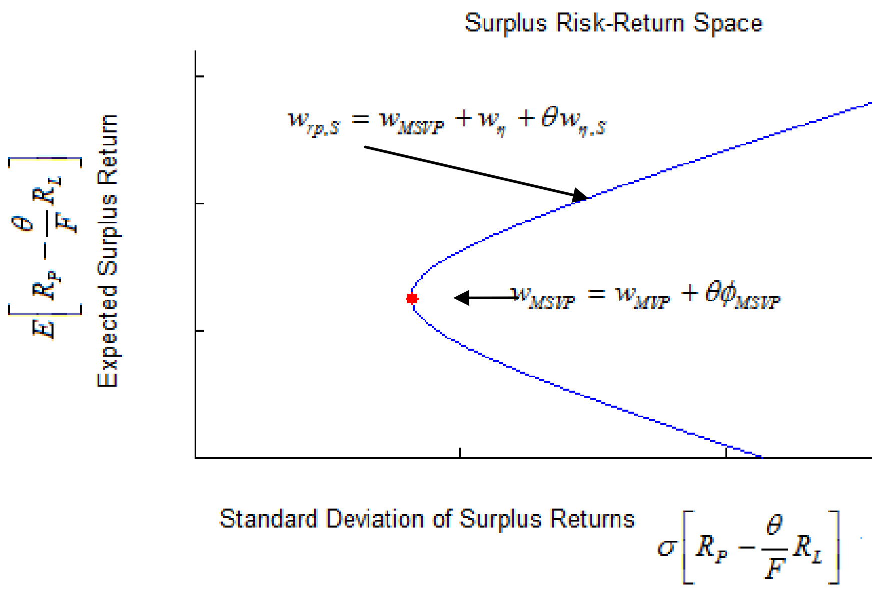


Figure 2.5: The surplus risk-return frontier. The MSVP has the minimum surplus variance of all optimal surplus return portfolios. Recall from figure 2.4 that the MSVP does not have the minimum return variance in the set of optimal surplus return portfolios in risk-return space.

2.10 Descriptive properties of the matrix *Q*.

As the portfolios in the optimal sets have been derived and their respective expected returns and variances of returns, it‘s now time to inspect the certain characteristics of the optimal sets in both risk-return and surplus risk-return spaces. In section 2.1, the matrix Q was introduced for convenience in expressing the optimal sets, the expected returns of the optimal sets and their respective return variances. As the watchful reader may have noted, the matrix Q and it‘s sub-matrices also shed light on more properties that have not been analyzed directly yet. These properties have to do with the existence of the optimal sets, expected returns, return variance shifts and the covariance between portfolio and liabilities returns.

In section 2.1, the matrix Q, its sub-matrices and determinants were introduced in accordance with equations (2.1.1) – (2.1.12). The optimal portfolios derived in theorems 1, 2, 5 and 7, are given by (2.2.6), (2.2.9), (2.6.6) and (2.6.16) respectively as

$$w_{MVP} = \frac{\Sigma_A^{-1}\upsilon}{Q_{22}},$$

$$w_{rp} = \frac{\Sigma_A^{-1}\upsilon}{Q_{22}} + \frac{r_P Q_{22} - Q_{12}}{|Q_{\#1}|}\left[\Sigma_A^{-1}\mu_A - \frac{Q_{12}}{Q_{22}}\Sigma_A^{-1}\upsilon\right],$$

$$w_{MSVP} = \frac{\Sigma_A^{-1}\upsilon}{Q_{22}} + \frac{\theta}{F}\left[\Sigma_A^{-1}\Sigma_{AL} - \frac{Q_{23}}{Q_{22}}\Sigma_A^{-1}\upsilon\right]$$

and

$$w_{rp,S} = \frac{\Sigma_A^{-1}\upsilon}{Q_{22}} + \frac{\theta}{F}\left[\Sigma_A^{-1}\Sigma_{AL} - \frac{Q_{23}}{Q_{22}}\Sigma_A^{-1}\upsilon\right] + \frac{r_P Q_{22} - Q_{12} + \theta F^{-1}\left|Q_{\#2}\right|}{\left|Q_{\#1}\right|}\left[\Sigma_A^{-1}\mu_A - \frac{Q_{12}}{Q_{22}}\Sigma_A^{-1}\upsilon\right]$$

Since both w_{rp} and $w_{rp,S}$ contain $\left|Q_{\#1}\right|$ in the last term, it can easily be seen that

$$\lim_{|Q\#1|\to 0} w_{rp} = w_{MVP} \quad \text{and} \quad \lim_{|Q\#1|\to 0} w_{rp,S} = w_{MSVP}$$

as the last terms in both w_{rp} and $w_{rp,S}$ are not defined when $\left|Q_{\#1}\right| = 0$. Therefore, the solutions to both optimization problems from sections 2.2 and 2.6 are limited if $\left|Q_{\#1}\right| = 0$ and accordingly, the optimal sets W and W_S consist out of only $W = \varnothing / \left\{w_{MVP}\right\}$ and $W_S = \varnothing / \left\{w_{MSVP}\right\}$ when $\left|Q_{\#1}\right| = 0$ as other solutions are undefined. This is also confirmed by the results of theorems 3 and 8, since the optimal portfolio return variances given by (2.2.18) and (2.6.30) as

$$VAR\left[R_P\left(w_{rp}\right)\right] = \frac{r_P^2 Q_{22} - 2r_P Q_{12} + Q_{11}}{\left|Q_{\#1}\right|}$$

and

$$VAR\left[R_P\left(w_{rp,S}\right)\right] = \frac{r_P^2 Q_{22} - 2r_P Q_{12} + Q_{11} + \theta^2 F^{-2}\left|Q\right|}{\left|Q_{\#1}\right|}$$

are not defined when $\left|Q_{\#1}\right| = 0$. The assumptions given in section 2.1 provide the conditions for $\left|Q_{\#1}\right| \neq 0$ as is generally assumed for the mean-variance framework. Recalling the expressions for the return variance on the MVP and MSVP, given by (2.2.8) and (2.6.15) as

$$VAR\left[R_P\left(w_{MVP}\right)\right] = \frac{1}{Q_{22}} \quad \text{and} \quad VAR\left[R_P\left(w_{MSVP}\right)\right] = \frac{1}{Q_{22}}\left(1 + \frac{\theta^2}{F^2}\left|Q_{\#4}\right|\right)$$

is can be seen that $\left|Q_{\#1}\right|$ is not included in these expressions. As a result from the above, it can be stated that the optimal sets W and W_S consist out of only $W = \varnothing / \left\{w_{MVP}\right\}$ and $W_S = \varnothing / \left\{w_{MSVP}\right\}$ when $\left|Q_{\#1}\right| = 0$ as other solutions are undefined. Also, it can be seen that the return variance for the optimal portfolios w_{rp} and $w_{rp,S}$ are proportional to $\left|Q_{\#1}\right|^{-1}$, i.e.

$$VAR\left[R_P\left(w_P\right)\right] \propto \left|Q_{\#1}\right|^{-1} \quad \text{and also that} \quad VAR\left[R_P\left(w_{MSVP}\right)\right] \propto \left|Q_{\#4}\right|$$

The effect due to that the return variances is inversely proportional to $\left|Q_{\#1}\right|$ is thus directly observable in both risk-return and surplus risk-return spaces. Keeping all other terms constant, as $\left|Q_{\#1}\right|$ decreases, the return variance of the portfolio increases for certain return requirement r_P. As $\left|Q_{\#1}\right|$ approaches zero, the curvature vanishes and the variance increases rapidly with infinitesimal increase in return until when $\left|Q_{\#1}\right| = 0$, the frontier cannot be defined as infinitesimal increase in return causes infinite variance. As this happens, the optimal sets consist only out of the MVP or MSVP, depending upon whether liabilities are included on not.

Furthermore, comparing (2.2.8) and (2.6.15), it can easily be seen that

$$VAR\left[R_P\left(w_{MSVP}\right)\right] = VAR\left[R_P\left(w_{MVP}\right)\right] \quad \text{if} \quad \left|Q_{\#4}\right| = 0$$

which happens if $\Sigma_{AL} = \mathbf{0}$. The return variance on the MSVP and MVP can also be equal if $F \to \infty$ or if the importance given to the liabilities via θ is zero. In all cases, the surplus optimization problem converges to the traditional problem and $W_S = W$. Similarly, from the return variance expressions for w_{rp} and $w_{rp,S}$, it can be seen that

$$VAR\left[R_P\left(w_{rp,S}\right)\right] = VAR\left[R_P\left(w_{rp}\right)\right] \quad \text{if} \quad \left|Q\right| = 0$$

which happens if $\Sigma_{AL} = \mathbf{0}$. The return variance on the $w_{rp,S}$ and w_{rp} can also be equal if $F \to \infty$ or the importance given to the liabilities via θ is zero. In both cases, the surplus optimization problem converges to the traditional problem and $W_S = W$.

In section 2.8, figure 2.4 allowed for a comparison of the traditional and surplus frontiers in risk-return space. On the figure, the expected return on the MSVP was not the same as for the MVP. The reason for that can be seen from equations (2.2.7) and (2.6.12), respectively

$$E\left[R_P\left(w_{MVP}\right)\right] = \frac{Q_{12}}{Q_{22}} \quad \text{and} \quad E\left[R_P\left(w_{MSVP}\right)\right] = \frac{1}{Q_{22}}\left(Q_{12} - \frac{\theta}{F}\left|Q_{\#2}\right|\right)$$

From the two equations above for expected returns, it can clearly be seen that if $\left|Q_{\#2}\right| = 0$ then $E\left[R_P\left(w_{MSVP}\right)\right] = E\left[R_P\left(w_{MVP}\right)\right]$ and also, if $\left|Q_{\#2}\right| > 0$, then $E\left[R_P\left(w_{MSVP}\right)\right] < E\left[R_P\left(w_{MVP}\right)\right]$ and vice versa if $\left|Q_{\#2}\right| < 0$. Accordingly, the sign of $\left|Q_{\#2}\right|$ thus tells whether the expected return on the MSVP is higher or lower than the expected return on the MVP.

The effects of the sub-determinants of Q on the return covariance between any portfolio and liabilities is not as directly observable. By expressing the return covariance between the MSVP and liabilities as

$$COV\left[R_P\left(w_{MSVP}\right)\ ,\ R_L\right]=\Sigma_{AL}^T w_{MSVP}=\frac{Q_{23}}{Q_{22}}+\frac{\theta}{F}\frac{|Q_{\#4}|}{Q_{22}}$$

It becomes clear that the MSVP return covariance value depends on the values for $|Q_{\#4}|,\theta$ and F. For the surplus optimal portfolio, the expression for the return covariance is

$$COV\left[R_P\left(w_{rp,S}\right)\ ,\ R_L\right]=\Sigma_{AL}^T w_{rp,S}$$

$$=\frac{Q_{23}}{Q_{22}}+\frac{\theta}{F}\frac{|Q_{\#4}|}{Q_{22}}-\frac{\left(r_P Q_{22}-Q_{12}+\theta F^{-1}|Q_{\#2}|\right)|Q_{\#2}|}{Q_{22}|Q_{\#1}|}$$

Now, it can be easily seen that the return covariance between $w_{rp,S}$ and liabilities is proportional to $|Q_{\#4}|$, $-r_P|Q_{\#2}|$, and $-|Q_{\#2}|^2$, but inversely proportional to $|Q_{\#1}|$ and F. This directly tells that by increasing r_P, the return covariance increases if the sign of $|Q_{\#2}|$ is negative and decreases if the sign of $|Q_{\#2}|$ is positive as the following derivative confirms;

$$\frac{\partial}{\partial r_P}COV\left[R_P\left(w_{rp,S}\right)\ ,\ R_L\right]=-\frac{|Q_{\#2}|}{|Q_{\#1}|}$$

For convenience, the results from this section are summarized in table 2.1. This table is useful for explaining the properties of the optimal solutions that have been covered in this section and certain structural properties that are observed for the risk-return frontiers. From the expressions that have been derived in sections 2.2 – 2.10, the convenience of the Q matrix notation should be clear for the purposes of expressing the optimal solutions for portfolios, their respective expected returns and return variances. Also, this notation proposes a method in explaining certain differences in the characteristics of the optimal sets, W and W_S.

Table 2.1: Explanatory factors for certain properties of the optimal portfolios.

Factors	**Proportionality**
$E\left[R_P\left(w_{MSVP}\right)\right]-E\left[R_P\left(w_{MVP}\right)\right]\propto$	$-\lvert Q_{\#2}\rvert$
$VAR\left[R_P\left(w_{MSVP}\right)\right]\propto$	$\lvert Q_{\#4}\rvert$, F^{-2}
$VAR\left[R_P\left(w_{rp,S}\right)\right]\propto$	$\lvert Q\rvert$, $\lvert Q_{\#1}\rvert^{-1}$, F^{-2}
$COV\left[R_P\left(w_{MSVP}\right), R_L\right]\propto$	$\lvert Q_{\#4}\rvert$, F^{-1}
$COV\left[R_P\left(w_{rp,S}\right), R_L\right]\propto$	$\lvert Q_{\#1}\rvert^{-1}$, $-\lvert Q_{\#2}\rvert^{2}$, $\lvert Q_{\#4}\rvert$, F^{-1}, $-r_P\lvert Q_{\#2}\rvert$

Table 2.1: The value of the sub-determinants of Q explains certain properties of the optimal solutions and the structure of the frontiers in risk-return space.

2.11 Minimization of surplus return variance

For the general surplus optimizer, the funding ratio at each point in time is assumed to be fixed as it represents the ratio of the values of assets to liabilities for the investor. Accordingly, initial funding ratio is not a variable although with time, the funding ratio is assumed to vary as the value of assets and liabilities change. Nevertheless, it is interesting to consider the effects of different funding ratios on surplus return variance from a mathematical point of view. For any given value of F, the minimum surplus return variance portfolio (MSVP) is the optimal portfolio for which the surplus return variance is at minimum as the name implies. Given any reasonable data for surplus optimization and assuming that F can be varied within the feasible range of funding ratios, $F \in (0,\infty]$, one might consider whether a unique funding ratio can be found which yields an absolute minimum surplus return variance for all feasible minimum surplus return variance portfolios. In order to find an answer to this consideration, the surplus return variance of the MSVP has to be analyzed and a condition for a minimum has to be found.

The MSVP was expressed according to (2.6.6)

$$w_{MSVP} = \frac{\Sigma_A^{-1}\upsilon}{Q_{22}} + \frac{\theta}{F}\left[\Sigma_A^{-1}\Sigma_{AL} - \frac{Q_{23}}{Q_{22}}\Sigma_A^{-1}\upsilon\right]$$

and the return variance on the MSVP was given by (2.6.15)

$$\sigma_{MSVP}^2 = \frac{1}{Q_{22}}\left(1 + \frac{\theta^2}{F^2}\left|Q_{\#4}\right|\right)$$

The general expression for normalized surplus return variance on the MSVP can be written as

$$\sigma_{S,n}^2\left(w_{MSVP}\right) = \sigma_{MSVP}^2 + \frac{\theta^2}{F^2}\sigma_L^2 - \frac{2\theta}{F}\Sigma_{AL}^T w_{MSVP} \tag{2.11.1}$$

where the return covariance between the MSVP and liabilities is

$$\Sigma_{AL}^T w_{MSVP} = \frac{Q_{23}}{Q_{22}} + \frac{\theta}{F}\frac{\left|Q_{\#4}\right|}{Q_{22}}$$

As a result, (2.11.1) can be rewritten as

$$\sigma_{S,n}^2\left(w_{MSVP}\right) = \frac{1}{Q_{22}} + \frac{\theta^2}{F^2}\left(\sigma_L^2 - \frac{\left|Q_{\#4}\right|}{Q_{22}}\right) - \frac{2\theta}{F}\frac{Q_{23}}{Q_{22}} \tag{2.11.2}$$

Let's consider equation (2.11.2) for all feasible values of F, i.e. $F \in [0,\infty]$ and reasonable values for the elements of the matrix Q and σ_L^2. If $F = \infty$, then $\sigma_{S,n}^2\left(w_{MSVP}\right) = \sigma^2\left(w_{MSVP}\right)$ as the MSVP equals the MVP. As the value for F is decreased from infinity, the third term in (2.11.2) grows faster than the middle term due to different power operators for F in the equation. Depending on the data used in each case, this is likely to result in decreasing surplus return variance on the MSVP as F decreases from infinity and for some value of F, the change in (2.11.2) becomes zero. High return covariance values between portfolio and liabilities results from asset data with favourable hedging characteristics and can have considerable effects in decreasing surplus return variance for low values of F. Accordingly, surplus return variance for MSVP's has a potential to reach a minimum for certain value of F. This minimum is found via the partial derivative of (2.11.2) with respect to F and a necessary second order condition for a minimum. Proposition 14 gives the expression for the funding ratio that yields the absolute minimum surplus variance and the expression for the absolute minimum surplus variance.

Proposition 14: The funding ratio for absolute minimum surplus return variance and the absolute minimum surplus return variance.

Given any reasonable data used for surplus optimization, the absolute minimum surplus return variance for this data can be found. The funding ratio that minimizes the surplus return variance is

$$F_{MSV} = \frac{\theta\left(Q_{22}\sigma_L^2 - \left|Q_{\#4}\right|\right)}{Q_{23}} \tag{2.11.3}$$

This funding ratio exists if the condition

$$\frac{3\theta}{F}\left(Q_{22}\sigma_L^2 - \left|Q_{\#4}\right|\right) - 2Q_{23} > 0 \tag{2.11.4}$$

is satisfied for $F_{\min} \le F \le F_{\max}$.

The absolute minimum surplus variance that can be achieved for any feasible surplus optimal portfolio is

$$\begin{aligned} VAR\left[R_P\left(w_{MSV}\right) - \frac{\theta}{F_{MSV}} R_L\right] &= \sigma_{S,\min}^2 \\ &= \sigma_{S,MSVP}^2\Big|_{F=F_{MSV}} \\ &= \frac{1}{Q_{22}}\left(1 - \frac{Q_{23}^2}{Q_{22}\sigma_L^2 - \left|Q_{\#4}\right|}\right) \end{aligned} \tag{2.11.5}$$

Proof: See Appendix 25.

The funding ratio that minimizes the surplus return variance on the MSVP is given by F_{MSV} (2.11.3), given that the data satisfy (2.11.4) for any $F_{\min} \le F \le F_{\max}$. Where (2.11.4) is not satisfied, the surplus return variance is an increasing function with decreasing F and a minimum does not exist.

As a result of proposition 14, the MSVP that minimizes the surplus return variance can be found by inserting (2.11.3) into the expression for MSVP (2.6.6), i.e.

$$w_{MSV} = \frac{\Sigma_A^{-1}\upsilon}{Q_{22}} + \frac{Q_{23}}{Q_{22}\sigma_L^2 - |Q_{\#4}|}\left[\Sigma_A^{-1}\Sigma_{AL} - \frac{Q_{23}}{Q_{22}}\Sigma_A^{-1}\upsilon\right] \tag{2.11.6}$$

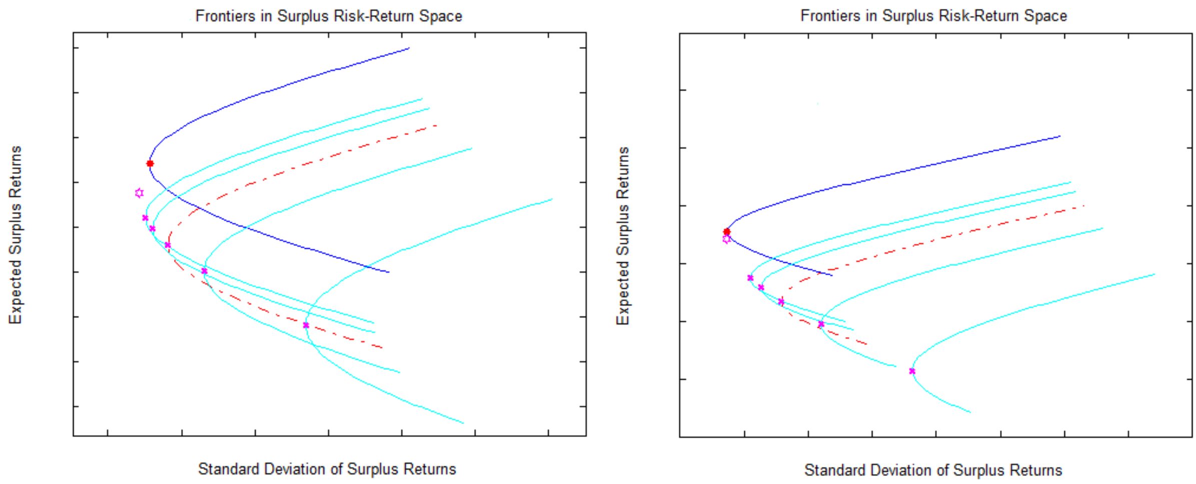
The expected return and return variance on this portfolio are found by inserting F_{MSV} into (2.6.12) and (2.6.15), respectively;

$$E\left[R_P\left(w_{MSV}\right)\right] = \frac{1}{Q_{22}}\left(Q_{12} - \frac{Q_{23}|Q_{\#2}|}{Q_{22}\sigma_L^2 - |Q_{\#4}|}\right) \tag{2.11.7}$$

and

$$\begin{aligned} VAR\left[R_P\left(w_{MSV}\right)\right] &= \sigma_{MSV}^2 \\ &= \frac{1}{Q_{22}}\left(1 + \frac{Q_{23}^2|Q_{\#4}|}{\left(Q_{22}\sigma_L^2 - |Q_{\#4}|\right)^2}\right) \end{aligned} \tag{2.11.8}$$

Figures 2.6 and 2.7 illustrate two examples of frontiers in surplus risk-return space and the absolute minimum surplus variance portfolio (MSV). In figure 2.6, the MSV (magenta coloured hexagram) is relatively far from the MVP (red dot on the blue/uppermost frontier) in comparison with the MSV in figure 2.7. This can be explained by the properties of the data; as the data provide more hedging characteristics against liabilities, the difference in surplus return variance between MVP and MSV tends to be greater as a consequence of the negative covariance term in the expression for surplus return variance (2.11.2). In figure 2.7, the MSV is very close to the MVP and therefore, F_{MSV} is relatively higher in figure 2.7 than in figure 2.6.



Figures 2.6 and 2.7: Surplus frontiers in surplus risk-return space for five different values of F where the blue/uppermost frontier has $F = \infty$ and the red dash-dotted frontier has $F = 1$. In both figures, the absolute minimum surplus return variance point (MSV) is shown by the magenta hexagram. In figure 2.6, the difference in expected surplus returns between the MSV and the MVP is greater than in figure 2.7. Apparently, F_{MSV} is high in figure 2.7 as the MSV is close to the MVP where the F is considered to be infinity.

2.12 The market portfolio in presence of liabilities

In section 2.3, it was assumed that a risk-free asset was available for investing, yielding the risk-free rate, r_f. Theorem 4 gave the allocation vector for the traditional risky assets market portfolio, which does not incorporate any consideration of liabilities at all since it is an optimal portfolio in the absence of liabilities. The market portfolio is extensively used as an optimal investment portfolio in asset allocation theory and is often referred to as the CAPM tangency portfolio since it is the optimal portfolio of risky assets where the capital market line (CML) is tangent to the set of optimal portfolios in risk-return space. At the CML, the investor invests in the risky asset market portfolio given by (2.3.3); w_0 in the risk-free asset and $(1 - w_0)w_{MKT}$ in the risky asset market portfolio. Accordingly, the asset allocation vector becomes

$$\tilde{w}_P = \left[w_0, w_P\right]^T, \ \tilde{w}_P \in \mathbb{R}^{n+1}$$

where w_P denotes the market portfolio of risky assets as before, $\tilde{\upsilon}^T \tilde{w}_P = 1$ where $\tilde{\upsilon} = \left[1_i\right]_{i=1,\dots,n+1}$

In the presence of liabilities, an investment in the risky asset market portfolio requires a liability hedge as to take account of liabilities in the asset allocation process. Just as for an optimal portfolio in presence of liabilities, a proper liability hedge with respect to the funding status and importance has to be established. The market portfolio in presence of liabilities is given in theorem 9 that follows.

Theorem 9: The market portfolio in presence of liabilities.

The risky asset market portfolio, given in theorem 4, is expressed as (2.3.3)

$$w_{MKT} = \frac{\Sigma_A^{-1}\left(\mu_A - r_f\upsilon\right)}{\upsilon^T\Sigma_A^{-1}\left(\mu_A - r_f\upsilon\right)}$$

The allocation vector for the risky assets market portfolio in presence of liabilities is

$$w_{MKT,S} = \frac{\Sigma_A^{-1}\left(\mu_A - r_f\upsilon\right)}{\upsilon^T\Sigma_A^{-1}\left(\mu_A - r_f\upsilon\right)} + \frac{\theta}{F}\left[\Sigma_A^{-1}\Sigma_{AL} - \frac{\upsilon^T\Sigma_A^{-1}\Sigma_{AL}}{\upsilon^T\Sigma_A^{-1}\left(\mu_A - r_f\upsilon\right)}\Sigma_A^{-1}\left(\mu_A - r_f\upsilon\right)\right] \tag{2.12.1}$$

The portfolio (2.12.1) *is composed out of two portfolios;*

$$w_{MKT,S} = w_{MKT} + \theta\phi_{MKT} \tag{2.12.2}$$

where w_{MKT} *is in accordance with theorem 4, the market portfolio correction component is*

$$\phi_{MKT} = \frac{1}{F}\left[\Sigma_A^{-1}\Sigma_{AL} - \frac{\upsilon^T\Sigma_A^{-1}\Sigma_{AL}}{\upsilon^T\Sigma_A^{-1}\left(\mu_A - r_f\upsilon\right)}\Sigma_A^{-1}\left(\mu_A - r_f\upsilon\right)\right] \tag{2.12.3}$$

and $\upsilon^T\phi_{MKT} = 0$.

Proof: See Appendix 26.

Equation (2.12.1) includes the traditional market portfolio (2.3.3) with a liability hedge in proportion to the funding status. Just as for the minimum surplus variance hedging component (2.6.8), the market portfolio liability hedging component is linear in Σ_{AL}, where the asset portfolio is rebalanced in accordance with the funding status. Similar to the minimum surplus variance portfolio, the market portfolio liability hedging component leads to a shift of the market portfolio, just as the minimum surplus variance correction component did for the whole optimal set. Here, ϕ_{MKT} has great similarities with minimum surplus variance correction, ϕ_{MSVP} although ϕ_{MKT} does not ensure minimum surplus variance correction directly. Instead, it provides liability correction via the maximum covariance component $\Sigma_A^{-1}\Sigma_{AL}$ and does therefore indirectly provide normalized surplus correction through θ/F in accordance with section 2.6.

In the presence of liabilities, $w_{MKT,S}$ is not a CAPM tangency portfolio as the w_{MKT} is in absence of liabilities, neither in risk-return nor surplus risk-return spaces. In a number of articles on asset allocation in presence of liabilities, $w_{MKT,S}$ appears in various forms derived from utility functions. Based on the work of Merton (1973), Rudolf & Ziemba (2004) presented an intertemporal portfolio selection model for pension funds that maximizes the intertemporal expected utility of surplus returns, using continuous-time model with HARA (Hyperbolic absolute risk aversion) utility. Their solution involves $w_{MKT,S}$ directly for log utility but indirectly as the investor‘s risk attitude changes. Martellini (2006) introduced a continuous-time intertemporal asset-liability model in the absence and presence of a constrained funding ratio where the optimal strategy also involves $w_{MKT,S}$ indirectly. With log utility, the optimal strategy is equivalent to $w_{MKT,S}$. Martellini and Milhau (2009) consider continuous-time dynamic asset allocation model for an investor facing liability commitments subject to inflation and interest rate risks. The optimal strategy involves $w_{MKT,S}$ indirectly in a slightly different form.

Interestingly, for a certain funding ratio, $w_{MKT,S}$ and w_{MSVP} converge to the unit normalized covariance portfolio from section 2.6, w_{COV} (2.6.10), i.e.

$$w_{MKT,S} = w_{MSVP} = w_{COV}$$

This happens for a unique funding ratio, F_{COV} which is

$$F_{COV} = \theta Q_{23} = \theta \upsilon^T \Sigma_A^{-1} \Sigma_{AL} \tag{2.12.4}$$

This can be easily seen by inserting (2.12.4) into the expression for the minimum surplus variance portfolio (2.6.6) and the expression for the market portfolio in presence of liabilities (2.12.1); in both cases, this results in the expression for w_{COV} (2.6.10). The expected return and return variance for this point of convergence where all three portfolios are equal, are

$$E\left[R_{P,S}\left(w_{COV}\right)\right] = \mu_A^T w_{COV} = \frac{Q_{13}}{Q_{23}} \tag{2.12.5}$$

and

$$VAR\left[R_{P,S}\left(w_{COV}\right)\right] = w_{COV}^T \Sigma_A w_{COV} = \frac{Q_{33}}{Q_{23}^2} \tag{2.12.6}$$

Figure 2.8 illustrates the traditional risk-return frontier and several surplus frontiers in risk-return space. The expected return on the market portfolio including the liability hedge decreases as F decreases and $w_{MKT,S}$ becomes inefficient in terms of surplus returns if

$$E\left[R_{P,S}\left(w_{MKT,S}\right) - \frac{\theta}{F} R_L\right] < E\left[R_{P,S}\left(w_{MSVP}\right) - \frac{\theta}{F} R_L\right] \quad \text{which happens for} \quad F < F_{COV} .$$

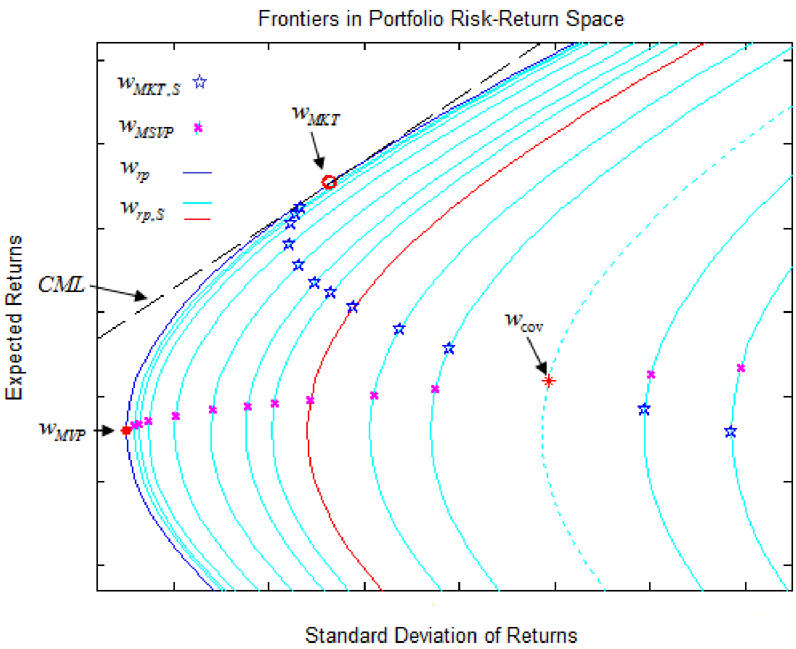


Figure 2.8: The traditional mean-variance frontier (blue/leftmost) with the classic market portfolio (red o) and surplus frontiers in risk-return space. The market portfolio in presence of liabilities (blue pentagrams) drifts away from the classic market portfolio as F decreases. The red asterisk illustrates the point where $w_{MSVP} = w_{MKT,S} = w_{COV}$. Similar pattern is observed in surplus risk return space.

2.13 Probability of assets covering liabilities.

In this section, convenient probability measure will be derived as this measure is related to the other material in this thesis. Current funding ratio along with present asset allocation can give probabilistic information on the on the ability of an investor to cover its liabilities.

Consider the simple question on how likely a pension fund is to cover its liabilities at a certain time point in future, given the necessary information and assumptions in section 2.1. More precisely, what is the probability that total assets value will exceed the value of the liabilities at certain time point in the future w.r.t. the importance parameter θ, i.e.

$$P\left(A(T) \geq \theta L(T) \;\middle|\; \mu_A(t), \Sigma_A(t), \mu_L(t), \sigma_L(t), \Sigma_{AL}(t), F(t)\right)$$

In section 2.4, surplus was defined as $S(T) = A(T) - \theta L(T)$, so accordingly;

$$P\left(A(T) \geq \theta L(T)\right) = P\left(A(T) - \theta L(T) \geq 0\right) = P\left(S(T) \geq 0\right) \tag{2.13.1}$$

As the funding ratio is defined as $F(T) = A(T)/L(T)$, the probability measure can be written as

$$P\left(A(T) \geq \theta L(T)\right) = P\left(A(T)/L(T) \geq \theta\right) = P\left(F(T) \geq \theta\right) \tag{2.13.2}$$

As a result of (2.13.1) and (2.13.2), two completely equivalent expressions for the probability of assets exceeding liabilities at time T can be written. The expressions (2.13.1) and (2.13.2) can be rewritten as

$$\begin{aligned} P\left(S(T) \geq 0\right) &= P\left(A(T) \geq \theta L(T)\right) \\ &= P\left(\ln A(T) \geq \ln \theta L(T)\right) \\ &= P\left(\ln A(T) - \ln \theta L(T) \geq 0\right) \\ &= P\left(\ln \frac{A(T)}{\theta L(T)} \geq 0\right) \\ &= P\left(\ln \frac{F(T)}{\theta} \geq 0\right) \end{aligned} \tag{2.13.3}$$

From (2.13.3), it can be seen that the surplus and the natural logarithm of the funding ratio are equivalent expressions when $\theta > 0$. In this thesis, it has been assumed that logarithmic returns are normally distributed which directly allows for the application of geometric Brownian motion processes for both asset portfolio and liabilities as, respectively

$$\frac{dP_{t,w}}{P_{t,w}} = w_P^T \mu_A dt + w_P^T \sigma_A dW_t^i \tag{2.13.4}$$

and

$$\frac{dL_t}{L_t} = \mu_L dt + \sigma_L dW_t^L + \sigma_{L,\varsigma} dW_t^{\varsigma} \tag{2.13.5}$$

with $\mu_A = [\mu_{A1},...,\mu_{An}]^T$, $\sigma_A = [\sigma_{A1},...,\sigma_{An}]^T$ and $\sigma_{L,\varsigma}$ is specific liability risk that cannot be related to market variables, i.e. $\rho_{Ai,\varsigma} = 0$.

The distribution of the logarithmic asset portfolio and liabilities prices (P_w and L) are, respectively,

$$\ln P_w(T) \sim N\left(\ln P_w(t) + \left(\mu_A^T w_P - \frac{1}{2}\sigma_P^2 \right)(T-t)\ ,\ \sigma_P^2 (T-t) \right) \tag{2.13.6}$$

and

$$\ln L(T) \sim N\left(\ln L(t) + \left(\mu_L - \frac{1}{2}\sigma_L^2 \right)(T-t)\ ,\ \sigma_L^2 (T-t) \right) \tag{2.13.7}$$

From (2.13.3), it can be seen that $\ln P_w(T) - \ln \theta L(T) = \ln F_w(T)/\theta$ and that the surplus and the logarithm of the funding ratio are equivalent processes with expected value of

$$E\left[\ln F_w(T)/\theta\right] = \ln F_w(t)/\theta + \left(\mu_P^T w_P - \theta\mu_L - \frac{1}{2}\left(\sigma_P^2 - \theta^2\sigma_L^2\right) \right)(T-t) \tag{2.13.8}$$

and variance

$$\begin{aligned} \sigma_{\ln F(T)}^2 = \sigma_{S(T)}^2 &= \left(\sigma_P^2 + \theta^2\sigma_L^2 - 2\theta\sigma_P\sigma_L\rho_{P,L}\right)(T-t) \\ &= \left(w_P^T \Sigma_A w_P + \theta^2\sigma_L^2 - 2\theta\Sigma_{AL}^T w_P\right)(T-t) \end{aligned} \tag{2.13.9}$$

as $dW_t^P dW_t^L = \rho_{P,L}$, $dW_t^L dW_t^{\varsigma} = 0$ and $dW_t^P dW_t^{\varsigma} = 0$.

Accordingly, the distribution of the logarithmic funding ratio can be expressed as

$$\ln F_w(T) \sim N\left(E\left[\ln F_w(T)/\theta\right],\ \sigma_{\ln F(T)}^2 \right) \tag{2.13.10}$$

Therefore, the probability measure on that total assets value will exceed the value of the liabilities or equivalently, gaining positive surplus at a certain time point at the end of a time horizon $(T-t)$, can be expressed as

$$
\begin{aligned}
P\left(P_w(T) \geq \theta L(T)\right) &= P\left(S(T) \geq 0\right) \\
&= P\left(\ln F_w(T)/\theta \geq 0\right) \\
&= \frac{1}{\sqrt{2\pi}} \int_{\left(\frac{E\left[\ln F_w(t)/\theta\right]}{\sigma_S}\right)}^{\infty} \exp\left(-\frac{u^2}{2}\right) du
\end{aligned}
$$

or in more convenient way as

$$
\begin{aligned}
P\left(P_w(T) \geq \theta L(T)\right) &= P\left(S(T) \geq 0\right) \\
&= 1 - \Phi\left(\frac{-E\left[\ln F_w(T)/\theta\right]}{\sigma\left[R_P(w_P) - R_L\right]\sqrt{T-t}}\right) \\
&= \Phi\left(\frac{E\left[\ln F_w(T)/\theta\right]}{\sigma\left[R_P(w_P) - R_L\right]\sqrt{T-t}}\right)
\end{aligned}
\tag{2.13.11}
$$

where $\sigma^2\left[R_P(w_P) - R_L\right] = \sigma_S^2$ in accordance with (2.5.2) and (2.13.9).

Equation (2.13.11) can be expressed explicitly as

$$
\begin{aligned}
&P\left(P_w(T) \geq \theta L(T)\right) \\
&\quad = P\left(S(T) \geq 0\right) \\
&\quad = 1 - \Phi\left(\frac{\ln\frac{\theta}{F(t)} + \left(\theta\mu_L - \mu_P^T w_P - \frac{1}{2}\left(\theta^2\sigma_L^2 - \sigma_P^2\right)\right)(T-t)}{\left(\left(w_P^T \Sigma_A w_P + \theta^2\sigma_L^2 - 2\theta\Sigma_{AL}^T w_P\right)(T-t)\right)^{1/2}}\right) \\
&\quad = \Phi\left(\frac{\ln\frac{F(t)}{\theta} + \left(\mu_P^T w_P - \theta\mu_L - \frac{1}{2}\left(\sigma_P^2 - \theta^2\sigma_L^2\right)\right)(T-t)}{\left(\left(w_P^T \Sigma_A w_P + \theta^2\sigma_L^2 - 2\theta\Sigma_{AL}^T w_P\right)(T-t)\right)^{1/2}}\right)
\end{aligned}
\tag{2.13.12}
$$

Equation (2.13.11)/(2.13.12) is applicable for no, partial or full consideration to surplus optimization, i.e. for $\theta, F \in [0, \infty]$. If no consideration is given to surplus optimization by $\theta = 0$ or by $F \to \infty$, then

$$P\left(P_w(T) \geq \theta L(T)\right) = \Phi(\infty) = 1$$

i.e. the probability of covering liabilities is 1. Conversely, if $\theta \to \infty$ or $F \to 0$, then

$$P\left(P_w(T) \geq \theta L(T)\right) = \Phi(-\infty) = 0$$

as a result of $\lim\limits_{x \to 0} \ln x = -\infty$.

2.14 Shortfall constraints and their properties

A commonly known concept in both practical asset management and academic research are shortfall constraints on limiting the probability on earning a return on a portfolio below some specified threshold level. Under the log-returns normality assumption, the continuously compounded returns are normally distributed. Therefore, the probability of portfolio return being less than or equal to some threshold return, R_{thr}, is given by multivariate normal CDF, $\mathbb{R}^n$, evaluated at R_{thr}. For the traditional risk-return optimizer, a typical shortfall constraint for limiting the probability of portfolio return being less than or equal to some threshold return, R_{thr}, can be written as

$$P\left(R_P\left(w_P\right)\leq R_{thr}\right)\leq \xi \tag{2.14.1}$$

or more precisely

$$P\left(R_P\left(w_P\right)\leq R_{thr}\right)=\Phi\left(\frac{R_{thr}-\left(E\left[R_P\left(w_P\right)\right]-\frac{1}{2}\sigma^2\left[R_P\left(w_P\right)\right]\right)\left(T-t\right)}{\sigma\left[R_P\left(w_P\right)\right]\sqrt{T-t}}\right)\leq \xi \tag{2.14.2}$$

with $E\left[R_P\left(w_P\right)\right]=\mu_A^T w_P$ and $\sigma^2\left[R_P\left(w_P\right)\right]=w_P^T\Sigma_A w_P$ as before.

This constraint can be added to both optimization problems from sections 2.2 and 2.6, rewritten as

$$\frac{R_{thr}-\left(E\left[R_P\left(w_P\right)\right]-\frac{1}{2}\sigma\left[R_P\left(w_P\right)\right]^2\right)\left(T-t\right)}{\sigma\left[R_P\left(w_P\right)\right]\sqrt{T-t}}\leq z_{\xi} \tag{2.14.3}$$

A commonly observed behaviour in relation with shortfall constraints is that the shortfall probability declines as the horizon $\left(T-t\right)$ increases and as a result, a portfolio that violates a shortfall constraint at a short horizon may become feasible at longer horizon. Also, for different time horizons and an active shortfall constraint, the shorter the horizon is, the less risky the feasible portfolio can be as a result of the active shortfall constraint.

Shortfall constraints can also be applied in ALM problems as in the surplus risk-return optimization under consideration in this thesis. The constraint (2.14.3) can be applied directly to the surplus optimization problem for the same purposes as in the absence of liabilities. It can also be applied whereas the threshold return, R_{thr}, is considered to be an expense ratio, i.e. per period liabilities paid during the time period as a ratio of assets. The constraint

(2.14.3) limits the probabilities that returns on the asset portfolio are less than or equal to the expense ratio for chosen time period.

A shortfall constraint can be in the form of limiting the probabilities of declining funding status, i.e. a constraint on the form

$$P\left(F_w(T) \le \bar{F}\right) \le \delta \tag{2.14.4}$$

Furthermore, under the assumptions and definitions in section 2.1, 2.2 and 2.6, an optimal surplus return variance portfolio w_P is chosen. After a time period $(T-t)$, the expected funding ratio is given by

$$E\left[F_w(T)/\theta\right] = \frac{F_w(t)}{\theta}\exp\left(\left(\mu_A^T w_P - \theta\mu_L - \frac{1}{2}\left(\sigma_P^2 - \theta^2\sigma_L^2\right)\right)(T-t)\right) \tag{2.14.5}$$

Equation (2.14.5) results directly from exponenting (2.13.8).

The probability distribution for $\{R_{A1},...,R_{An},R_L\}$ is multivariate normal, $\mathbb{R}^{n+1}$. From (2.14.5) and (2.13.8), the shortfall constraint (2.14.4) can be written as

$$P\left(\ln F_w(T)/\theta \le \ln \bar{F}\right) \le \delta$$

or more precisely as

$$\frac{\ln\dfrac{\theta\bar{F}}{F(t)} - \left(\mu_A^T w_P - \theta\mu_L - \dfrac{1}{2}\left(\sigma_P^2 - \theta^2\sigma_L^2\right)\right)(T-t)}{\sigma\left[R_P(w_P) - \dfrac{\theta\bar{F}}{F(t)}R_L\right]\sqrt{T-t}} \le z_\delta \tag{2.14.6}$$

with the cumulative distribution function

$$\frac{1}{\sqrt{2\pi}}\int_{z_\delta}^{\infty}\exp\left(-\frac{u^2}{2}\right)du = 1 - \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{-z_\delta}\exp\left(-\frac{u^2}{2}\right)du = \delta$$

Under the surplus optimization model (2.6.1) - (2.6.3) and the shortfall constraint (2.14.6), any optimal portfolio $w_{rp,S}$ (2.6.16) that does not violate (2.14.6) makes the optimal set $W_{E,S}$, nonempty. Therefore, for the portfolio return coefficient α (2.6.22), i.e.

$$\alpha = \alpha_\eta + \theta\alpha_{\eta,S} = \frac{r_P Q_{22} - Q_{12} + \theta F^{-1}\left|Q_{\#2}\right|}{\left|Q_{\#1}\right|}$$

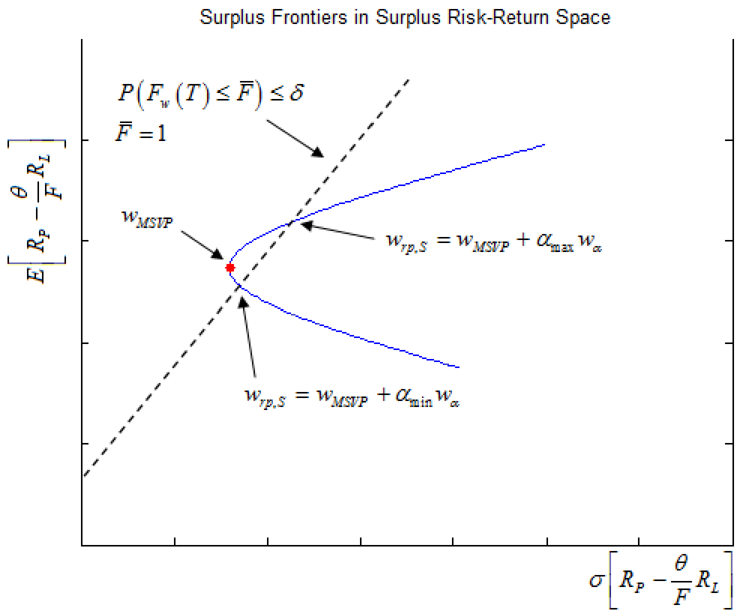
there exist some $\alpha_{\min}(r_{\min})$, $\alpha_{\max}(r_{\max})$ where $0 \le \alpha_{\min} \le \alpha_{\max}$ so that for every w_P that does not violate (2.14.6);

$$w_P \in W_{E,S} \quad \text{with} \quad w_P = w_{MSVP} + \alpha w_\alpha, \quad \alpha \in \{\alpha_{\min}, \alpha_{\max}\}$$

Furthermore, from the above, the surplus optimization model (2.6.1) - (2.6.3) and the shortfall constraint (2.14.6), fixing the threshold funding ratio as $\bar{F}=1$, the funding ratio shortfall constraint (2.14.6) has the interesting property of being linear in the surplus risk-return space

$$E\left[R_P\left(w_P\right)-\frac{\theta}{F}R_L\right],\quad \sigma\left[R_P\left(w_P\right)-\frac{\theta}{F}R_L\right]$$

The linearity makes the constraint easily applicable as if the portfolio returns associated with $\alpha_{\min}$ and $\alpha_{\max}$ have been found. By choosing a suitable value for δ and fixing $\bar{F}=1$, the funding ratio shortfall constraint provides convenience via linearity; otherwise the constraint is not linear in the surplus risk-return space as defined above.



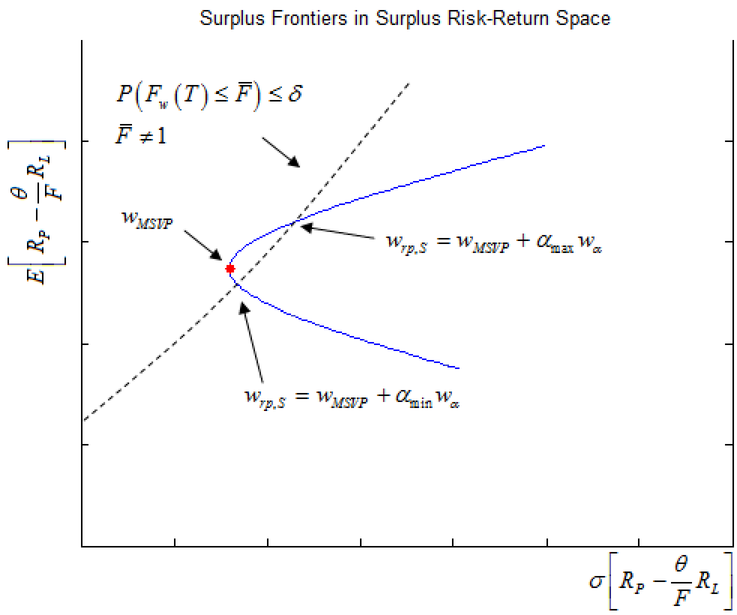


Figure 2.9 and 2.10: In figure 2.9, the funding ratio shortfall constraint is linear in surplus risk-return space due to $\bar{F}=1$ If $\bar{F}\neq1$ as in figure. 2.10, the constraint is curved as for shortfall constraints in risk-return space.

3. Numerical Example

In this chapter, the methods from chapter 2 are applied to a hypothetical Icelandic pension fund where the value of its liabilities is assumed to follow the pension obligation index (POI) for employees in the Icelandic public sector. Through all this chapter, full consideration is given to liabilities by setting the importance parameter $\theta = 1$. Initially, return data are analyzed by descriptive statistics of logarithmic returns and return correlation analysis. Also, the values of the determinant and sub-determinants of Q show the characteristics of the optimal sets in accordance with section 2.10. Optimal allocations in the absence of liabilities are analyzed in parallel with surplus optimization for different funding ratios to demonstrate how the funding status of a pension fund affects the optimal asset allocation decisions and the characteristics of the liability hedge. From the optimal sets in absence and presence of liabilities, the risk-return frontiers and the surplus risk return frontiers are constructed. Market portfolio in presence of liabilities is added to the surplus risk-return analysis and the theoretical absolute minimum surplus return variance is found. Probabilities positive surplus and shortfall constraints are analyzed. and this chapter ends with a comparative analysis of the ability of asset allocation strategies in absence and presence of liabilities in generating surplus.

The optimization models from chapter 2 are not constrained to long-only positions and the models are applied unchanged in this numerical example allowing short positions which provide more risk diversification and better hedge against the liability index. In many countries, short positions are not allowed for the purposes of pension management. This has been criticized for the fact that short positions result in more market efficiency and greater hedging ability of portfolios against negative effects. On the other hand, pension funds are large scale investors and relatively large as such in many economies. Therefore, large scale shorting of assets is often not possible as there might not be enough securities available for shorting on the market and many other arguments exist in limiting shorting in pension fund management. Despite of this, the models used in this thesis are applied here without any changes to prevent short positions as to observe the full functionality of the model and to observe how large the short positions are. The size of short positions in the optimal portfolios depends on the data used and the return preferences of the investor which is decisive for the optimal allocation as shown in chapter 2.

Matlab was used for all numerical calculations and programming. All programming codes were written from scratch and no predefined optimization functions were used since the methods from chapter 2, supported by the derivations in appendices 1 – 26, provide all necessary information on solving the problems and ideas stated in this thesis.

3.1 Liability and asset class data analysis

The data chosen for this numerical example consist out of the POI as an index for liabilities and eight asset class indices that are assumed to be representatives for the investments available for the pension fund at the time of the analysis. The pension obligation index (POI) for employees in the Icelandic public sector is calculated and published on monthly basis by Statistics Iceland, starting with base value of 100 in December 1996. The index is calculated according to the index of fixed salaries for day time work of employees in the public sector according to act no.1/1997 on pension rights for state employees (Statistics Iceland, 2013).

The eight asset class indices used in this numerical example consist of four domestic (Icelandic) indices and four foreign indices. Four out of these eight asset class indices are bond indices; one foreign and three domestic indices. The other four are indices for riskier investments; domestic stocks, foreign stocks, hedge funds and private equities. The returns of the foreign indices are originally all in USD. The returns are corrected for currency returns so that all returns are in the domestic currency of ISK. Table 3.1 provides information on the asset classes and the representative indices.

Table 3.1: Indices for Liabilities and Asset Classes

Liabilites	**Representative Index**	**Currency**	**Source**
L: Liability Index	Pension obligation index for employees in the Icelandic public sector	ISK	Statistics Iceland
Asset classes			
A1: Domestic Stock Market Index	Icelandic All Share Price Index	ISK	Nasdaq OMX Nordic exchange
A2: Foreign Stock Market Index	MSCI World Developed Markets standard	USD	MSCI Inc.
A3: Domestic Short Mat. Bond Index	50/50 OMXI3MNI/OMXI1YNI Bond Indices	ISK	Nasdaq OMX Nordic exchange
A4: Domestic Long Mat. Bond Index	OMX5YNI Bond Index	ISK	Nasdaq OMX Nordic exchange
A5: Domestic Indexed Bond Index	50/50 OMXI5YI/OMXI10YI Bond Indices	ISK	Nasdaq OMX Nordic exchange
A6: Hedge Fund Index	Barclay Hedge Fund Index	USD	BarclayHedge Ltd.
A7: Foreign Bond Index	Vanguard Total Bond Market Index	USD	The Vanguard Group, Inc.
A8: Private Equity Index	GLPE Private Equity Index	USD	Red Rocks Capital, LLC.

Table 3.1: Information on liability index, asset classes and representative indices for asset classes.

The OMX Iceland All-Share Index includes all the shares listed on Nasdaq OMX Nordic Exchange Iceland with a base date of December 31, 1997, and a base value of 1000 (Bloomberg, 2013). The MSCI World Index is a stock market index, including approximately

2000 securities from 23 countries but excluding stocks from emerging and frontier markets. The index includes a collection of stocks of all the developed markets in the world, as defined by MSCI Inc. Due to the index construction rules and company related events, the number of constituents significantly fluctuates while increasing over time (MSCI Inc., 2013). The domestic short maturity bond index constructed for this analysis was composed out of equally weighted returns of two non-indexed bond indices from Nasdaq OMX Nordic; the 3 month OMXI3MNI and the 1 year OMX1YNI indices. The non-indexed OMX5YNI bond index is used as domestic long maturity bond index as information on other indices with longer maturity was not available before 2005. The inflation index linked bond index used was composed out of equally weighted returns of two indexed bond indices from Nasdaq OMX Nordic; the 5 year OMXI5YI and the 10 year OMX10YI indices. The hedge fund index data are provided by Barclay Hedge Ltd. (USA) database; Barclay Hedge Ltd. is a privately owned corporation and is not affiliated with Barclays Bank or any of its affiliated entities. The Barclay Hedge Fund Index is a measure of the average return of all hedge funds (excepting Funds of Funds) in the Barclay database. The index is based on the average of the net returns of all the funds that have reported that month, generally in between 1,000 – 3,000 funds (Barclay Hedge Ltd., 2013). The foreign bond index is the Vanguard Total Bond Market index. The index is designed to provide broad exposure to U.S. investment grade bonds and consists of approximately 30% in corporate bonds and 70% in U.S. government bonds of all maturities (Yahoo finance, 2013). Finally, the private equity index is the Red Rocks Capital Global Listed Private Equity (GLPE) index. The GLPE index is designed to track the performance of private equity firms which are publicly traded on any nationally recognized exchange worldwide. These companies invest in, lend capital to, or provide services to privately held businesses. The index is comprised of 40 - 75 public companies and the securities of the index are selected and rebalanced quarterly per modified market capitalization weights (Red Rocks Capital LLC., 2013).

The return data for the asset classes and liabilities consist out of 65 monthly data points from January 2003 to and including June 2008. Figures 3.1 and 3.2 illustrate normalized prices for the indices where all indices have initial price 100 in the beginning of January 2003.
Figure 3.1 illustrates the four fixed income securities (A3, A4, A5, A7) and the liability index and figure 3.2 illustrates the other asset classes (A1, A2, A6, A8) and the liability index.

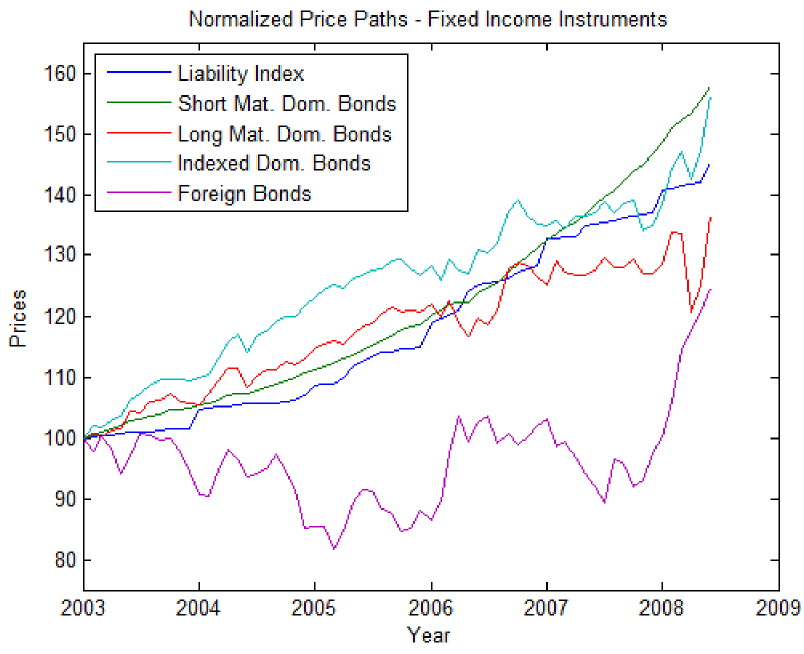


Figure 3.1: Price paths for the four fixed income securities and the liabiltiy index, starting at 100 in January 2003.

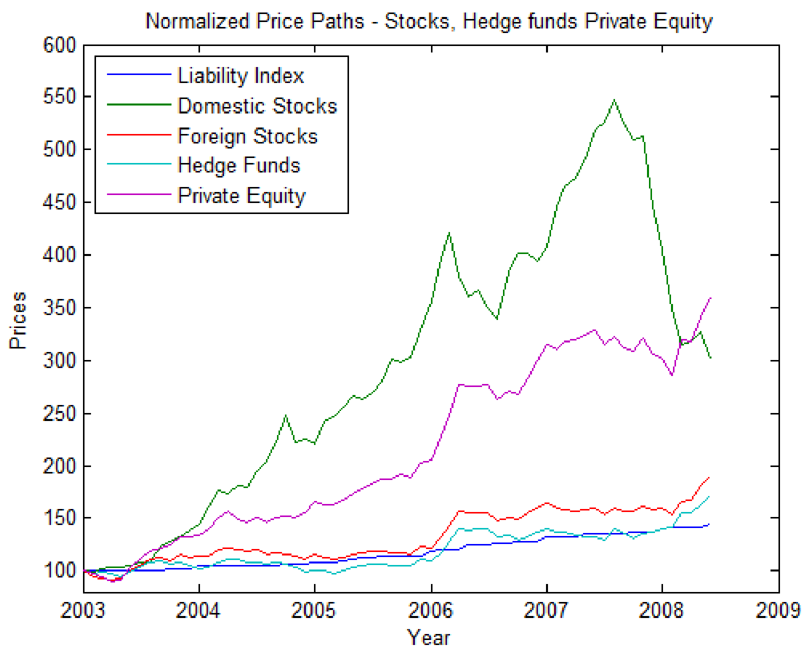


Figure 3.2: Price paths for liability, stock, hedge fund and private equity indices, starting at 100 in January 2003.

For further analysis of the return data, table 3.2 provides descriptive statistics for the logarithmic returns of the nine time series. The liability returns have a mean growth rate of 6.9% with volatility of 2.9%, indicating a steady growth with small variation. The price path for the liability index in figure 3.1 rises in levels, frequently in the beginning of each year as changes of pension rights come into effect. In fact, none of the return values is negative in the liability return data series. Accordingly, skewness and excess kurtosis suggest a leptokurtic distribution with positively skewed returns. This is also confirmed by the Jarque-Bera (JB) test result which clearly rejects the null hypothesis of a normal distribution for the liabilities returns. The data analysis suggests a leptokurtic distribution for all the asset classes with positively skewed returns, except for domestic stocks and domestic long maturity bonds where the returns are negatively skewed. The JB test rejects normality hypothesis for the domestic short and long bond indices (A3, A4) as for the liability index.

Table 3.2: Descriptive statistics for logarithmic returns

# Obs: 65	L	A1	A2	A3	A4	A5	A6	A7	A8
Mean	0.069	0.204	0.118	0.084	0.057	0.082	0.100	0.041	0.236
Std. dev.	0.029	0.213	0.118	0.012	0.079	0.057	0.115	0.119	0.145
Skewness	2.393	-0.661	0.422	0.772	-0.714	0.111	0.675	0.262	0.333
Exc. Kurtosis	7.541	3.352	2.606	3.360	11.015	4.260	3.024	2.447	2.649
JB tstat	115.041	4.846	2.261	6.515	179.237	4.426	4.709	1.540	1.482
JB crit. tstat	5.179	5.179	5.179	5.179	5.179	5.179	5.179	5.179	5.179

Table 3.2: If JB tstat > JB critical tstat, the null hypothesis of normally distributed returns can be rejected

Return correlation estimates can be seen in table 3.3 where return correlation between the asset classes and liabilities indicates that returns on all asset classes are positively correlated with the liabilities, except for domestic stocks and foreign bonds where returns are negatively correlated with the liabilities. This readily indicates that six out of eight assets classes have some liability hedging properties, individually. The highest return correlation with liabilities is from short maturity bonds, twice as good as for indexed bonds. Foreign stocks and long maturity bonds have the third and fourth highest correlation with liabilities. Foreign bonds have the second lowest return correlation with liabilities which can be largely explained by currency returns. This is confirmed by a simple linear regression where more than 90% of the variability in ISK returns of the foreign bond index is explained by currency returns.

The value of the ISK against the USD rose considerably during the period from January 2003 – January 2008, although it decreased to almost the same level by the end of June 2008 as figure 3.3 illustrates. The returns on foreign securities suffered due to the increase in USD/ISK and low yielding foreign bonds yielded negative returns in ISK for the years 2003 – 2008 as figure 3.1 implies. During the largest part of the period under consideration, currency returns made most foreign investments unattractive in terms of ISK returns. Nevertheless, the lowest return correlation is between the liabilities and domestic stocks.

Table 3.3: Estimates for return correlation, return covariance between asset classes and liabilities ($\sigma_i\sigma_L\rho_{iL}$) and hedging ability ($\sigma_i\rho_{iL}$)									
# Obs: 65	**L**	**A1**	**A2**	**A3**	**A4**	**A5**	**A6**	**A7**	**A8**
L	1								
A1	-0.113	1							
A2	0.079	0.036	1						
A3	0.212	-0.224	-0.021	1					
A4	0.068	0.046	-0.042	0.477	1				
A5	0.105	-0.080	-0.046	0.464	0.830	1			
A6	0.017	-0.061	0.857	0.088	0.025	0.042	1		
A7	-0.059	-0.156	0.582	0.191	0.023	0.103	0.850	1	
A8	0.059	0.162	0.898	-0.101	0.020	-0.021	0.753	0.462	1
$\sigma_i\sigma_L\rho_{i,L}$	8.51.E-04	-7.06.E-04	2.74.E-04	7.67.E-05	1.57.E-04	1.76.E-04	5.75.E-05	-2.06.E-04	2.50.E-04
$\sigma_i\rho_{i,L}$	0.029	-0.024	0.009	0.003	0.005	0.006	0.002	-0.007	0.009
σ_i	0.029	0.213	0.118	0.012	0.079	0.057	0.115	0.119	0.145

Table 3.3: $\sigma_i\,\sigma_L\,\rho_{i,L}$ are the elements of the asset-liability return covariance vector Σ_{AL} from chapter 2. The combined effect of individual security risk and correlation of individual asset with liabilities is obtained by $\sigma_i\,\rho_{i,L}$.

The bottom line in table 3.3 shows the standard deviation of returns on liabilities and individual assets where domestic stocks are the riskiest asset class followed by the private equity asset class. The foreign bond index which contains 70% AAA rated U.S. government bonds is the third riskiest asset class; this is mainly explained by currency returns as written before. Hedging ability (Sharpe & Tint, 1990), i.e. the combined effect of correlation and risk is obtained by multiplying the correlation coefficient by the standard deviation $\sigma_i\rho_{i,L}$, is in the second row from the bottom in table 3.3. The highest values are for foreign stocks and private equity, followed by indexed and long maturity domestic bonds. The lowest value is for domestic stocks, showing the least preferable hedging characteristics of this relatively volatile and negatively correlated asset. The return covariance vector, Σ_{AL} , where the elements of the vector are the return covariance estimate between asset i and liabilities, $\sigma_i\sigma_L\rho_{i,L}$, is found in the third row from the bottom in table 3.3.

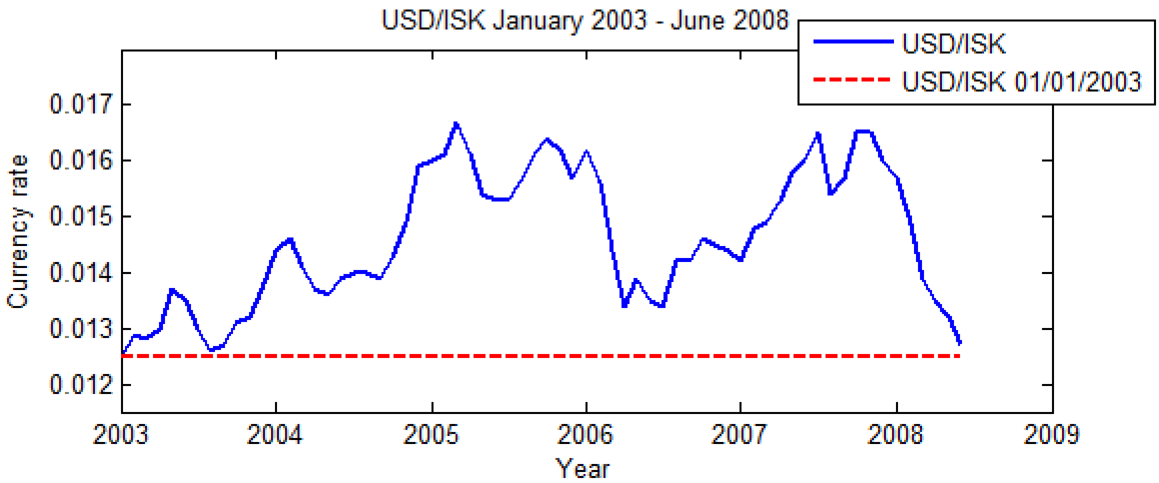


Figure 3.3: The USD/ISK currency rate from January 2003 – June 2008 (Central bank of Iceland, 2013).

Table 3.4 shows the determinant and sub-determinants of the matrix Q in accordance with section 2.10. The determinants $|Q|$, $|Q_{\#1}|$ and $|Q_{\#4}|$, imply that the optimal sets in absence and presence of liabilities are not the same; the surplus optimal frontiers have higher return volatility than frontiers where liabilities are ignored. Furthermore, the expected returns on the MSVPs are higher than on the MVP since $|Q_{\#2}| < 0$ and increase with decreasing funding ratio. For the same reason, return covariance between efficient portfolios and liabilities increases as the return requirement on any portfolio is increased.

Table 3.4: The determinants and sub-determinants of the matrix Q

Term	Value	Interpretation
$\lvert Q\rvert$	1.96	$\lvert Q\rvert > 0 \Rightarrow VAR\left[R_p\left(w_{rp,S}\right) \mid r_p\right] > VAR\left[R_p\left(w_{rp}\right) \mid r_p\right]$
$\lvert Q_{\#1}\rvert$	40832.14	$\lvert Q_{\#1}\rvert > 0$ The optimal sets W/W_S are not limited to w_{MVP} / w_{MSVP}
$\lvert Q_{\#2}\rvert$	-13.42	$\lvert Q_{\#2}\rvert < 0 \Rightarrow E\left[R_p\left(w_{MSVP}\right)\right] > E\left[R_p\left(w_{MVP}\right)\right]$, $\lvert Q_{\#2}\rvert < 0 \Rightarrow \frac{\partial}{\partial r_p} COV\left[R_p\left(w_{rp,S}\right), R_L\right] > 0$
$\lvert Q_{\#4}\rvert$	0.43	$\lvert Q_{\#4}\rvert > 0 \Rightarrow W \neq W_S \Rightarrow VAR\left[R_p\left(w_{MSVP}\right) \mid r_p\right] > VAR\left[R_p\left(w_{MVP}\right) \mid r_p\right]$

Table 3.4: Values for the determinants and sub-determinants of Q with interpretation of the values.

As a result of section 2.10, it can readily be read from table 3.4 that the surplus frontiers examined in this chapter are characterized by a rising return on the MSVP as F decreases, since $|Q_{\#2}| < 0$ and

$$E\left[R_P\left(w_{MVP}\right)\right] = \frac{Q_{12}}{Q_{22}} \quad \text{and} \quad E\left[R_P\left(w_{MSVP}\right)\right] = E\left[R_P\left(w_{MVP}\right)\right] - \frac{\theta}{F}\frac{|Q_{\#2}|}{Q_{22}}$$

This is confirmed by figure 3.4 that illustrates the risk-return frontiers in the absence and presence of liabilities. Figure 3.4 illustrates the well-known risk-return space where the leftmost frontier is the traditional risk-return frontier with no consideration on liabilities, i.e. where $F = \infty$ $\rho_{P,L} = 0$ or $\mu_L = 0$. The frontiers to the right of the traditional frontier are risk-return surplus frontiers with funding ratios F = [1.5, 1.25, 1, 0.75, 0.5], where the rightmost frontier has the lowest funding ratio. Also, a frontier for F_{COV} = 0.6164 (see section 2.6 and 2.12) lies between the frontiers with F = 0.75 and F = 0.5.

As can be observed from figure 3.4, the MSVPs are not on the minimum variance point on their respective risk-return frontiers. As shown in section 2.10 and by table 3.4, this is due to the fact that

$$E\left[R_P\left(w_{MSVP}\right)\right] > E\left[R_P\left(w_{MVP}\right)\right] \quad \text{when} \quad |Q_{\#2}| < 0$$

as can be seen from the equations above. The MSVP is a portfolio that minimizes surplus return variance for a given funding ratio. Expected return on a MSVP equals the expected return on MVP if $|Q_{\#2}| = 0$, $\theta = 0$ or $F = \infty$.

The return variance for the MVP and the MSVP are given by

$$VAR\left[R_P\left(w_{MVP}\right)\right] = \frac{1}{Q_{22}} \quad \text{and} \quad VAR\left[R_P\left(w_{MSVP}\right)\right] = VAR\left[R_P\left(w_{MVP}\right)\right] + \frac{\theta^2}{F^2}\frac{|Q_{\#4}|}{Q_{22}}$$

and return variance on any portfolio on respective frontiers is expressed as

$$VAR\left[R_P\left(w_{rp}\right)\right] = \frac{r_P^2 Q_{22} - 2r_P Q_{12} + Q_{11}}{|Q_{\#1}|}$$

and

$$VAR\left[R_P\left(w_{rp,S}\right)\right] = \frac{r_P^2 Q_{22} - 2r_P Q_{12} + Q_{11} + \theta^2 F^{-2}|Q|}{|Q_{\#1}|}$$

The determinants $|Q_{\#4}|$ and $|Q|$ explain the difference in return variance between the frontiers in absence and presence of liabilities as observed in figure 3.4. The positive value of $|Q_{\#4}|$ affects the return variance shift form the MVP to the MSVP whereas for any other portfolios on the surplus frontier, $|Q|$ explains the higher return variance. In both cases, $\theta^2 F^{-2}$ acts as a scaling factor on the return variance shift and if $\theta = 0$ or $F = \infty$, the surplus frontier converges with the traditional mean variance frontier.

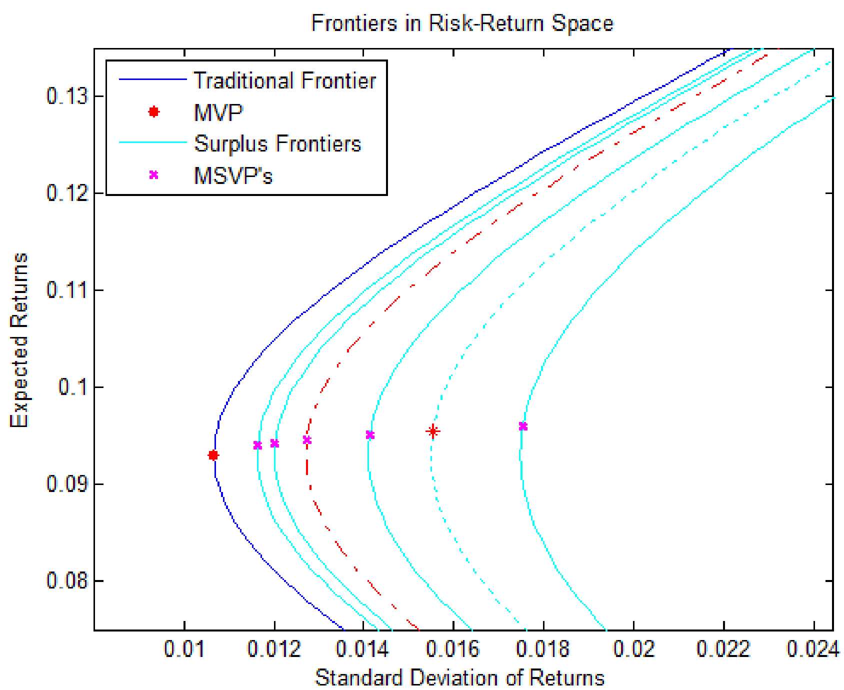


Figure 3.4: Risk-return frontiers in absence of liabilities (leftmost frontier, blue with w_{MVP} as a red dot) and presence of liabilities (red with w_{MSVP}'s as magenta colored x's) for F = [1.5, 1.25, 1.0, 0.75, 0.5] and θ = 1. The red dash-dotted surplus frontier is for F = 1, the surplus frontiers to the left of the red dash-dotted one have $F > 1$ and the frontiers to the right of the red dash-dotted one have $F < 1$. The unit normalized return covariance portfolio (see sections 2.6 and 2.12), w_{COV}, is shown by a red asterisk on the dashed surplus frontier for F_{COV} = 0.6164.

3.2 Minimum variance and minimum surplus variance portfolios

Table 3.5 shows the asset allocations for the MVP and the MSVP's using the previously mentioned set of funding ratios and table 3.6 shows risk-return values for these portfolios, expressed as

$$w_{MVP} = \frac{\Sigma_A^{-1}\upsilon}{Q_{22}} \quad \text{and} \quad w_{MSVP} = w_{MVP} + \theta\phi_{MSVP} \quad \text{with} \quad \phi_{MSVP} = \frac{1}{F}\left[\Sigma_A^{-1}\Sigma_{AL} - \frac{Q_{23}}{Q_{22}}\Sigma_A^{-1}\upsilon\right]$$

As the hedging ability in table 3.3 suggests, the MSVP portfolios have a short position in domestic stocks (A1) although a low one. Similarly, negative return correlation and low return estimate explains the shorting of foreign bonds (A7). The short positions increase as the funding ratio decreases as to hedge against negative return correlation between those asset classes and liabilities. Also, domestic long maturity bonds (A4) are shorted although having positive correlation, as a result of relatively low return estimate against volatility for domestic long maturity bonds. Although the foreign stocks (A2) and private equities (A8) have the highest hedging ability, the MSVP allocations in those asset classes are relatively low.

Table 3.5: Asset allocation; w_{MVP} ; ϕ_{MSVP} ; w_{MSVP} and w_{COV}

	A1	A2	A3	A4	A5	A6	A7	A8	Total
w_{MVP}	0.014	-0.020	1.040	-0.083	0.033	0.014	-0.025	0.028	1.000
ϕ_{MSVP}									
F = 1.50	-0.014	0.031	-0.012	-0.009	0.040	0.015	-0.046	-0.005	0.000
F = 1.25	-0.017	0.038	-0.014	-0.011	0.048	0.018	-0.055	-0.006	0.000
F = 1.00	-0.021	0.047	-0.018	-0.013	0.060	0.022	-0.069	-0.007	0.000
F = 0.75	-0.028	0.063	-0.024	-0.018	0.080	0.029	-0.092	-0.010	0.000
F = 0.50	-0.042	0.094	-0.036	-0.027	0.120	0.044	-0.138	-0.015	0.000
w_{MSVP}									
F = 1.50	0.000	0.011	1.028	-0.092	0.072	0.028	-0.070	0.023	1.000
F = 1.25	-0.003	0.018	1.025	-0.094	0.080	0.031	-0.080	0.022	1.000
F = 1.00	-0.007	0.027	1.021	-0.096	0.092	0.036	-0.093	0.021	1.000
F = 0.75	-0.015	0.043	1.015	-0.101	0.112	0.043	-0.116	0.018	1.000
F = 0.50	-0.029	0.074	1.003	-0.110	0.152	0.058	-0.162	0.013	1.000
w_{COV}									
F = 0.6164	-0.021	0.056	1.010	-0.105	0.130	0.049	-0.136	0.016	1.000

Table 3.5: Asset class allocations for w_{MVP} , ϕ_{MSVP}, w_{MSVP} and w_{COV} with $\theta = 1$; $w_{MSVP} = w_{MSVP} + \theta\phi_{MSVP}$, $w_{COV} = w_{MSVP}$ for $F = F_{COV}$.

As the funding ratio decreases, allocation in foreign stocks increases while allocation in private equities decreases. This may be explained by the lower risk and higher return correlation of foreign stocks than for private equities. The allocation in domestic short maturity bonds (A3) is higher than 1 for all funding ratios as a result of low volatility and a high return correlation between this asset class and liabilities.

Table 3.6 shows further results for the MVP and MSVPs. The expected return on the MSVP increases as F decreases as table 3.4 suggests. Accordingly, the return on the liability hedge portfolio, ϕ_{MSVP}, is positive. The covariance between portfolio and liabilities returns increases as funding ratio decreases, resulting from increased concentration in the liability hedging portfolio. The liability hedging credit and the hedging ability increase for the same reason and indicate increasing hedging characteristics of the MSVP as the funding ratio decreases.

The risk-return space representations of the portfolios given in tables 3.5 and 3.6 are shown in figure 3.4.

Table 3.6: Results from the optimization model for w_{MVP} and w_{MSVP} for selected funding ratios F, and w_{COV} for F_{COV}

Portfolio; P	w_{MVP}	w_{MSVP}					w_{COV}
Variable	$F=\infty$	$F=1.50$	$F=1.25$	$F=1.00$	$F=0.75$	$F=0.50$	$F=0.6164$
$E[R_P(\phi_{MSVP})]$		0.001	0.001	0.002	0.002	0.003	
$E[R_P(w_P)]$	0.093	0.094	0.094	0.095	0.095	0.096	0.096
$\sigma[R_P(w_P)]$	0.011	0.012	0.012	0.013	0.014	0.018	0.016
$E[R_S(w_P)]$		0.048	0.039	0.026	0.003	-0.042	-0.016
$\sigma[R_S(w_P)]$		0.019	0.023	0.028	0.037	0.055	0.045
$E[R_P(w_P)]$ - $E[R_P(w_{MVP}$		0.001	0.001	0.002	0.003	0.007	0.002
$\sigma[R_P(w_P)]$ - $\sigma[R_P(w_{MVP})]$		1.0.E-03	1.2.E-03	1.5.E-03	2.0.E-03	3.1.E-03	4.9.E-03
$Cov[R_P, R_L]$	7.0.E-05	1.0.E-04	1.1.E-04	1.2.E-04	1.3.E-04	1.7.E-04	1.5.E-04
$Cov[R_P, R_L]$ - $Cov[R_{MVP}, R_L]$		3.2.E-05	3.9.E-05	4.8.E-05	6.5.E-05	9.7.E-05	7.9.E-05
LHC_P		1.4.E-04	1.7.E-04	2.4.E-04	3.6.E-04	6.7.E-04	4.8.E-04
$\sigma_P \rho_{P,L}$	2.4.E-03	3.5.E-03	3.7.E-03	4.1.E-03	4.6.E-03	5.7.E-03	5.1.E-03

Table 3.6: Results for the w_{MVP}, w_{MSVP} and w_{COV} according to sections 2.3 and 2.6 with $\theta = 1$.
$E[R_P(w_{MSVP})] = E[R_P(w_{MVP})] + \theta E[R_P(\phi_{MVSP})]$, $E[R_S(w_P)]$ - expected returns on surplus, $\sigma[R_S(w_P)]$ - standard deviation of surplus returns, $Cov[R_P, L] - Cov[R_{MVP}, L]$ confirms an increase in return covariance between optimal portfolio and liabilities as F decreases. Liability hedging credit: $LHC_P = 2\theta\, \sigma_P\, \sigma_L\, \rho_{P,L} / F$.

3.3 Optimal portfolios in absence and presence of liabilities with return preferences

Figure 3.5 illustrates the same risk-return frontiers as in figure 3.4 with optimal portfolios added, having return requirement parameters r_P = [0.10, 0.11, 0.12]. These portfolios are analyzed on following pages. Tables 3.7 – 3.12 show the results for the specified values of r_P and F.

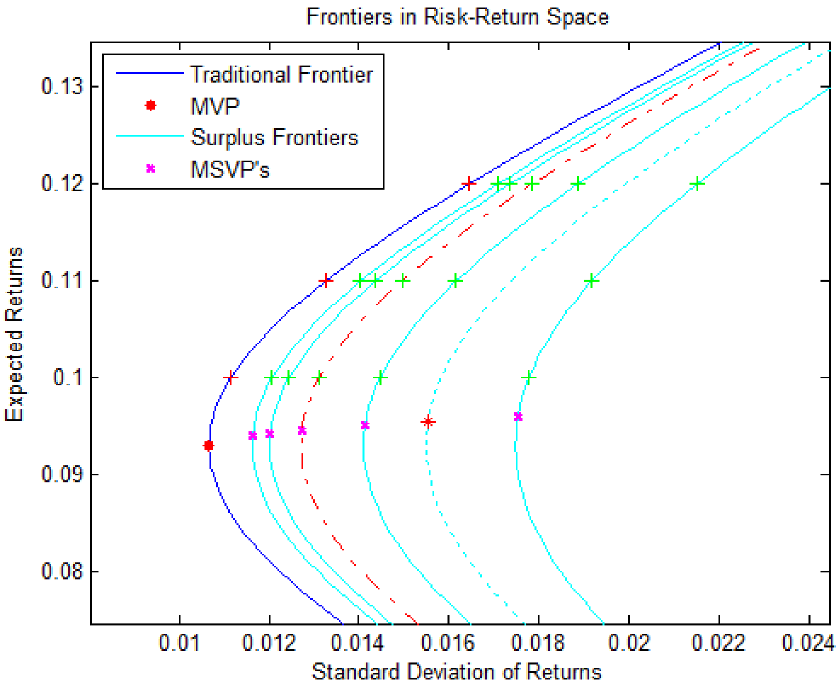


Figure 3.5: This is the same figure as 3.4 where optimal portfolios with return requirement r_P = [0.10, 0.11, 0.12] (red and green +'s) have been added.

As the return requirement is raised above MVP/MSVP level, the magnitudes of the short and long positions change as indicated by tables 3.7 – 3.12. The optimal portfolios in absence and presence of liabilities with any return requirement of r_P are given by, respectively;

$$w_{rp} = w_{MVP} + w_\eta = \frac{\Sigma_A^{-1}\upsilon}{Q_{22}} + \frac{r_P Q_{22} - Q_{12}}{|Q_{\#1}|}\left[\Sigma_A^{-1}\mu_A - \frac{Q_{12}}{Q_{22}}\Sigma_A^{-1}\upsilon\right]$$

and

$$\begin{aligned} w_{rp,S} &= w_{MVP} + \theta\phi_{MSVP} + w_\eta + \theta w_{\eta,S} \\ &= \frac{\Sigma_A^{-1}\upsilon}{Q_{22}} + \frac{\theta}{F}\left[\Sigma_A^{-1}\Sigma_{AL} - \frac{Q_{23}}{Q_{22}}\Sigma_A^{-1}\upsilon\right] \\ &+ \frac{r_P Q_{22} - Q_{12} + \theta F^{-1}|Q_{\#2}|}{|Q_{\#1}|}\left[\Sigma_A^{-1}\mu_A - \frac{Q_{12}}{Q_{22}}\Sigma_A^{-1}\upsilon\right] \end{aligned}$$

For a return requirement of 10%, allocation in domestic stocks (A1) is nearly unchanged. Allocation in foreign stocks (A2) and bonds (A7) as well as domestic long maturity bonds (A4) decreases while an increased allocation in domestic short maturity (A3) and indexed bonds (A5), hedge funds (A6) and private equities (A8) is observed. The excess long position in domestic short maturity bonds is increased from the MVP/MSVP level as a result of high return vs. low risk characteristics of this asset class.

Table 3.7: Asset allocation; w_{MVP}, w_η, $w_{\eta,S}$, w_{rp}, $w_{rp,S}$ with return requirement r_P = 10%	A 1	A 2	A 3	A 4	A 5	A 6	A 7	A 8	Total
w_{MVP}	0.014	-0.020	1.040	-0.083	0.033	0.014	-0.025	0.028	1.000
w_η	0.000	-0.055	0.007	-0.035	0.034	0.027	-0.021	0.043	0.000
w_{rp}	0.014	-0.075	1.047	-0.118	0.066	0.041	-0.046	0.071	1.000
$w_{\eta,S}$									
F = 1.50	0.000	0.008	-0.001	0.005	-0.005	-0.004	0.003	-0.006	0.000
F = 1.25	0.000	0.010	-0.001	0.006	-0.006	-0.005	0.004	-0.008	0.000
F = 1.00	0.000	0.012	-0.002	0.008	-0.007	-0.006	0.005	-0.009	0.000
F = 0.75	0.000	0.016	-0.002	0.010	-0.010	-0.008	0.006	-0.013	0.000
F = 0.50	0.000	0.024	-0.003	0.015	-0.015	-0.012	0.009	-0.019	0.000
$w_{rp,S}$									
F = 1.50	0.000	-0.036	1.034	-0.122	0.101	0.051	-0.089	0.060	1.000
F = 1.25	-0.003	-0.028	1.031	-0.123	0.108	0.054	-0.097	0.058	1.000
F = 1.00	-0.007	-0.016	1.027	-0.124	0.119	0.057	-0.110	0.054	1.000
F = 0.75	-0.014	0.004	1.020	-0.126	0.136	0.062	-0.131	0.049	1.000
F = 0.50	-0.028	0.043	1.007	-0.129	0.171	0.073	-0.174	0.038	1.000

Table 3.7: Asset class allocations for portfolios with r_P = 10%, 0.7% above the w_{MVP} return and θ = 1. The portfolios shown in the table are: $w_{rp} = w_{MVP} + \theta w_\eta$, $w_{rp,S} = w_{MVP} + \theta\phi_{MSVP} + w_\eta + \theta w_{\eta,S}$. The liability hedge portfolio allocation ϕ_{MSVP} is found in table 3.5.

The first two rows in table 3.8 are the same as the first two rows in table 3.6. The third row provides the expected return on the return generating and return generating correction components, $E\ [R_P\ (w_\eta)]$ and $E\ [R_P\ (w_{\eta,S})]$. The return covariance increase, resulting from increased return requirement and suggested by table 3.4, is hardly noticeable for this small increase in return requirement above the MVP/MSVP level, but will become clearer in tables 3.10 and 3.12 where the return requirement on the portfolios is increased further. The liability hedging credits and the hedging ability increase a little compared with the MVP/MSVPs.

Table 3.8: Results from the optimization model for portfolios with return requirement r_P = 10%						
Portfolio; P	w_{rp}	$w_{rp,S}$				
Variable	$F = \infty$	F = 1.50	F = 1.25	F = 1.00	F = 0.75	F = 0.50
$E\,[R_P\,(\phi_{MSVP})]$		0.001	0.001	0.002	0.002	0.003
$E\,[R_P\,(w_{MVP})]$, $E\,[R_P\,(w_{MSVP})]$	0.093	0.094	0.094	0.095	0.095	0.096
$E\,[R_P\,(w_\eta)]$, $E\,[R_P\,(w_{\eta,S})]$	0.007	-0.001	-0.001	-0.002	-0.002	-0.003
$E\,[R_P\,(w_{rp})]$, $E\,[R_P\,(w_{rp,S})]$	0.100	0.100	0.100	0.100	0.100	0.100
$\sigma\,[R_P\,(w_{rp})]$, $\sigma\,[R_P\,(w_{rp,S})]$	0.011	0.012	0.012	0.013	0.014	0.018
$E\,[R_S\,(w_{rp,S})]$		0.054	0.045	0.031	0.008	-0.038
$\sigma\,[R_S\,(w_{rp,S})]$		0.020	0.023	0.028	0.037	0.055
$Cov\,[R_{rp}, R_L]$, $Cov\,[R_{rp,S}, R_L]$	7.2.E-05	1.0.E-04	1.1.E-04	1.2.E-04	1.4.E-04	1.7.E-04
$Cov\,[R_{rp,S}, R_L]$ - $Cov\,[R_{MSVP}, R_L]$		2.0.E-06	1.9.E-06	1.8.E-06	1.6.E-06	1.3.E-06
LHC_P		1.4.E-04	1.8.E-04	2.4.E-04	3.6.E-04	6.7.E-04
$\sigma_P\rho_{P,L}$	2.5.E-03	3.6.E-03	3.8.E-03	4.1.E-03	4.7.E-03	5.8.E-03

Table 3.8: Results for portfolios with r_P = 10% and θ = 1. $E\ [R_P\ (w_{MSVP})] = E\ [R_P\ (w_{MVP})] + \theta E\ [R_P\ (\phi_{MVSP})]$, $E\ [R_P\ (w_{rp})] = E\ [R_P\ (w_{MVP})] + E\ [R_P\ (w_\eta)]$, $E\ [R_P\ (w_{rp,S})] = E\ [R_P\ (w_{MVP})] + \theta E\ [R_P\ (\phi_{MVSP})] + E\ [R_P\ (w_\eta)] + \theta E\ [R_P\ (w_{\eta,S})]$, $E\ [R_S\ (w_P)]$ - expected return on surplus, $\sigma\ [R_S\ (w_P)]$ - standard deviation of surplus return. $Cov[R_{rp,S}\ , L]$ - $Cov[R_{MSVP}\ , L]$ confirms an increase in return covariance as $r_P > r_{MVP}$ as table 3.4 suggests.

In tables 3.9 – 3.10, the return requirement on the optimal portfolios has been increased to 11%. By increasing the return requirement from 10% to 11%, the same pattern is observed as when it was increased from the MVP/MSVP level to 10%. Allocation in domestic stocks (A1) is increases a little as a result of high historical return estimate. Allocation in foreign stocks (A2) and bonds (A7) as well as domestic long maturity bonds (A4) decreases while a still increased allocation in domestic short maturity (A3) and indexed bonds (A5), hedge funds (A6) and private equities (A8) is observed. The excess long position in domestic short maturity bonds is increased further from the MVP/MSVP allocation.

Table 3.9: Asset allocation; w_{MVP}, w_{η}, $w_{\eta,S}$, w_{rp}, $w_{rp,S}$ with return requirement r_P = 11%

	A 1	A 2	A 3	A 4	A 5	A 6	A 7	A 8	Total
w_{MVP}	0.014	-0.020	1.040	-0.083	0.033	0.014	-0.025	0.028	1.000
w_{η}	0.001	-0.134	0.017	-0.085	0.082	0.066	-0.052	0.105	0.000
w_{rp}	0.015	-0.154	1.057	-0.168	0.114	0.080	-0.076	0.133	1.000
$w_{\eta,S}$									
F = 1.50	0.000	0.008	-0.001	0.005	-0.005	-0.004	0.003	-0.006	0.000
F = 1.25	0.000	0.010	-0.001	0.006	-0.006	-0.005	0.004	-0.008	0.000
F = 1.00	0.000	0.012	-0.002	0.008	-0.007	-0.006	0.005	-0.009	0.000
F = 0.75	0.000	0.016	-0.002	0.010	-0.010	-0.008	0.006	-0.013	0.000
F = 0.50	0.000	0.024	-0.003	0.015	-0.015	-0.012	0.009	-0.019	0.000
$w_{rp,S}$									
F = 1.50	0.001	-0.115	1.044	-0.172	0.149	0.090	-0.119	0.122	1.000
F = 1.25	-0.002	-0.107	1.041	-0.173	0.156	0.092	-0.128	0.120	1.000
F = 1.00	-0.006	-0.095	1.037	-0.174	0.167	0.096	-0.140	0.116	1.000
F = 0.75	-0.014	-0.075	1.030	-0.176	0.184	0.101	-0.162	0.111	1.000
F = 0.50	-0.028	-0.036	1.017	-0.180	0.219	0.112	-0.205	0.099	1.000

Table 3.9: Asset class allocations for portfolios with r_P = 11%, 1.7% above the w_{MVP} return and θ = 1. The portfolios shown in the table are: $w_{rp} = w_{MVP} + \theta w_{\eta}$, $w_{rp,S} = w_{MVP} + \theta\phi_{MSVP} + w_{\eta} + \theta w_{\eta,S}$. The liability hedge portfolio allocation ϕ_{MSVP} is found in table 3.5.

The expected return on surplus along with the risk and surplus risk values increase as the return requirement on the portfolios is increased. The covariance increase is hardly noticeable and as a consequence, the values for the liability hedging credit and the hedging ability do not change in the tables although a very small increase is noticed by increasing the number of digits.

Table 3.10: Results from the optimization model for portfolios with return requirement r_P = 11%						
Portfolio; P	w_{rp}	$w_{rp,S}$				
Variable	$F = \infty$	$F = 1.50$	$F = 1.25$	$F = 1.00$	$F = 0.75$	$F = 0.50$
$E\,[R_P\,(\phi_{MSVP})]$		0.001	0.001	0.002	0.002	0.003
$E\,[R_P\,(w_{MVP})]$, $E\,[R_P\,(w_{MSVP})]$	0.093	0.094	0.094	0.095	0.095	0.096
$E\,[R_P\,(w_\eta)]$, $E\,[R_P\,(w_{\eta,S})]$	0.017	-0.001	-0.001	-0.002	-0.002	-0.003
$E\,[R_P\,(w_{rp})]$, $E\,[R_P\,(w_{rp,S})]$	0.110	0.110	0.110	0.110	0.110	0.110
$\sigma\,[R_P\,(w_{rp})]$, $\sigma\,[R_P\,(w_{rp,S})]$	0.013	0.014	0.014	0.015	0.016	0.019
$E\,[R_S\,(w_{rp,S})]$		0.064	0.055	0.041	0.018	-0.028
$\sigma\,[R_S\,(w_{rp,S})]$		0.021	0.024	0.029	0.037	0.056
$Cov\,[R_{rp}\,, R_L]$, $Cov\,[R_{rp,S}\,, R_L]$	7.6.E-05	1.1.E-04	1.1.E-04	1.2.E-04	1.4.E-04	1.7.E-04
$Cov\,[R_{rp,S}\,, R_L]$ - $Cov\,[R_{MSVP}\,, R_L]$		5.2.E-06	5.2.E-06	5.1.E-06	4.9.E-06	4.6.E-06
LHC_P		1.4.E-04	1.8.E-04	2.5.E-04	3.7.E-04	6.9.E-04
$\sigma_P \rho_{P,L}$	2.6.E-03	3.7.E-03	3.9.E-03	4.2.E-03	4.8.E-03	5.9.E-03

Table 3.10: Results for portfolios with r_P = 11% and θ = 1. $E\,[R_P\,(w_{MSVP})] = E\,[R_P\,(w_{MVP})] + \theta E\,[R_P\,(\phi_{MVSP})]$, $E\,[R_P\,(w_{rp})] = E\,[R_P\,(w_{MVP})] + E\,[R_P\,(w_\eta)]$, $E\,[R_P\,(w_{rp,S})] = E\,[R_P\,(w_{MVP})] + \theta E\,[R_P\,(\phi_{MVSP})] + E\,[R_P\,(w_\eta)] + \theta E\,[R_P\,(w_{\eta,S})]$, $E\,[R_S\,(w_P)]$ - expected return on surplus, $\sigma\,[R_S\,(w_P)]$ - standard deviation of surplus return. $Cov[R_{rp,S}\,, L]$ - $Cov[R_{MSVP}\,, L]$ confirms an increase in return covariance as $r_P > r_{MVP}$ as table 3.4 suggests.

Increasing the return requirement to 12% as shown in tables 3.11 – 3.12, the same allocation change pattern is observed as before. Slight increase and decrease in the same asset classes as before serves the role of minimizing surplus return risk with respect to the increased return requirement on the optimal portfolio. Interestingly, the optimal portfolios in the presence of liabilities follow the same allocation change pattern with increased return requirement as the optimal portfolios in absence of liabilities. The presence of liabilities does not seem to change this pattern. Instead, the liabilities only affect allocation in order to increase the correlation between portfolio and liabilities returns as the objective of the surplus optimization model implies via minimizing surplus return variance.

Table 3.11: Asset allocation; w_{MVP}, w_η, $w_{\eta,S}$, w_{rp}, $w_{rp,S}$ with return requirement r_P = 12%									
	A 1	**A 2**	**A 3**	**A 4**	**A 5**	**A 6**	**A 7**	**A 8**	**Total**
w_{MVP}	0.014	-0.020	1.040	-0.083	0.033	0.014	-0.025	0.028	1.000
w_η	0.002	-0.213	0.027	-0.135	0.130	0.105	-0.082	0.167	0.000
w_{rp}	0.015	-0.233	1.067	-0.218	0.163	0.118	-0.107	0.195	1.000
$w_{\eta,S}$									
$F = 1.50$	0.000	0.008	-0.001	0.005	-0.005	-0.004	0.003	-0.006	0.000
$F = 1.25$	0.000	0.010	-0.001	0.006	-0.006	-0.005	0.004	-0.008	0.000
$F = 1.00$	0.000	0.012	-0.002	0.008	-0.007	-0.006	0.005	-0.009	0.000
$F = 0.75$	0.000	0.016	-0.002	0.010	-0.010	-0.008	0.006	-0.013	0.000
$F = 0.50$	0.000	0.024	-0.003	0.015	-0.015	-0.012	0.009	-0.019	0.000
$w_{rp,S}$									
$F = 1.50$	0.001	-0.194	1.054	-0.222	0.198	0.129	-0.150	0.184	1.000
$F = 1.25$	-0.002	-0.186	1.051	-0.223	0.205	0.131	-0.158	0.181	1.000
$F = 1.00$	-0.006	-0.174	1.047	-0.224	0.215	0.134	-0.171	0.178	1.000
$F = 0.75$	-0.013	-0.154	1.041	-0.226	0.233	0.140	-0.192	0.172	1.000
$F = 0.50$	-0.027	-0.115	1.027	-0.230	0.267	0.151	-0.235	0.161	1.000

Table 3.11: Asset class allocations for portfolios with r_P = 12%, 2.7% above the w_{MVP} return and θ = 1. The portfolios shown in the table are: $w_{rp} = w_{MVP} + \theta w_\eta$, $w_{rp,S} = w_{MVP} + \theta\phi_{MSVP} + w_\eta + \theta w_{\eta,S}$. The liability hedge portfolio allocation ϕ_{MSVP} is found in table 3.5.

Table 3.12: Results from the optimization model for portfolios with return requirement r_P = 12%						
Portfolio; P	w_{rp}	$w_{rp,S}$				
Variable	$F=\infty$	$F=1.50$	$F=1.25$	$F=1.00$	$F=0.75$	$F=0.50$
$E[R_P(\phi_{MSVP})]$		0.001	0.001	0.002	0.002	0.003
$E[R_P(w_{MVP})]$, $E[R_P(w_{MSVP})]$	0.093	0.094	0.094	0.095	0.095	0.096
$E[R_P(w_\eta)]$, $E[R_P(w_{\eta,S})]$	0.027	-0.001	-0.001	-0.002	-0.002	-0.003
$E[R_P(w_{rp})]$, $E[R_P(w_{rp,S})]$	0.120	0.120	0.120	0.120	0.120	0.120
$\sigma[R_P(w_{rp})]$, $\sigma[R_P(w_{rp,S})]$	0.016	0.017	0.017	0.018	0.019	0.022
$E[R_S(w_{rp,S})]$		0.074	0.065	0.051	0.028	-0.018
$\sigma[R_S(w_{rp,S})]$		0.023	0.026	0.030	0.039	0.056
$Cov[R_{rp}, R_L]$, $Cov[R_{rp,S}, R_L]$	7.9.E-05	1.1.E-04	1.2.E-04	1.3.E-04	1.4.E-04	1.7.E-04
$Cov[R_{rp,S}, R_L]$ - $Cov[R_{MSVP}, R_L]$		8.5.E-06	8.5.E-06	8.4.E-06	8.2.E-06	7.9.E-06
LHC_P		1.5.E-04	1.9.E-04	2.5.E-04	3.8.E-04	7.0.E-04
$\sigma_P \rho_{P,L}$	2.7.E-03	3.8.E-03	4.0.E-03	4.4.E-03	4.9.E-03	6.0.E-03

Table 3.12: Results for portfolios with r_P = 12% and θ = 1. $E[R_P(w_{MSVP})] = E[R_P(w_{MVP})] + \theta E[R_P(\phi_{MVSP})]$, $E[R_P(w_{rp})] = E[R_P(w_{MVP})] + E[R_P(w_\eta)]$, $E[R_P(w_{rp,S})] = E[R_P(w_{MVP})] + \theta E[R_P(\phi_{MVSP})] + E[R_P(w_\eta)] + \theta E[R_P(w_{\eta,S})]$, $E[R_S(w_P)]$ - expected return on surplus, $\sigma[R_S(w_P)]$ - standard deviation of surplus return. $Cov[R_{rp,S}, L]$ - $Cov[R_{MSVP}, L]$ confirms an increase in return covariance as $r_P > r_{MVP}$ as table 3.4 suggests.

As this example points out, full surplus optimization involves short sales of assets for diversification and to select the assets in such a way as to gain surplus on the liability benchmark with as little surplus volatility as possible, with respect to funding status and return preferences. The funding status determines the surplus return hedge, i.e. how much risk the plan can afford to take in gaining returns and how strong the hedge against surplus volatility should be in order to maintain surplus return at acceptable levels. As the funding ratio decreases, the ability to take risk decreases and so should the willingness to take risk also. Lower funding ratio increases the emphasis on hedging against changes in liabilities by increasing the allocation in assets that provide similarly behaving returns as the liabilities do and give the highest potential on receiving positive surplus returns with as little volatility as possible via diversification.

3.4 Market portfolios in absence and presence of liabilities

Since optimal portfolios with certain return requirement goals have been studied to some extent, the market portfolios in absence and presence of liabilities from section 2.12 remain untouched. Assuming a risk-free rate of 5.5%, little lower than the return estimate on domestic long term bonds, the market portfolios in absence (MKT) and presence (MKTS) of liabilities have been plotted on figure 3.6. Their expressions are, respectively,

$$w_{MKT,S} = \frac{\Sigma_A^{-1}\left(\mu_A - r_f \upsilon\right)}{\upsilon^T \Sigma_A^{-1}\left(\mu_A - r_f \upsilon\right)}$$

and

$$w_{MKT,S} = w_{MKT} + \theta\phi_{MKT} \quad \text{where} \quad \phi_{MKT} = \frac{1}{F}\left[\Sigma_A^{-1}\Sigma_{AL} - \frac{\upsilon^T\Sigma_A^{-1}\Sigma_{AL}}{\upsilon^T\Sigma_A^{-1}\left(\mu_A - r_f\upsilon\right)}\Sigma_A^{-1}\left(\mu_A - r_f\upsilon\right)\right]$$

The capital market line (CML) and the market portfolios are illustrated in figure 3.6. The decreasing return on the market portfolio in presence of liabilities as F decreases can be observed, as well as the expected return on the MSVP, the MKT and MKTS are the same at $F = F_{COV} = 0.6164$, denoted by the asterisk in figure 3.6. For a funding ratio below F_{COV}, the MKTS becomes surplus inefficient, i.e. is below the MSVP on the respective frontier.

The MKT and MKTS asset allocations are found in table 3.13. As the allocations in MKT and MKTS are compared, the effects of the liability hedging component decrease the allocation in five out of eight asset classes as F decreases. The most significant increase in allocation as F decreases is in foreign stocks (A2).

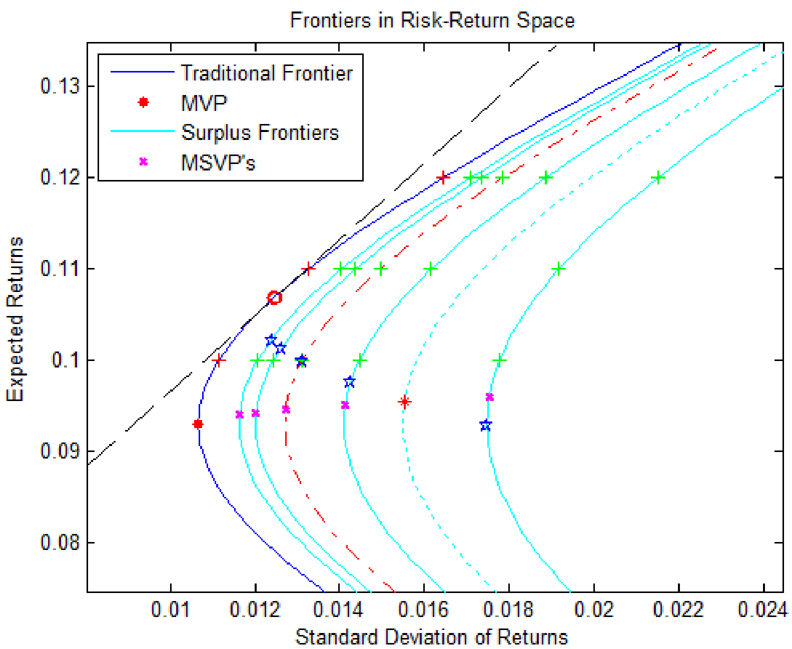


Figure 3.6: This is the same figure as 3.5 where market portfolio in absence of liabilities (red o) and market portfolios in presence of liabilities (blue pentagrams) with F = [1.5, 1.25, 1.0, 0.75, 0.5] have been added. Risk-free rate is assumed 5.5%, CML is the black dashed tangency line and portfolios with return requirement r_P = [0.10, 0.11, 0.12] are marked with red and green +'s. The red asterisk denotes the point in risk-return space where $w_{MKT,S} = w_{MSVP} = w_{COV}$ for $F = F_{COV} = 0.6164$

Table 3.13: Asset allocation; w_{MKT}, ϕ_{MKT}, $w_{MKT,S}$, and w_{COV}; r_f = 5,5%	A1	A2	A3	A4	A5	A6	A7	A8	Total
w_{MKT}	0.015	-0.130	1.054	-0.153	0.100	0.068	-0.067	0.114	1.000
ϕ_{MKT}									
F = 1.50	-0.014	0.076	-0.018	0.020	0.012	-0.008	-0.028	-0.040	0.000
F = 1.25	-0.017	0.092	-0.021	0.024	0.015	-0.009	-0.034	-0.048	0.000
F = 1.00	-0.022	0.115	-0.027	0.030	0.018	-0.011	-0.043	-0.060	0.000
F = 0.75	-0.029	0.153	-0.036	0.039	0.025	-0.015	-0.057	-0.080	0.000
F = 0.50	-0.043	0.229	-0.053	0.059	0.037	-0.023	-0.085	-0.121	0.000
$w_{MKT,S}$									
F = 1.50	0.000	-0.053	1.036	-0.133	0.112	0.060	-0.095	0.074	1.000
F = 1.25	-0.003	-0.038	1.032	-0.129	0.114	0.059	-0.101	0.066	1.000
F = 1.00	-0.007	-0.015	1.027	-0.123	0.118	0.056	-0.110	0.054	1.000
F = 0.75	-0.014	0.023	1.018	-0.113	0.124	0.053	-0.124	0.034	1.000
F = 0.50	-0.029	0.100	1.000	-0.094	0.137	0.045	-0.152	-0.007	1.000
w_{COV}									
F = 0.6164	-0.021	0.056	1.010	-0.105	0.130	0.049	-0.136	0.016	1.000

Table 3.13: Asset class allocation for w_{MKT}, ϕ_{MKT}, $w_{MKT,S}$ and w_{COV} with $\theta = 1$; $w_{MKT,S} = w_{MKT} + \theta\phi_{MKT}$, $w_{COV} = w_{MKT,S}$ for $F = F_{COV}$.

As can be seen in table 3.14, the same liability hedging tendencies apply in the case of market portfolio in presence of liabilities as with other surplus optimal portfolios. Lower funding ratio increases the need for hedging against liabilities and optimal allocation selects assets in such a way as to increase the return covariance between the asset portfolio and the liabilities. The last four rows in table 3.14 confirm this fact when compared with tables 3.6, 3.8, 3.10 and 3.12.

Table 3.14: Results from the optimization model for w_{MKT} and $w_{MKT,S}$ for selected funding ratios F, and w_{COV} for F_{COV}; r_f = 5,5%							
Portfolio; P	w_{MKT}	$w_{MKT,S}$					w_{COV}
Variable	$F = \infty$	F = 1.50	F = 1.25	F = 1.00	F = 0.75	F = 0.50	F = 0.6164
$E[R_P(\phi_{MKT})]$		-0.005	-0.006	-0.007	-0.009	-0.014	
$E[R_P(w_P)]$	0.107	0.102	0.101	0.100	0.098	0.093	0.096
$\sigma[R_P(w_P)]$	0.012	0.012	0.013	0.013	0.014	0.017	0.016
$E[R_S(w_P)]$		0.056	0.046	0.031	0.006	-0.045	-0.016
$\sigma[R_S(w_P)]$		0.020	0.023	0.028	0.037	0.055	0.045
$E[R_P(w_{MKT})]$ - $E[R_P(w_P)]$		0.005	0.006	0.007	0.009	0.014	0.002
$\sigma[R_P(w_{MKT})]$ - $\sigma[R_P(w_P)]$		8.1.E-05	-1.5.E-04	-6.5.E-04	-1.8.E-03	-5.0.E-03	3.1.E-03
$Cov[R_P, R_L]$	7.5.E-05	1.1.E-04	1.1.E-04	1.2.E-04	1.4.E-04	1.7.E-04	1.5.E-04
$Cov[R_P, R_L]$ - $Cov[R_{MKT}, R_L]$		3.0.E-05	3.7.E-05	4.6.E-05	6.1.E-05	9.1.E-05	7.4.E-05
LHC_P		1.4.E-04	1.8.E-04	2.4.E-04	3.6.E-04	6.6.E-04	4.8.E-04
$\sigma_P \rho_{P,L}$	2.6.E-03	3.6.E-03	3.8.E-03	4.1.E-03	4.6.E-03	5.7.E-03	5.1.E-03

Table 3.14: Results for w_{MKT}, $w_{MKT,S}$ and w_{COV} according to sections 2.6 and 2.12 with $\theta = 1$.
$E[R_P(w_{MKT,S})] = E[R_P(w_{MKT})] + \theta E[R_P(\phi_{MKT})]$, $E[R_S(w_P)]$ - expected returns on surplus, $\sigma[R_S(w_P)]$ - standard deviation of surplus returns, $Cov[R_P, L] - Cov[R_{MKT}, L]$ confirms an increase in return covariance between optimal portfolio and liabilities as F decreases due to increased hedging demand. LHC – Liability hedging credit.

3.5 Absolute minimum surplus variance

As shown in section 2.11, a unique funding ratio exists where the surplus return variance on the MSVP for that particular *F*, has an absolute minimum for all funding ratios. This funding ratio, F_{MSV}, has a relatively high value in this numerical example; F_{MSV} = 11.4371. since it is unlikely that any pension fund has the ability to cover its liabilities more than eleven times, this analysis is not practical in terms of real asset management, Nevertheless, it sheds a light on certain properties of the surplus optimization model and the data from a theoretical point of view. Table 3.15 shows the asset allocation for the absolute minimum surplus variance portfolio (MSV), expressed as

$$\begin{aligned} w_{MSV} &= w_{MSVP}\big|_{F=F_{MSV}} \\ &= \frac{\Sigma_A^{-1}\upsilon}{Q_{22}} + \frac{Q_{23}}{Q_{22}\sigma_L^2 - |Q_{\#4}|}\left[\Sigma_A^{-1}\Sigma_{AL} - \frac{Q_{23}}{Q_{22}}\Sigma_A^{-1}\upsilon\right] \end{aligned}$$

with

$$F_{MSV} = \frac{\theta\left(Q_{22}\sigma_L^2 - |Q_{\#4}|\right)}{Q_{23}}$$

Table 3.15: Asset allocation; w_{MSV}, risk and surplus risk decomposition, F_{MSV} = 11,4371									
	A 1	**A 2**	**A 3**	**A 4**	**A 5**	**A 6**	**A 7**	**A 8**	**Total**
w_{MSV}	0.012	-0.016	1.038	-0.084	0.038	0.016	-0.031	0.027	1.000

Table 3.15: Asset class allocation for w_{MSV} with θ = 1: Absolute minimum surplus variance portfolio. $w_{MSV} = w_{MSVP}$ for $F = F_{MSV}$.

Table 3.16: Minimum surplus variance comparison; w_{MSV} vs. w_{MVP}		
Variable	w_{MSV}	w_{MVP}
$E[R_P(w_P)]$	0.093	0.093
$\sigma[R_P(w_P)]$	0.011	0.011
$E[R_S(w_P)]$	0.087	
$\sigma[R_S(w_P)]$	0.010	
$Cov[R_P, R_L]$	7.4.E-05	7.0.E-05
LHC_P	1.3.E-05	
$\sigma_P\rho_{PL}$	2.5.E-03	2.4.E-03

Table 3.16: Comparison of values; w_{MSV} vs. w_{MVP}

The absolute minimum surplus variance was expressed as

$$\begin{aligned} VAR\left[R_P\left(w_{MSV}\right)\right] &= \sigma_{MSV}^2 \\ &= \frac{1}{Q_{22}}\left(1 + \frac{Q_{23}^2|Q_{\#4}|}{\left(Q_{22}\sigma_L^2 - |Q_{\#4}|\right)^2}\right) \end{aligned}$$

Table 3.16 shows numerical values for the MSV portfolio. As the F_{MSV} is high, the MSV portfolio is very similar to the MVP since concentration in liability hedge is weak due to the high funding ratio. The difference in expected returns and standard deviation of returns cannot be noticed using three digits in respective numerical results for these portfolios. Small difference in return covariance between the portfolios and liabilities can though be observed from table 3.16.

Figure 3.7 shows surplus return variance as a function of funding ratio. From the figure, the absolute minimum surplus return variance resulting from w_{MSV} can be observed.

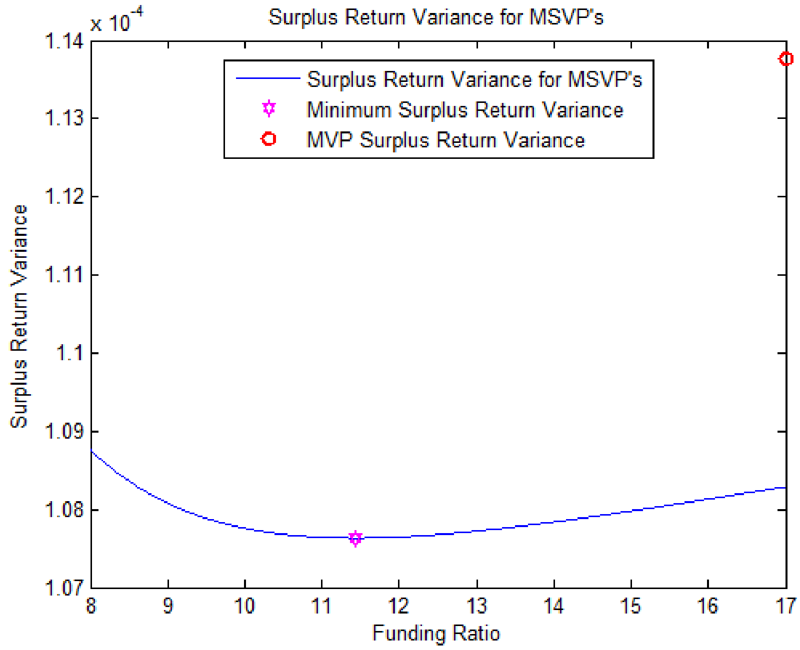


Figure 3.7 : An absolute minimum of surplus return variance for all optimal portfolios is the surplus return variance of w_{MSV} (magenta colored hexagram). For comparison, the surplus return variance for w_{MVP} is shown in the upper right corner (red o). The surplus return risk of these two portfolios is also shown in figure 3.8.

3.6 Surplus risk return space

Since the MSV portfolio has been added to this analysis, it‘s now timely to illustrate the surplus risk-return space. The classic risk-return space illustrates the quadratic relationship between expected return, $E\left[R_P\left(w_P\right)\right]$, and standard deviation of returns, $\sigma\left[R_P\left(w_P\right)\right]$, whereas in the surplus risk-return space, the quadratic relationship between expected surplus return and standard deviation of surplus returns are illustrated, i.e.

$$E\left[R_P\left(w_P\right)-\frac{\theta}{F}R_L\right],\quad \sigma\left[R_P\left(w_P\right)-\frac{\theta}{F}R_L\right]$$

Figure 3.8 illustrates the surplus risk-return space, where all the portfolios in absence and presence of liabilities already analyzed in this chapter are shown. Figure 3.8 is the surplus risk-return counterpart of figure 3.6 where expected surplus returns and standard deviations of surplus returns for all previously analyzed portfolios are projected into surplus risk-return space. The blue leftmost frontier is the traditional asset-only frontier with the MVP as a red dot. Six surplus optimal frontiers for funding ratios of F = [1.5, 1.25, 1.0, 0.75, 0.5] and for F_{COV} = 0.6164 are shown with respective MSVPs. The portfolios with return requirement

r_P = [0.10, 0.11, 0.12] and market portfolios in absence and presence of liabilities are shown on respective frontiers.

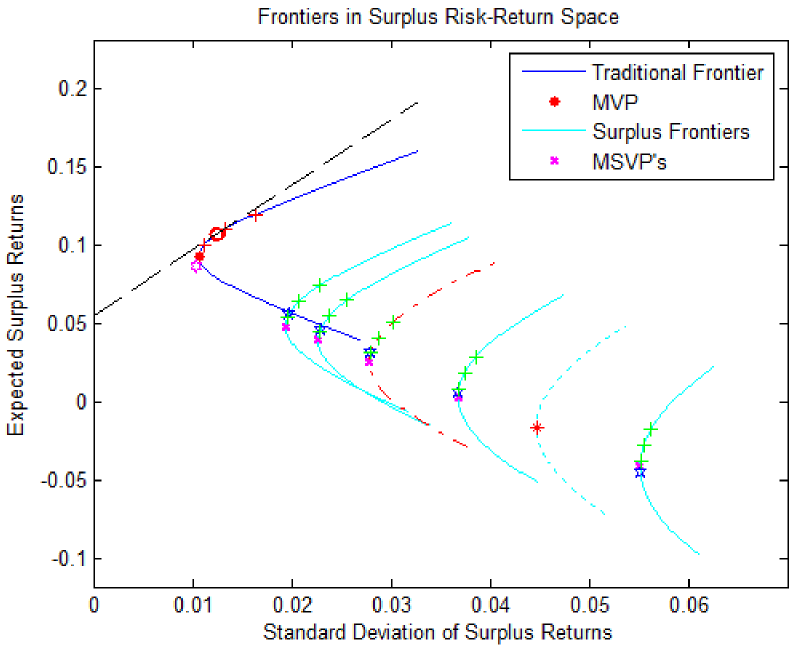


Figure 3.8: Surplus risk-return frontiers in absence of liabilities (leftmost frontier, blue with w_{MVP} as a red dot) and presence of liabilities (red/cyan with w_{MSVP}'s as magenta coloured x's) for F = [1.5, 1.25, 1.0, 0.75, 0.5] and for F_{COV} = 0.6164. The red dash-dotted surplus frontier is for F = 1, the surplus frontiers to the left of the red dash-dotted one have $F > 1$ and the frontiers to the right of the red dash-dotted one have $F < 1$. The unit normalized return covariance portfolio, w_{COV}, is shown by a red asterisk on the dashed surplus frontier for F_{COV} = 0.6164. Portfolios with return requirement r_P = [0.10, 0.11, 0.12] are marked with red and green +'s, the market portfolio in absence of liabilities (red o) and market portfolios in presence of liabilities (blue pentagrams) with F = [1.5, 1.25, 1.0, 0.75, 0.5]. Risk-free rate is assumed 5.5% and CML is the black dashed tangency line. The magenta coloured hexagram represents w_{MSV} with F_{MSV} = 11.43, having almost the same risk and return as the w_{MVP}. This figure and figure 3.6 are counterparts; figure 3.6 in risk-return space and figure 3.7 in surplus risk return space.

3.7 Probabilities of assets covering liabilities

In section 2.13, a simple probability measure on gaining surplus, i.e. whether assets will cover liabilities in the end of a specified horizon was derived. As already discussed section 2.13, current funding ratio and present asset allocation can give probabilistic information on the on the ability of a pension fund to cover its liabilities. For the hypothetical pension fund in this numerical example, the probabilities of covering liabilities at a certain time point in future are analyzed in tables 3.17 – 3.20. Since the MSVPs and optimal portfolios with return requirement of 10%, 11% and 12% were analyzed before in this chapter, it's convenient to observe the probabilistic measure for these portfolios. The funding ratios used in this probability analysis are F = [1, 0.75, 0.5] and the probability values in tables 3.17 – 3.20 are found by using the results of section 2.13. As can be seen from the tables, higher funding ratios are quite likely to cover liabilities at the end of short horizons as the growth rate of the asset portfolio is higher than of the liabilities. Also, higher return requirement on the portfolios results here in higher probabilities of covering liabilities. It should be noted that surplus return volatility plays a great role here. Where the surplus return volatility values for the optimal portfolios are low and F is fairly close to 1, the probability values are high. For low values of surplus return volatility, changes in F and the importance parameter θ can lead to large changes in this probability measure, given by

$$
\begin{aligned}
&P\left(A_w(T) \geq \theta L(T)\right) \\
&\qquad = P\left(S(T) \geq 0\right) \\
&\qquad = \Phi\left(\frac{\ln \frac{F(t)}{\theta} + \left(\mu_P^T w_P - \theta \mu_L - \frac{1}{2}\left(\sigma_P^2 - \theta^2 \sigma_L^2\right)\right)(T-t)}{\left(\left(w_P^T \Sigma_A w_P + \theta^2 \sigma_L^2 - 2\theta \Sigma_{AL}^T w_P\right)(T-t)\right)^{1/2}} \right)
\end{aligned}
$$

Table 3.17: Probabilities $P(A > \theta L)$ for w_{MSVP}

	w_{MSVP}		
$T - t$	$F = 1.00$	$F = 0.75$	$F = 0.50$
1	0.826	5.E-21	3.E-119
2	0.908	1.E-09	6.E-56
3	0.948	9.E-06	6.E-35
4	0.970	0.001	2.E-24
5	0.982	0.007	3.E-18

Table 3.17: $P(A > \theta L)$, $t = 0$, w_{MSVP}.

Table 3.18: Probabilities $P(A > \theta L)$ for portfolios with return requirement $r_P = 10\%$

	$w_{rp,S}$		
$T - t$	$F = 1.00$	$F = 0.75$	$F = 0.50$
1	0.865	3.E-20	3.E-117
2	0.941	6.E-09	2.E-54
3	0.972	3.E-05	1.E-33
4	0.986	0.002	2.E-23
5	0.993	0.017	3.E-17

Table 3.18: $P(A > \theta L)$, $t = 0$, $w_{rp,S}$, $r_P = 10\%$.

Table 3.19: Probabilities $P(A > \theta L)$ for portfolios with return requirement $r_P = 11\%$

	$w_{rp,S}$		
$T - t$	$F = 1.00$	$F = 0.75$	$F = 0.50$
1	0.922	6.E-18	5.E-108
2	0.978	2.E-07	1.E-48
3	0.993	5.E-04	4.E-29
4	0.998	0.016	2.E-19
5	0.999	0.099	8.E-14

Table 3.19: $P(A > \theta L)$, $t = 0$, $w_{rp,S}$, $r_P = 11\%$.

Table 3.20: Probabilities $P(A > \theta L)$ for portfolios with return requirement $r_P = 12\%$

	$w_{rp,S}$		
$T - t$	$F = 1.00$	$F = 0.75$	$F = 0.50$
1	0.954	3.E-15	2.E-95
2	0.991	7.E-06	1.E-41
3	0.998	0.005	5.E-24
4	1	0.083	2.E-15
5	1	0.314	1.E-10

Table 3.20: $P(A > \theta L)$, $t = 0$, $w_{rp,S}$, $r_P = 12\%$.

3.8 Shortfall constraints

In section 2.14, shortfall constraints were introduced as an additional tool for risk-return and surplus risk-return analysis. Conventional shortfall constraints limit the probability on earning a return on a portfolio below some specified threshold level. Under the log-returns normality assumption, the continuously compounded returns are normally distributed. As the returns here are logarithmic, an example of two shortfall constraints is added to this numerical example. The constraints are expressed as

Table 3.21: Shortfall constraints $P(R_P < R_{thr}) < \xi$		
$T = 1$	$\xi = 0{,}01$	
R_{thr}	0.07	0.08
$E[R_P]_{min}$	0.095	0.113
$\sigma[R_P]_{min}$	0.011	0.014
$E[R_P]_{max}$	0.304	0.207
$\sigma[R_P]_{max}$	0.099	0.054

Table 3.21: Traditional shortfall constraints $P(R_P \leq R_{thr}) \leq \xi$.

$$\frac{R_{thr} - \left(E\left[R_P\left(w_P\right)\right] - \frac{1}{2}\sigma\left[R_P\left(w_P\right)\right]^2 \right)(T-t)}{\sigma\left[R_P\left(w_P\right)\right]\sqrt{T-t}} \leq z_\xi$$

with $E\left[R_P\left(w_P\right)\right] = \mu_A^T w_P$ and $\sigma^2\left[R_P\left(w_P\right)\right] = w_P^T \Sigma_A w_P$

Table 3.21 and figure 3.9 show the two shortfall constraints and approximate values of expected returns and standard deviation of returns on optimal portfolios when the constraints are active. The constraint with higher threshold return narrows the set of feasible portfolios as can be observed form table 3.21. This is confirmed by the approximated constraint lines in figure 3.9.

The two dotted lines in figure 3.9 represent the shortfall constraints having threshold returns of 0.08 and 0.07, respectively. The constraint with higher threshold return drives the shortfall constrained feasible set very close to the frontier. As a consequence, very limited flexibility in allocation away from optimal portfolios is allowed if the shortfall constraint is to be satisfied. Only one of the portfolios analyzed before in this chapter is feasible; the optimal portfolio in absence of liabilities with return requirement of 12%. Raising the return requirement of the surplus optimal portfolios to about 15% allows the surplus optimal portfolio with F = 1 to satisfy the constraint. Reducing the threshold return down to 7% increases the feasible area under the efficient frontier considerably and several of the surplus optimal portfolio analyzed before satisfy the shortfall constraint. Increasing the shortfall probability level above 1% also allows for larger area of feasibility under the frontiers.

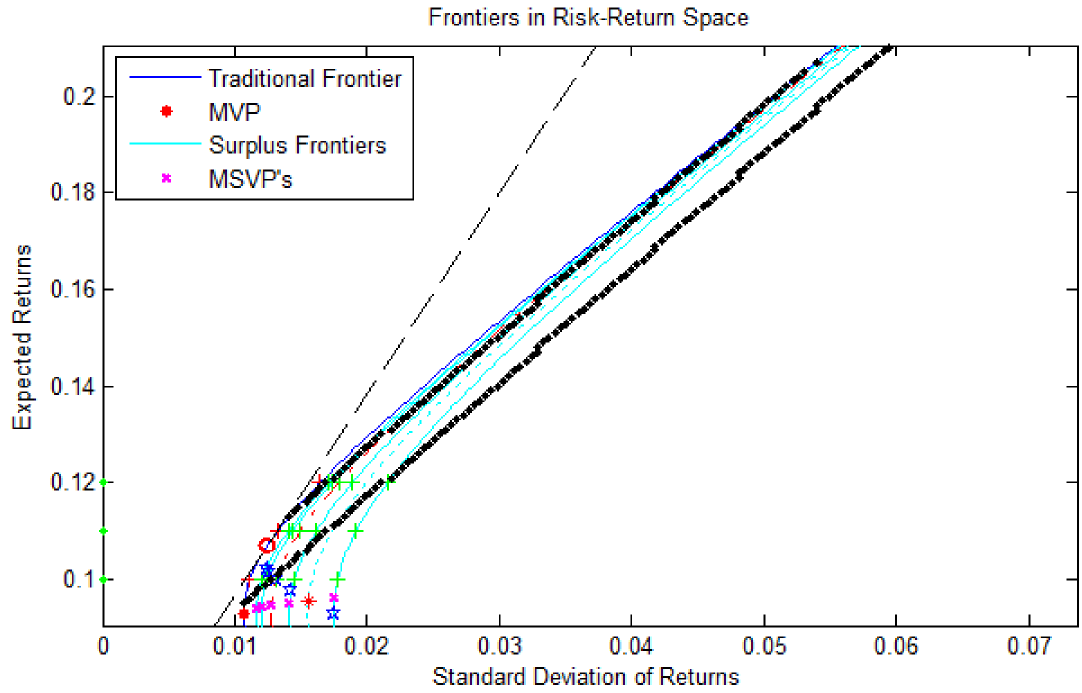


Figure 3.9: The same figure as figure 3.6 where traditional shortfall constraint lines are approximated: $P(R_P \leq R_{thr}) \leq \xi$, ξ = 0.01,
$t = 0$, $T = 1$, $R_{thr} = [0.07, 0.08]$.

The second type of shortfall constraints from section 2.14 limits the probability of the funding ratio being below some threshold funding ratio at the end of a specified horizon. If this funding ratio is 1, the shortfall constraints are linear in the surplus risk return space. Tables 3.22 and 3.23 provide examples of such shortfall constraints, expressed as

$$\frac{\ln\frac{\theta\bar{F}}{F(t)} - \left(\mu_A^T w_P - \theta\mu_L - \frac{1}{2}\left(\sigma_P^2 - \theta^2\sigma_L^2\right)\right)(T-t)}{\sigma\left[R_P\left(w_P\right) - \frac{\theta\bar{F}}{F(t)}R_L\right]\sqrt{T-t}} \leq z_\delta$$

with $z_\delta = \Phi^{-1}(0,1)$

Table 3.22: Shortfall constraints $P(F(T)<1)<\delta,\ \delta=0.1$			
$T=5$	$F=1.25$	$F=1.00$	$F=0.75$
$E[R_P]_{min}$	0.043	0.085	0.152
$\sigma[R_P]_{min}$	0.026	0.013	0.031
$E[R_S]_{min}$	-0.012	0.016	0.061
$\sigma[R_S]_{min}$	0.033	0.028	0.045

Table 3.22: Funding ratio shortfall constraints
$P(F(T) \leq 1) \leq \delta$, varying funding ratio, $F(t)$.

Table 3.23: Shortfall constraints $P(F(T)<1)<\delta,\ F=1.00$			
$T=3$	$\delta=0.01$	$\delta=0.05$	$\delta=0.10$
$E[R_P]_{min}$	0.107	0.095	0.089
$\sigma[R_P]_{min}$	0.014	0.013	0.013
$E[R_S]_{min}$	0.038	0.026	0.020
$\sigma[R_S]_{min}$	0.028	0.028	0.028

Table 3.23: Funding ratio shortfall constraints
$P(F(T) \leq 1) \leq \delta$, varying shortfall probabilities, δ.

The approximated shortfall constraint lines accompanied with table 3.22 are illustrated in figures 3.10 and 3.11. Similarly, the approximated shortfall constraint lines accompanied with table 3.23 are illustrated in figures 3.12 and 3.13. Figures 3.10 and 3.12 show nonlinear constraint lines in risk-return space whereas figures 3.11 and 3.13 show linear constraint lines in surplus risk-return space. In table 3.22 and figures 3.10 – 3.11, the funding ratio is varied while the horizon and shortfall probability are held fixed. In table 3.23 and figures 3.12 – 3.13, the shortfall probability is varied while the horizon and the funding ratio are held fixed. As shown by table 3.22, lower funding ratio decreases the feasible area under the frontiers in both risk-return spaces by raising the constraint lines in figures 3.10 – 3.11. For table 3.23 and figures 3.12 – 3.13, decreasing the shortfall probability level also decreases the feasible area under the frontiers in both risk-return spaces by raising the constraint lines in figures 3.12 – 3.13.

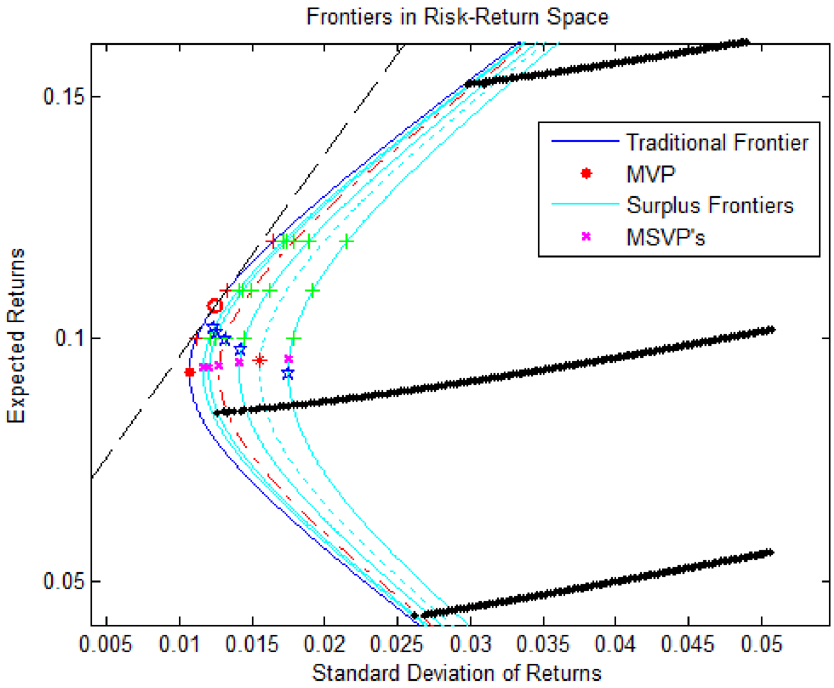


Figure 3.10: The same figure as figure 3.6 where funding ratio shortfall constraint lines are approximated: $P(F\,(T) \leq 1) \leq \delta$, $\delta = 0.1$, $t = 0$, $T = 5$, $F(t) = [0.75, 1.00, 1.25]$. The lowest shortfall constraint line on the figure is for $F(t) = 1.25$, the middle one is for $F(t) = 1$ and the uppermost one is for $F(t) = 0.75$.

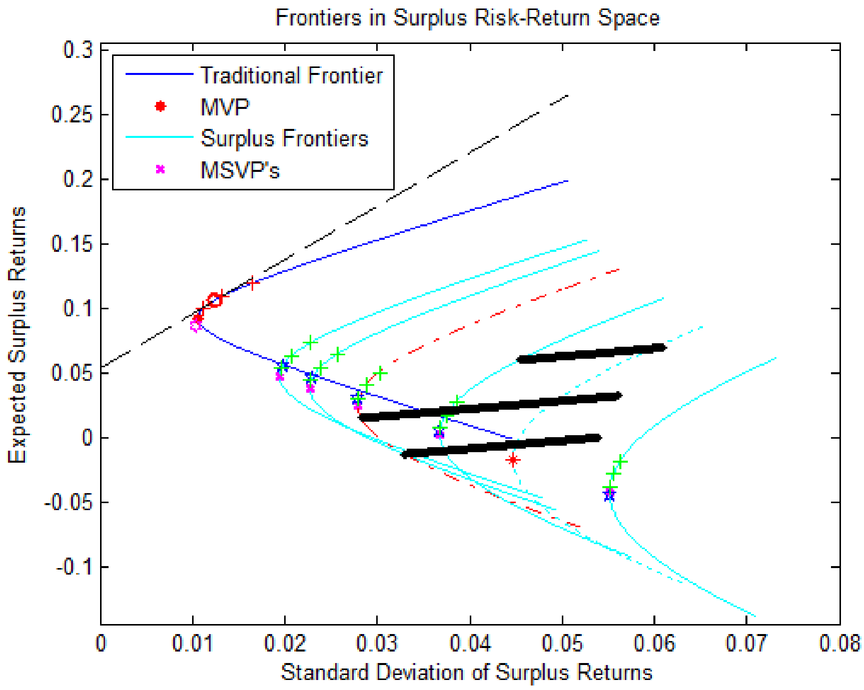


Figure 3.11: The surplus risk return space counterpart to figure 3.10 where funding ratio shortfall constraint lines are approximated: $P(F\,(T) \leq 1) \leq \delta$, $\delta = 0.1$, $t = 0$, $T = 5$, $F(t) = [0.75, 1.00, 1.25]$. The lowest shortfall constraint line on the figure is for $F(t) = 1.25$, the middle one is for $F(t) = 1$ and the uppermost one is for $F(t) = 0.75$.

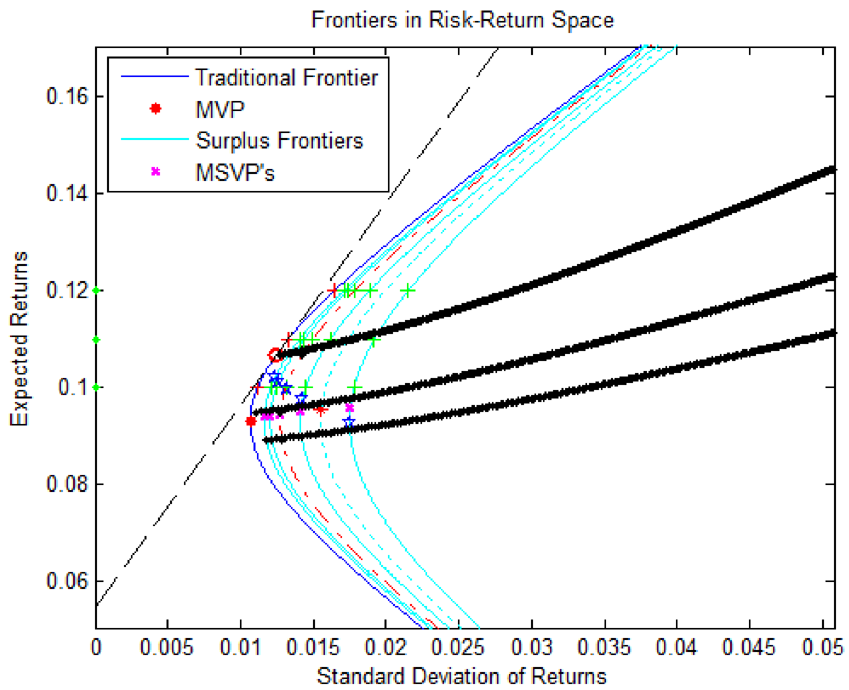


Figure 3.12: The same figure as figure 3.6 where funding ratio shortfall constraint lines are approximated: $P(F\ (T) \leq 1) \leq \delta$, $\delta\ = [0.01, 0.05, 0.1]$, $t = 0$, $T = 5$, $F(t) = 1.00$. The lowest shortfall constraint line on the figure is for $\delta = 0.1$, the middle one is for $\delta = 0.05$ and the uppermost one is for $\delta = 0.01$.

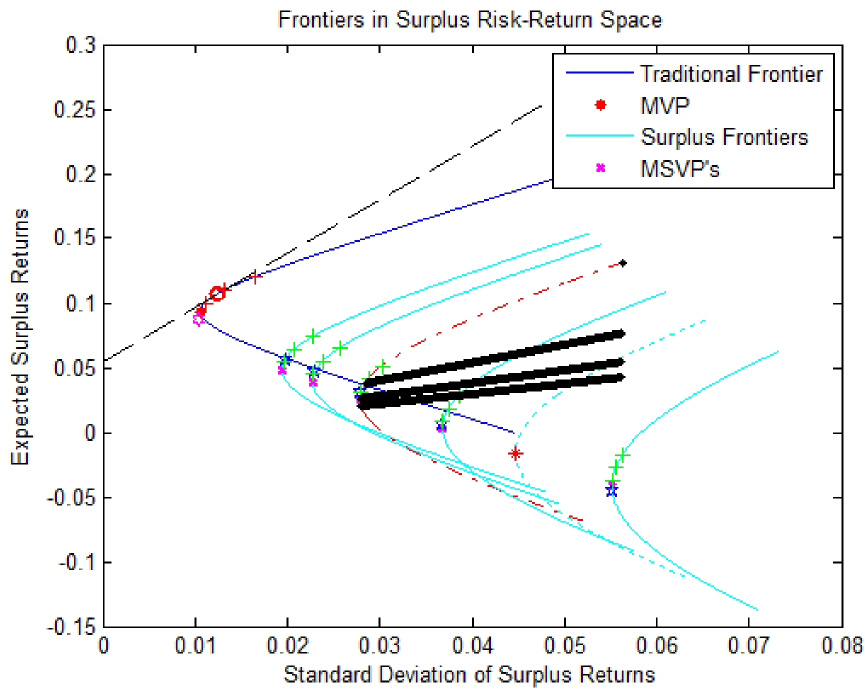


Figure 3.13: The surplus risk return space counterpart to figure 3.12 where funding ratio shortfall constraint lines are approximated: $P(F\ (T) \leq 1) \leq \delta$, $\delta\ = [0.01, 0.05, 0.1]$, $t = 0$, $T = 5$, $F(t) = 1.00$. The lowest shortfall constraint line on the figure is for $\delta = 0.1$, the middle one is for $\delta = 0.05$ and the uppermost one is for $\delta = 0.01$.

3.9 Surplus analysis for optimal allocations in presence versus absence of liabilities

In these final pages of the numerical example, optimal allocations are tested on historical data for the purposes of comparing the performance of and the surplus generated by optimal portfolios derived in chapter 2 and analyzed earlier in this section. In order to compare the performance of the optimal asset allocations in absence and presence of liabilities in parallel with the liabilities and their surplus generating ability, four allocation strategies are analyzed along with the liability index. To take account of different funding status for the hypothetical pension fund, the comparison is made for three different initial funding ratios. Also, three different rebalancing frequencies are implemented into the performance comparison and accordingly, the asset allocation process can be considered as semi-dynamic since the optimal portfolios are rebalanced during the historical test period. A detailed description of this surplus and performance analysis follows below.

The four optimal allocation strategies under consideration are the minimum variance portfolio (MVP) and the minimum surplus variance portfolio (MSVP) and optimal portfolios in absence (w_{rp}) and presence ($w_{rp,S}$) of liabilities, with a return requirement of 12%. Initial funding ratios for the strategies are F = [0.75, 1, 1.25]. The same data for liabilities and asset classes is used as introduced in section 3.1 and accordingly, initial allocations for all strategies are the same as those given in tables 3.5 and 3.11. Initially, the index level of liabilities in June 2008 is set as 100 and the asset value for all portfolio strategies is set accordingly with respect to the funding ratio. Using additional historical return data from July 2008 to January 2012 for the liabilities and the asset classes, the portfolio and the liability indices are plotted in parallel for a comparison between the strategies and comparing the strategies with the liabilities. This is done for portfolio rebalancing on annual, semi-annual and monthly basis as to compare the performance with respect to rebalancing frequency. When portfolios are rebalanced, the most recent historical data is added to the dataset for estimating new inputs into the optimization. The allocation process takes account of the funding status for the strategies when portfolios are rebalanced and allocates the assets accordingly.

The performance of the allocation strategies and the liability index is illustrated on even numbered figures from 3.14 to 3.30. In the figures legends, „Optimal Portfolio“ and „Surplus Optimal Portfolio“ refer to the optimal portfolio in absence (w_{rp}) and presence ($w_{rp,S}$) of liabilities, respectively. The surplus associated with the allocation strategies is shown on odd numbered figures from 3.15 to 3.31. The first six figures (3.14 – 3.19) illustrate the performance of the allocation strategies with rebalancing on an annual basis and associated surplus. The Next six figures (3.20 – 3.25) illustrate the same with rebalancing on a semi-annual basis and the last six figures (3.26 – 3.31) illustrate performance and surplus with rebalancing on a monthly basis. For a numerical comparison between strategies from July 2008 to January 2012 and rebalancing frequencies, tables 3.24 – 3.26 show average surplus associated with the strategies for rebalancing frequency of one year, half year and one month, respectively.

The performance patterns for the four strategies are similar in all cases, i.e. for all three funding ratios and all three rebalancing frequencies, with few exemptions. In the beginning of the time period under consideration, just before the 2008 market collapse, the performance and surplus of the MVP indices are just a little above the MSVPs. When the market downswing starts in the last quarter of 2008, the MSVP indices show better performance than their MVP counterparts and drop less in value. In all cases, the growth of the MSVPs are a little higher on average than for the MVPs during 2009 which results in higher index value in the end of 2009. For the remainder of the time period, the MSVP indices grow a little faster than the MVP indices. From this, one can conclude that based on this data, the MSVPs outperforms the MVPs in terms of surplus and portfolio value during the test period. This is also confirmed by tables 3.24 – 3.26 where the average surplus for the MSVPs is higher than for the MVPs. For a funding ratio of 0.75, where both the MVPs and the MSVPs start with a negative surplus (deficit) of - 25 in June 2008, the average surplus values in tables 3.24 – 3.26 associated with the MSVP are higher than the initial value, despite the severe market downswing in 2008.

The same characteristics are observed when comparing the optimal portfolios in absence w_{rp} and presence $w_{rp,S}$ of liabilities with a return requirement of 12%. When the markets collapse in October 2008, the index values of $w_{rp,S}$ exceed those of the w_{rp}, they drop less during the crisis and grow faster on average from March 2009 to January 2012. By comparing the surplus of the strategies, the odd numbered surplus figures and table 3.24 – 3.26 indicate a better performance in terms of surplus for the portfolios taking liabilities into account. Furthermore, the $w_{rp,S}$ performance and surplus indices ascend above the respective MVP indices temporarily in the years 2010 or 2011, except for funding ratios of 1 and 1.25 and semi-annual rebalancing and also for funding ratio of 1.25 and monthly rebalancing. The effects of increased funding ratio result in that the $w_{rp,S}$ index and surplus values exceed the

Table 3.24: Average Surplus (AS), r_P = 12%, Annual Rebalancing

	Jun. 08 - Jan. 12		
Portfolio	F = 0.75	F = 1.00	F = 1.25
w_{MVP}	-24.55	3.65	31.85
w_{MSVP}	-21.49	6.69	34.88
w_{rp}	-27.87	-0.78	26.31
$w_{rp,S}$	-24.45	2.62	29.70

Table 3.25: Average Surplus (AS), r_P = 12%, Semi-Annual Rebalancing

	Jun. 08 - Jan. 12		
Portfolio	F = 0.75	F = 1.00	F = 1.25
w_{MVP}	-23.79	4.66	33.11
w_{MSVP}	-22.27	6.18	34.63
w_{rp}	-28.14	-1.14	25.86
$w_{rp,S}$	-26.22	0.78	27.78

Table 3.26: Average Surplus (AS), r_P = 12%, Monthly Rebalancing

	Jun. 08 - Jan. 12		
Portfolio	F = 0.75	F = 1.00	F = 1.25
w_{MVP}	-23.35	5.24	33.84
w_{MSVP}	-20.42	8.17	36.76
w_{rp}	-28.04	-1.00	26.03
$w_{rp,S}$	-24.84	2.19	29.21

Tables 3.24, - 3.25: Average surplus for the four strategies, three funding ratios and three various rebalancing frequencies.

respective MVP values later in the year 2010 or 2011, which results from lower concentration in liability hedge portfolio.Nevertheless, for all three funding ratios and rebalancing frequencies, the strategies taking liabilities into account outperform their asset-only counterparts in terms of average surplus and also in terms of portfolio values.

By comparing the values for average surplus in tables 3.24 – 3.26 to analyze the effects of varying the rebalancing frequency, it can be seen that the MVP benefits in terms of average surplus from more frequent rebalancing whereas for the w_{rp}, average surplus is highest for annual rebalancing and lowest for semi-annual rebalancing. The average surplus for the MSVPs is highest for monthly rebalancing but lowest for semi-annual rebalancing. For the $w_{rp,S}$, annual rebalancing results in the highest average surplus. As more frequent updating of information and allocating accordingly is generally considered to result in better performance, this time period under consideration is relatively volatile in historical comparison and includes the greatest market collapse in the history of domestic (Icelandic) markets. As a result, rebalancing on the basis of updated information may not have given the best results in the unstable environment of the three years that followed the market collapse in 2008.

The same performance pattern is observed in all cases as the MSVPs and MVPs strategies are superior to the strategies with the return requirement preferences through the most volatile period. Increased funding ratio affects performance which can be observed by that the index value of the $w_{rp,S}$ does not exceed those of the MVPs until in the last quarter of 2010 in the occasions where that happens. Higher funding ratio seems to delay the index value of the $w_{rp,S}$ in exceeding the MVPs, due to lower concentration in liability hedge. Nevertheless, the optimal strategies taking liabilities into account outperform their asset-only counterparts in all cases in terms of surplus. Therefore, using this data and considering the ability of the strategies in keeping pace with liabilities for the years 2010 - 2012, the strategies where liabilities are considered are superior to their classic asset-only counterparts for all funding ratios and all rebalancing frequencies.

The sharp increase in liabilities in 2011 has a negative effect on surplus and can be observed in respective figures as a sharp drop in surplus for all strategies during in the latter half of the year 2011. Nevertheless, the strategies considering liabilities maintain higher surplus than their classic asset-only counterparts as indicated by the odd numbered figures from 3.15 to 3.31.

Given the data used in this analysis, the purpose of this section is to compare the surplus associated with the strategies where liabilities are taken into account with the surplus of those who don‘t on the basis of different funding ratios and rebalancing frequencies, irrespective of the market downturn in 2008. As already discussed, the superiority of the strategies considering liabilities over their asset-only counterparts in gaining surplus is confirmed by tables 3.24 – 3.26 and the odd numbered figures from 3.15 – 3.31. Interestingly, it can also be observed from the even numbered figures that a noticeable difference between the strategies in absence on one hand and presence of liabilities on the other, is that the portfolios taking liabilities into account seem to recover faster after the 2008 crisis and also, they grow a little faster on average than the asset-only strategies in the post-crisis era. Therefore, given the data used for this analysis, the surplus optimal strategies outperform the traditional mean-variance strategies in terms of surplus and provide better hedge against liabilities. Additionally, they seem to outperform the traditional strategies in terms of recovery after the 2008 crisis which can be largely explained with very little or no investment at all in domestic stocks that collapsed in value in October 2008 and equities in many firms became worthless. Also, greater exposure in domestic inflation indexed bonds provides partial explanation of the faster recovery as domestic inflation increased dramatically after the market collapse.

Given the data used in this numerical example, three cases have been analyzed in this section where the asset allocation strategies taking liabilities into account are found by using surplus optimization. The results indicate that the strategies where liabilities are taken into account in the asset allocation process – surplus return variance optimal portfolios – are superior in keeping pace with liabilities than their classic mean-variance counterparts and are thus have the potential of providing higher returns on surplus by benchmarking asset allocation with liabilities. Using other data might show the opposite since for normal market conditions, surplus optimal strategies are more conservative and allocate higher proportions in bonds and asset classes with growth similar to the growth of liabilities. Also, in normal market conditions, one might expect that the order of portfolios in terms of performance would be the other way around, i.e. the portfolios without a liability benchmark would have a superior performance to those who have such a benchmark. Furthermore, in a stable environment, one might expect that portfolios with higher return requirement would have a better performance than minimum variance portfolios. The explanation for the opposite pattern observed in the previous comparison can be seen directly from the different objectives of the two models. The traditional mean-variance optimization model aims at minimizing the portfolio return variance with respect to return preferences and selects the assets accordingly. Conversely, the objective of surplus optimization is to minimize surplus return variance, i.e. the return variance of assets net of liabilities, with respect to funding status, return preferences and importance given to liabilities. In other words, positive surplus return is desirable with as little surplus return volatility as possible. Since the liability index in this analysis had a stable growth with a low volatility, benchmarking portfolio returns with this liability index resulted in an investment strategy that suffered less from the 2008 market downswing than the traditional mean-variance strategies. This might suggest that for periods of market volatility, benchmarking portfolios with steady-growth and low-volatile indices could reduce the negative effects of market downturns.

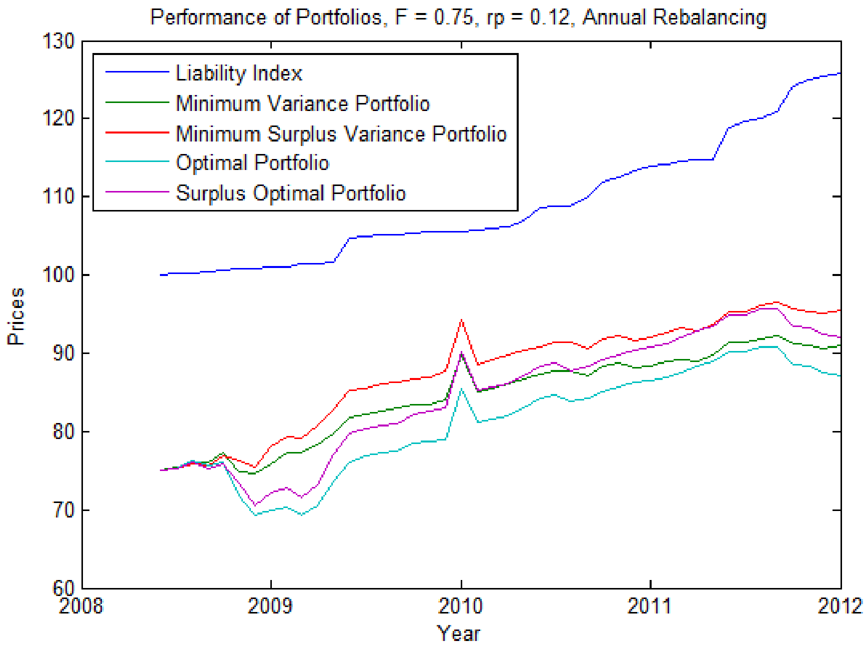


Figure 3.14: Four asset allocation strategies versus the liability index during the period June 2008 to January 2012, where initial funding status is 0.75 and the asset portfolios are rebalanced on an annual basis. Initial index level of liabilities in June 2008 is 100. The allocations for the minimum variance portfolio and minimum surplus variance portfolio found in table 3.5 and table 3.11 show the allocations for the optimal portfolios in absence and presence of liabilities with a return requirement of 12%.

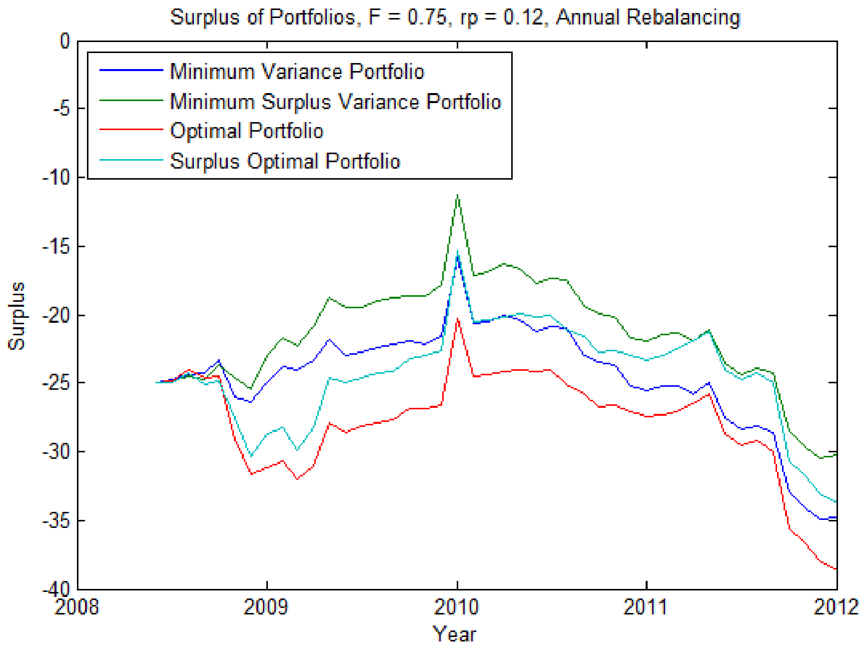


Figure 3.15: The surplus counterpart to figure 3.14 where surplus for the portfolio strategies is shown.

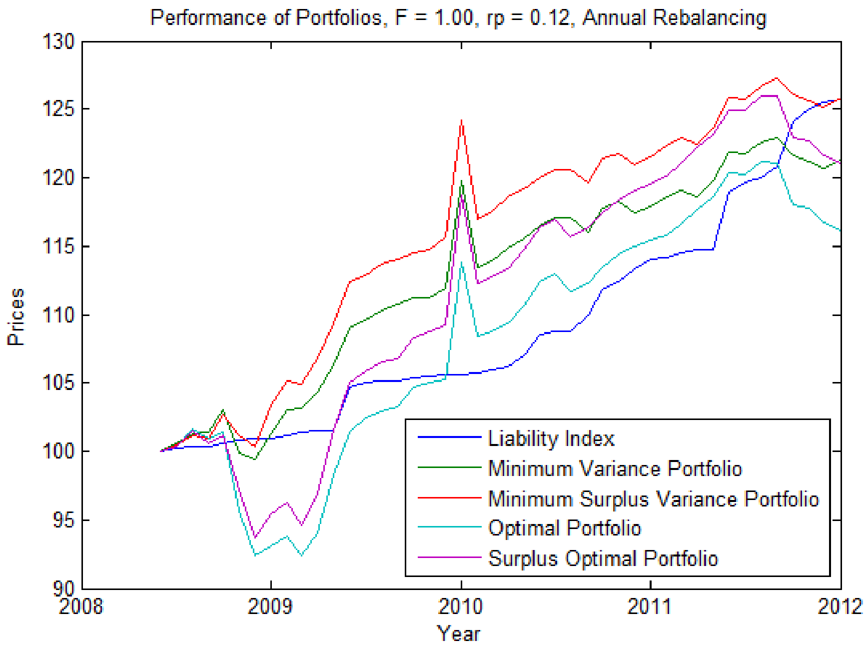


Figure 3.16: Four asset allocation strategies versus the liability index during the period June 2008 to January 2012, where initial funding status is 1 and the asset portfolios are rebalanced on an annual basis. Initial index level of liabilities in June 2008 is 100. The allocations for the minimum variance portfolio and minimum surplus variance portfolio found in table 3.5 and table 3.11 show the allocations for the optimal portfolios in absence and presence of liabilities with a return requirement of 12%.

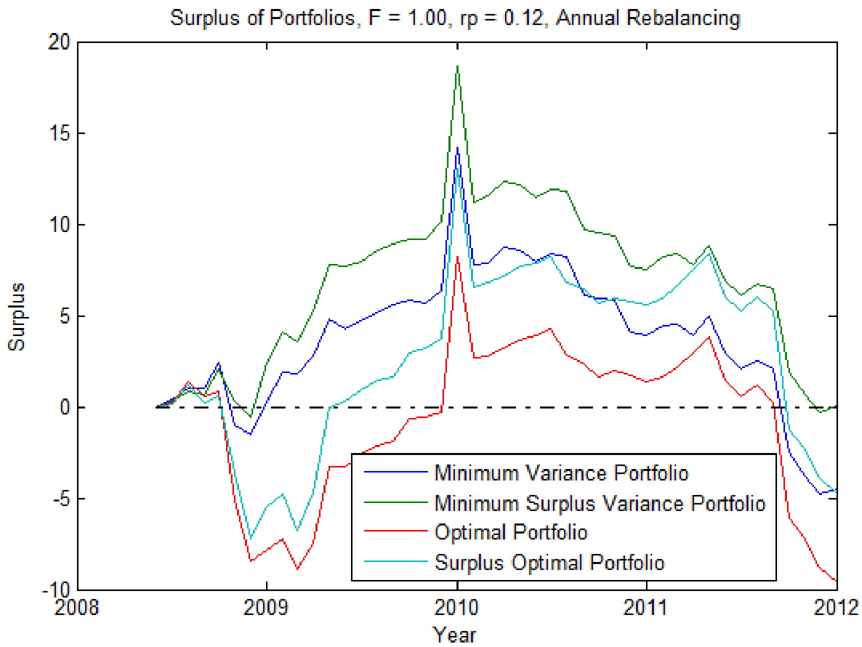


Figure 3.17: The surplus counterpart to figure 3.16 where surplus for the portfolio strategies is shown. The horizontal dash-dotted line is shown for clarity as the line for zero surplus.

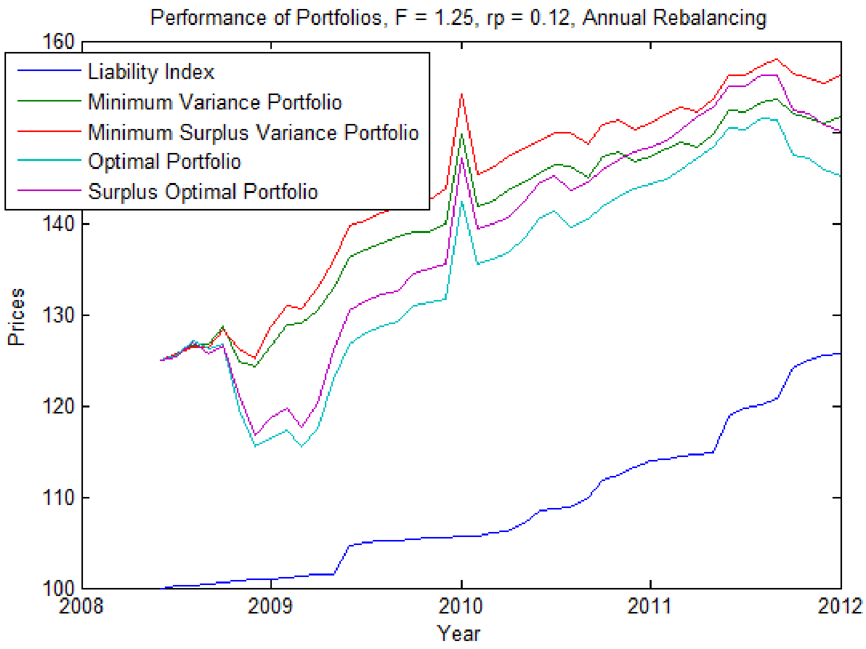


Figure 3.18: Four asset allocation strategies versus the liability index during the period June 2008 to January 2012, where initial funding status is 1.25 and the asset portfolios are rebalanced on an annual basis. Initial index level of liabilities in June 2008 is 100. The allocations for the minimum variance portfolio and minimum surplus variance portfolio found in table 3.5 and table 3.11 show the allocations for the optimal portfolios in absence and presence of liabilities with a return requirement of 12%.

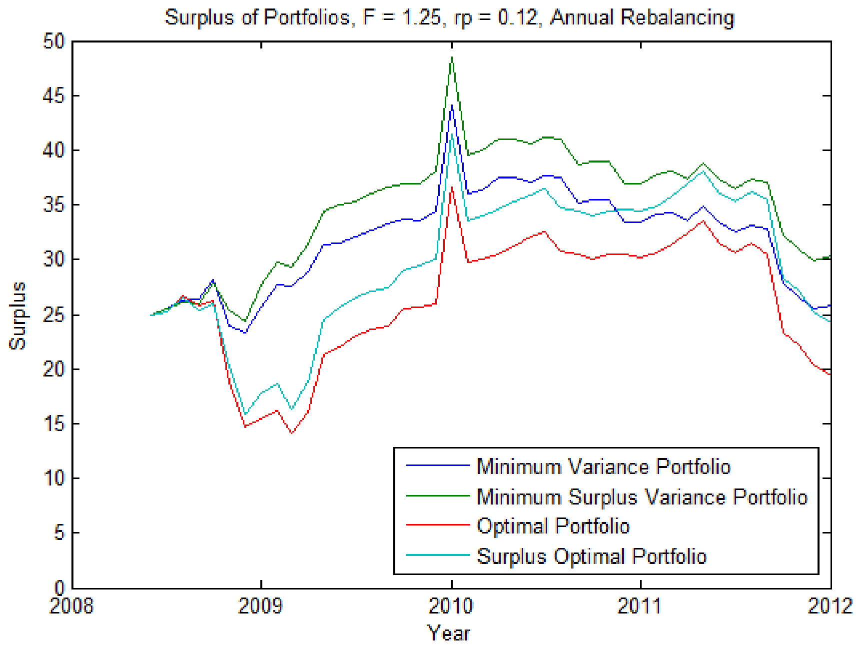


Figure 3.19: The surplus counterpart to figure 3.18 where surplus for the portfolio strategies is shown.

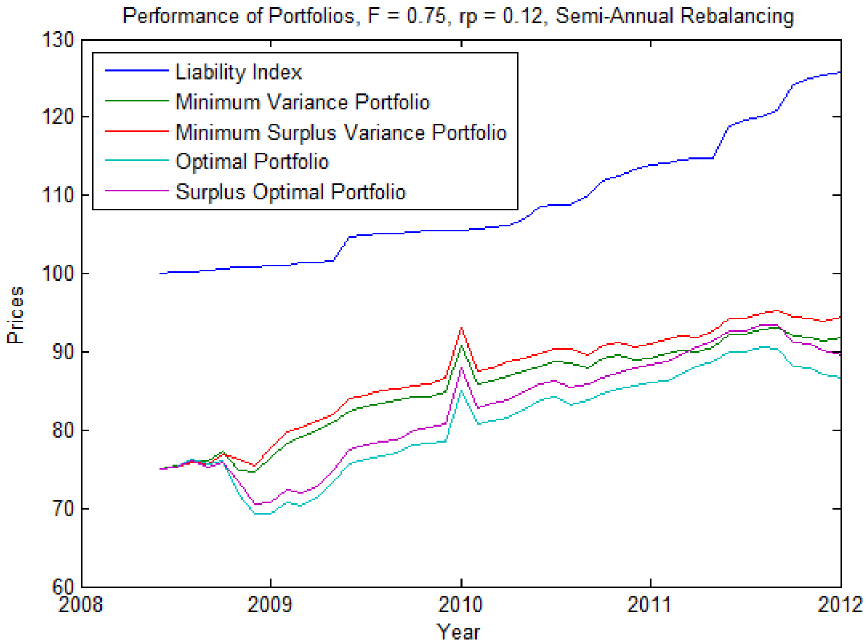


Figure 3.20: Four asset allocation strategies versus the liability index during the period June 2008 to January 2012, where initial funding status is 0.75 and the asset portfolios are rebalanced on a semi-annual basis. Initial index level of liabilities in June 2008 is 100. The allocations for the minimum variance portfolio and minimum surplus variance portfolio found in table 3.5 and table 3.11 show the allocations for the optimal portfolios in absence and presence of liabilities with a return requirement of 12%.

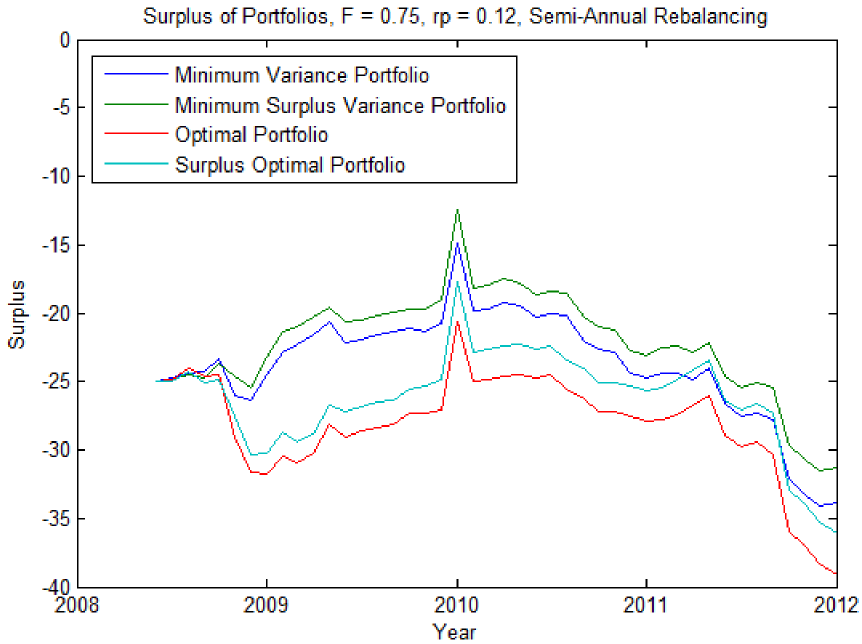


Figure 3.21: The surplus counterpart to figure 3.20 where surplus for the portfolio strategies is shown.

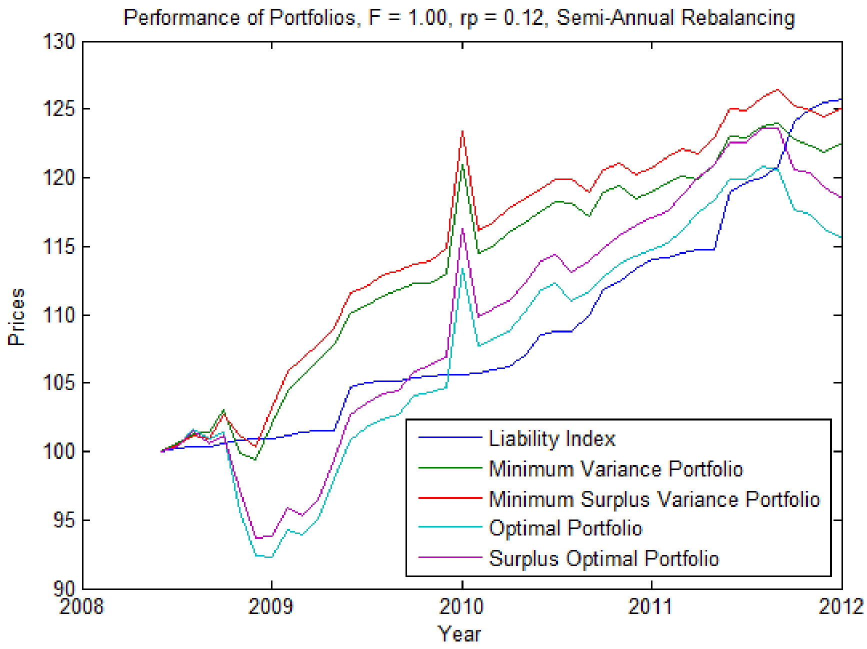


Figure 3.22: Four asset allocation strategies versus the liability index during the period June 2008 to January 2012, where initial funding status is 1 and the asset portfolios are rebalanced on a semi-annual basis. Initial index level of liabilities in June 2008 is 100. The allocations for the minimum variance portfolio and minimum surplus variance portfolio are in table 3.5 and table 3.11 show the allocations for the optimal portfolios in absence and presence of liabilities with a return requirement of 12%.

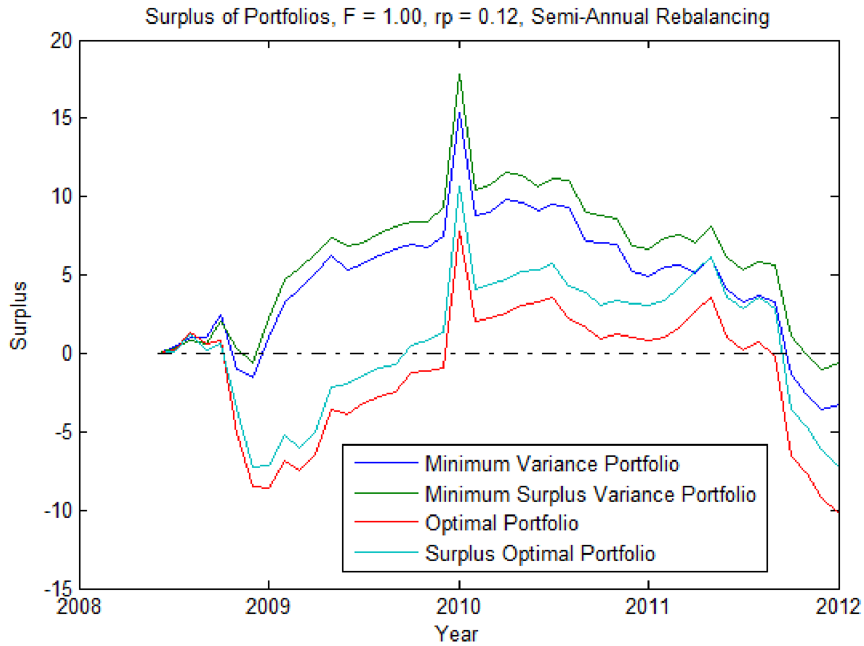


Figure 3.23: The surplus counterpart to figure 3.22 where surplus for the portfolio strategies is shown. The horizontal dash-dotted line is shown for clarity as the line for zero surplus.

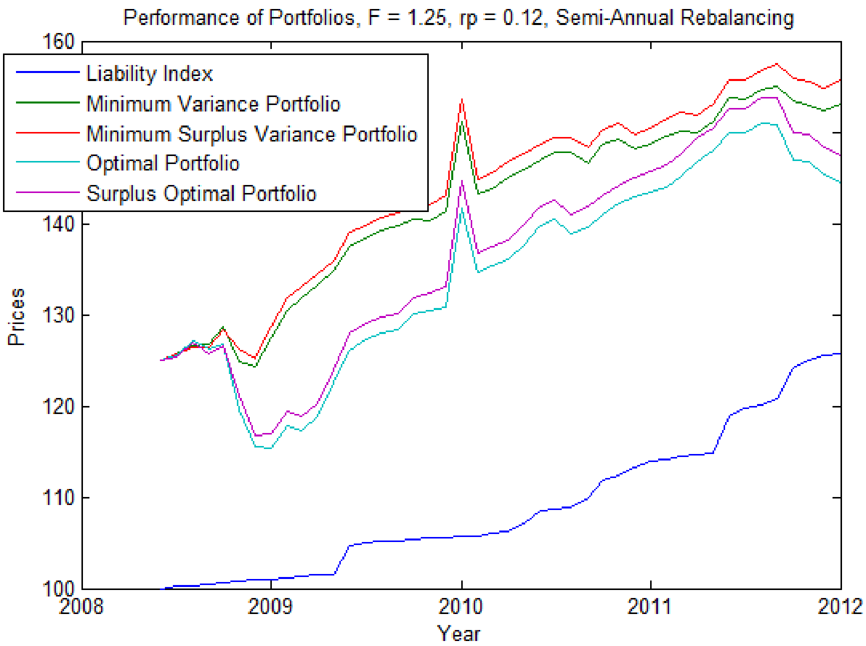


Figure 3.24: Four asset allocation strategies versus the liability index during the period June 2008 to January 2012, where initial funding status is 1.25 and the asset portfolios are rebalanced on a semi-annual basis. Initial index level of liabilities in June 2008 is 100. The allocations for the minimum variance portfolio and minimum surplus variance portfolio found in table 3.5 and table 3.11 show the allocations for the optimal portfolios in absence and presence of liabilities with a return requirement of 12%.

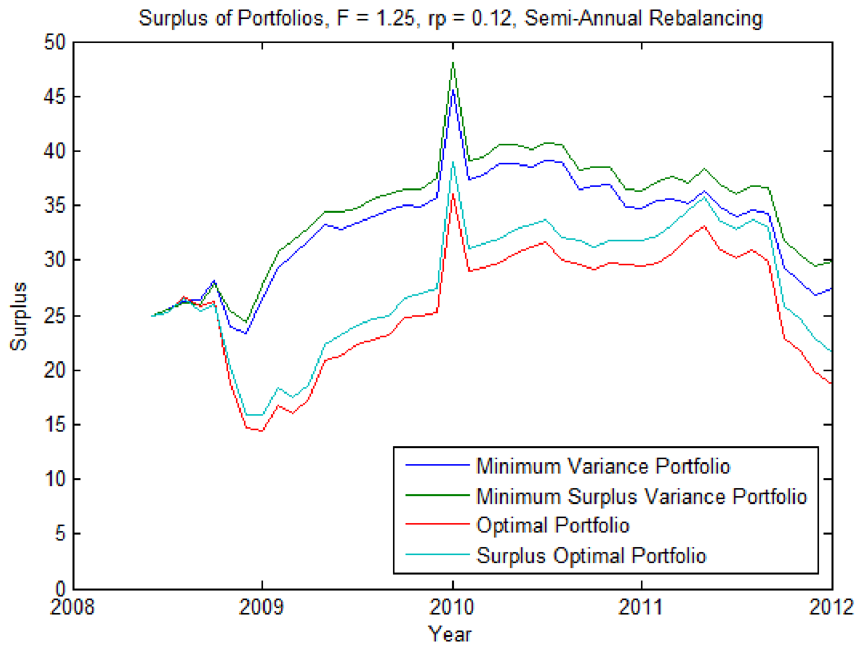


Figure 3.25: The surplus counterpart to figure 3.24 where surplus for the portfolio strategies is shown.

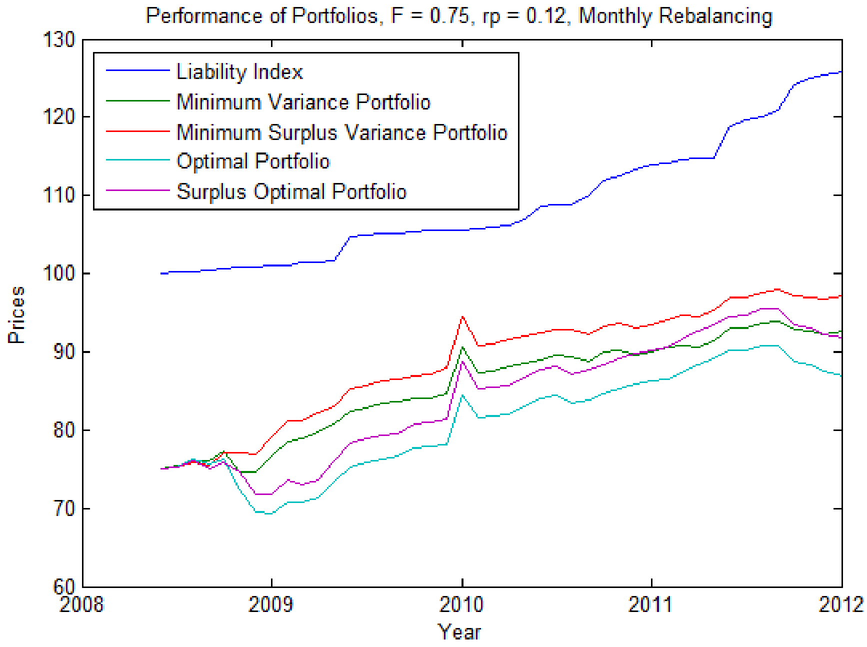


Figure 3.26: Four asset allocation strategies versus the liability index during the period June 2008 to January 2012, where initial funding status is 0.75 and the asset portfolios are rebalanced on a monthly basis. Initial index level of liabilities in June 2008 is 100. The allocations for the minimum variance portfolio and minimum surplus variance portfolio found in table 3.5 and table 3.11 show the allocations for the optimal portfolios in absence and presence of liabilities with a return requirement of 12%.

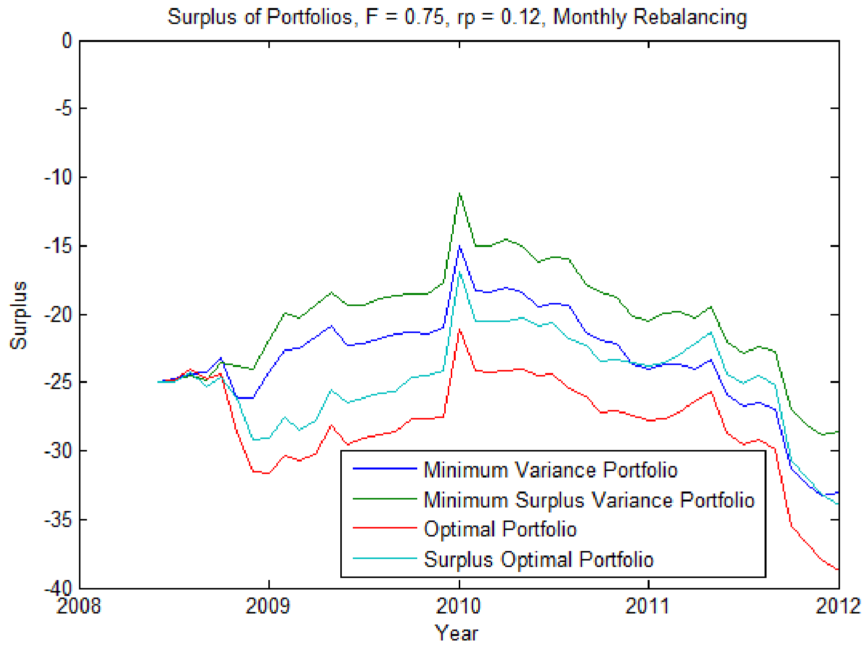


Figure 3.27: The surplus counterpart to figure 3.26 where surplus for the portfolio strategies is shown.

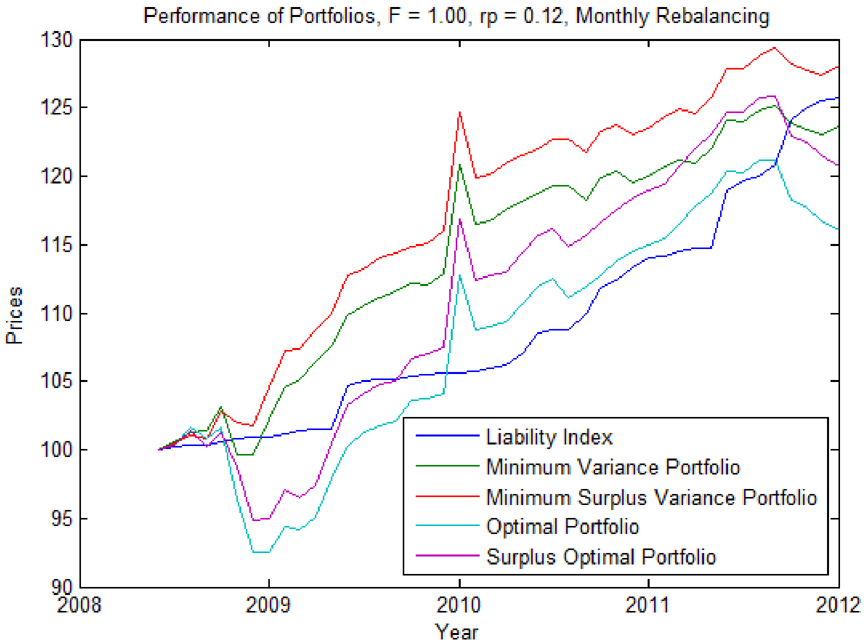


Figure 3.28: Four asset allocation strategies versus the liability index during the period June 2008 to January 2012, where initial funding status is 1 and the asset portfolios are rebalanced on a monthly basis. Initial index level of liabilities in June 2008 is 100. The allocations for the minimum variance portfolio and minimum surplus variance portfolio found in table 3.5 and table 3.11 show the allocations for the optimal portfolios in absence and presence of liabilities with a return requirement of 12%.

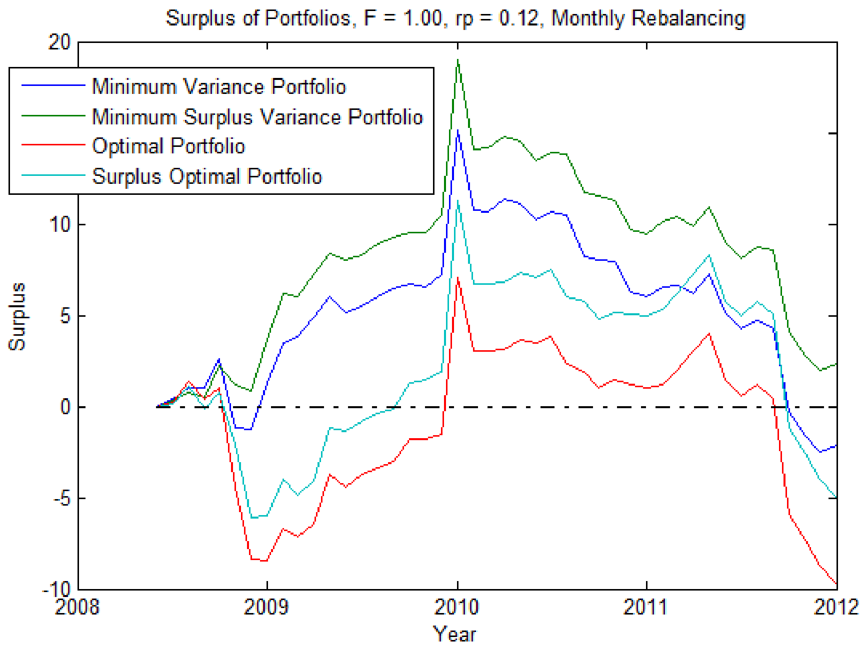


Figure 3.29: The surplus counterpart to figure 3.28 where surplus for the portfolio strategies is shown. The horizontal dash-dotted line is shown for clarity as the line for zero surplus.

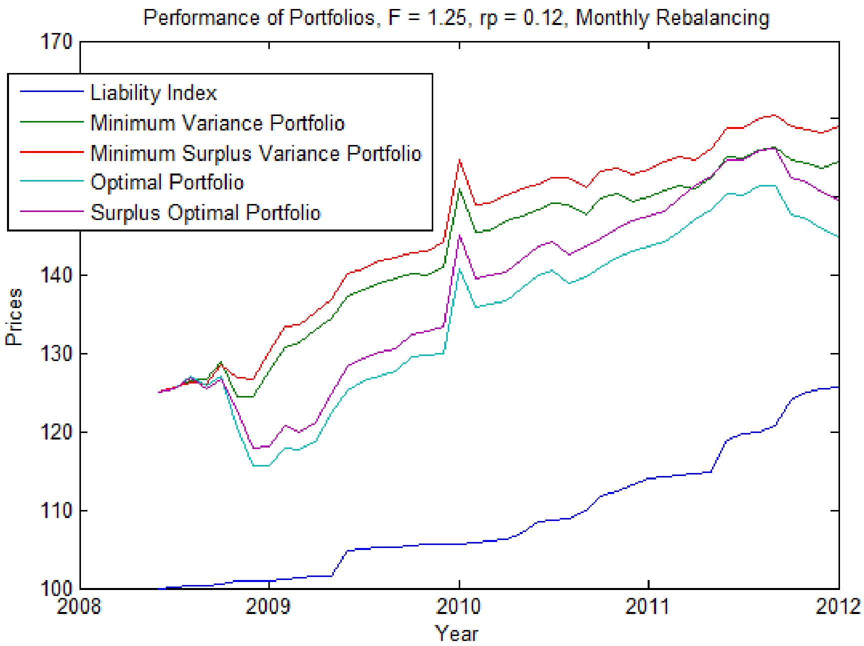


Figure 3.30: Four asset allocation strategies versus the liability index during the period June 2008 to January 2012, where initial funding status is 1.25 and the asset portfolios are rebalanced on a monthly basis. Initial index level of liabilities in June 2008 is 100. The allocations for the minimum variance portfolio and minimum surplus variance portfolio found in table 3.5 and table 3.11 show the allocations for the optimal portfolios in absence and presence of liabilities with a return requirement of 12%.

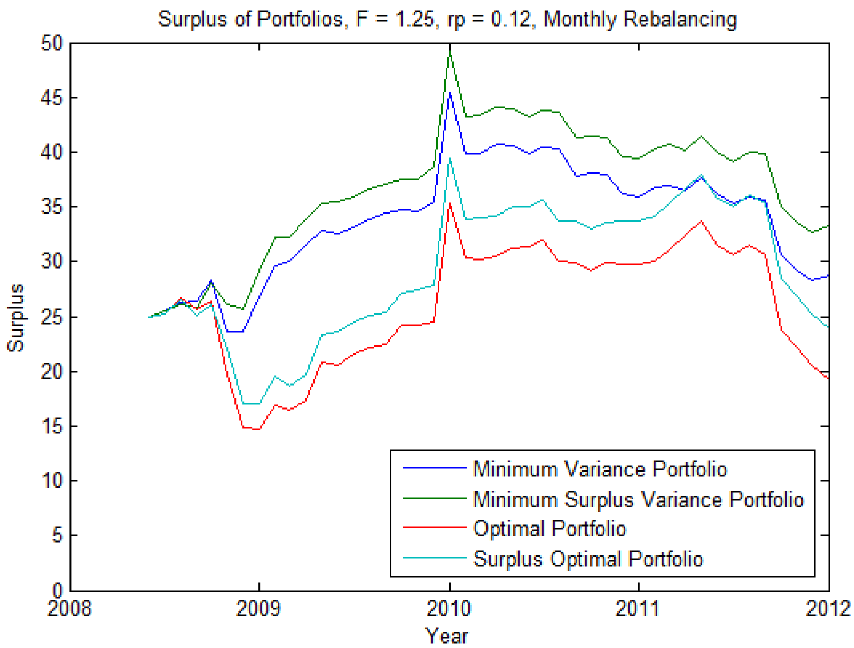


Figure 3.31: The surplus counterpart to figure 3.30 where surplus for the portfolio strategies is shown.

4. Conclusion

The material under consideration in this thesis is based on Keel and Muller (1995) that expressed the optimal sets for the surplus optimization problem in terms of Lagrangian parameters. In this thesis, their expressions are modified as to express the optimal solutions in terms of return requirements. By introducing a specific notation, the solutions associated with the return requirement parameter can be simplified and certain properties of the optimal solutions can be explained in a simple, yet efficient way.

Initially, a 3×3 symmetric matrix Q is introduced whose elements are composed from the inputs to the surplus optimization model. The matrix determinant and sub-determinants are used throughout the thesis to give relatively simple expressions for the optimal sets and the associated return variances. Also, the determinants provide an efficient way of understanding the difference between the traditional mean-variance frontier and surplus frontiers in risk return space. The determinants are helpful in explaining shifts in expected return, return variance and return covariance change from the classic mean-variance frontier to its surplus optimization counterpart.

An importance parameter introduced by Sharpe and Tint (1990) allows for a surplus optimization approach that avoids either asset only or full surplus optimization, as the consideration level given to liabilities can be set in the asset allocation process. This allows for flexibility in asset allocation and for partial ALM consideration.

The solution of the traditional mean-variance problem is found by constrained Lagrangian optimization. Removing the return requirement constraint, results in the minimum return variance portfolio. Solving the optimization problem for other portfolios results in the general solution where the optimal set in absence of liabilities can be decomposed into the minimum return variance component and an additional return generating component. The latter component serves the role of tuning the expected return of the portfolio with return preferences. Interestingly, the return variance on the optimal portfolio is the sum of the two components return variances since the return covariance between the components is zero.

Normalized surplus return variance is used as an objective function for surplus optimization, solved by constrained Lagrangian optimization as before. Solving the surplus return optimization problem without a return requirement constraint, results in the minimum surplus return variance portfolio, composed of the traditional minimum variance portfolio and an additional liability hedging component. The concentration in the liability hedge is determined by the ratio of importance to funding ratio and the liability hedging component leads to a shift of the optimal set. The general solution of the surplus optimization can be decomposed into four components where the first two components are the same as in the minimum surplus return variance portfolio. The third component is the same return generating component as in the traditional mean-variance optimal portfolio. The fourth component corrects the expected return on the optimal portfolio to the preferred level, by neutralizing the shift in expected return generated by the liability hedging component. Using the Q matrix notation, the optimal solutions are proposed with relatively simple expressions. Furthermore, the return variance associated with the optimal solutions can be written as a relatively simple expression, similar to its asset-only counterpart, with a small term added stemming from the liabilities.

Returns on liabilities can be assumed to be linearly dependent on several factors, e.g. economic growth, wage growth, inflation etc. A multiple linear regression model can be used to estimate coefficients for those factors, where the coefficients can be used in decomposing the return covariance elements between individual assets and liabilities. As a result, the liability correction component can be decomposed into separate components associated with each of these factors. This approach allows for multiple benchmarking in surplus optimization instead of using only one benchmark and increases the potentials in applying the surplus optimization model to a variety of problems with return benchmarking.

The relationship between expected return and return variance for the traditional mean-variance optimal set is quadratic as shown in section 2.8, commonly known as the risk-return frontier. Similarly, it is shown that the relationship between expected return and return variance for the surplus problem is also of quadratic nature. Since the optimal solutions for the surplus problem can be transferred into the surplus risk-return space, the quadratic relationship between expected return on surplus and surplus return variance is shown in section 2.9.

The classic market portfolio can be modified to account for liabilities by including a separate market portfolio liability correction component. This portfolio appears in various forms in the continuous-time literature considering asset allocation in presence of liabilities and thus strongly connects the discrete-time surplus optimization problem with the continuous-time literature.

From the normality assumption of logarithmic returns on assets and liabilities, the prices of assets and liabilities can be assumed to follow geometric Brownian motion processes. Accordingly, a simple probability measure on gaining positive surplus at the end of a specific horizon is derived. This probability measure is easily applied where logarithmic returns on assets and liabilities are assumed to be normally distributed.

In the final section in chapter 2, shortfall constraints of two types are analyzed. Traditional shortfall constraints are applicable in surplus optimization without any changes and such constraints were illustrated in risk-return space. Shortfall constraints on funding ratio are expressed and illustrated in risk-return and surplus risk-return spaces. Under certain conditions on the funding ratio, the shortfall constraints are linear in surplus risk return space

In chapter 3, the methods derived in chapter 2 are applied on a hypothetical pension fund where the value of its liabilities was assumed to follow the pension obligation index (POI) for employees in the Icelandic public sector. Despite of the long-only restrictions on pension funds in general, the optimization model is applied unchanged as to observe the full functionality of the model and to observe how large the short positions became. The asset and liability data used for this analysis span the period from January 2003 to June 2008, just before the downturn of global markets. Currency returns affected the returns on foreign asset classes considerably during this period since the domestic currency of ISK strengthened against the USD. Correlation analysis shows that only two out of eight asset classes were negatively correlated with liabilities and thus the assets had some liability hedging characteristics. Analysis of the determinants and sub-determinants of the matrix Q shows that the expected returns on minimum surplus variance portfolio are higher than on the classic minimum variance portfolio and that return covariance between asset portfolio and liabilities increase with increasing return requirement.

The optimal portfolios in presence of liabilities are analyzed for funding ratios of 1.5, 1.25, 1, 0.75 and 0.5 along with their asset-only counterparts; both optimal strategies are analyzed for minimum variance and for a return requirement of 10%, 11% and 12%. The analysis confirms the previously noted characteristics of the optimal sets expressed in terms of the sub-determinants of Q. The optimal strategies in presence of liabilities hold lower portions of assets with negative asset-liability return correlation than their classic mean-variance counterparts. As the return requirement on the optimal portfolios is increased, positive return correlation between assets and liabilities is sacrificed in order to achieve the preferred return requirement, resulting in weaker liability hedge.

The market portfolios in absence and presence of liabilities from sections 2.3 and 2.12 are analyzed assuming a risk-free rate of 5.5%. The liability correction component for the market portfolio in presence of liabilities has the same liability hedging characteristics as for other surplus optimal portfolios. With declining funding status, the need for hedging increases and the hedging component selects assets as to increase the return covariance between the asset portfolio and the liabilities.

The theoretical funding ratio which minimizes surplus return variance is found to be little less than 11.5 which can be considered an unlikely multiplier for asset value to liabilities for a typical pension fund. Nevertheless, this funding ratio provides the theoretical minimum surplus return variance for this data, a little lower than for the classic minimum variance portfolio.

Probabilities of assets covering liabilities for the horizon of 1 – 5 years are calculated for funding ratios of 1, 0.75 and 0.5. The asset allocation strategies under consideration are the same as analyzed earlier, i.e. the minimum surplus variance portfolio and optimal portfolios with return requirement of 10%, 11% and 12%. As expected, shorter horizon and lower funding ratio decrease the probabilities of covering liabilities.

An example of a traditional shortfall constraint is considered for two different levels of threshold return which show, as expected, that higher threshold return tightens the feasible set of portfolios satisfying the constraint. Two cases of shortfall constraints on the funding ratio are analyzed; one with varying funding ratio and another with varying shortfall probabilities. In the former case, lowering the initial funding ratio results in a tighter constraint that narrows the range of portfolios satisfying the constraint. Decreasing the shortfall probabilities affects on the feasible range of portfolios in a similar way. For both cases, the approximation lines for the shortfall constraints on the funding ratio are linear in surplus risk return space, since the threshold funding ratio is 1.

In the final section of the numerical example, four asset allocation strategies are tested on historical data for the purposes of comparing the surplus generated by and the performance of optimal portfolios in presence and absence of liabilities. This is done for three values of funding ratios and three portfolio rebalancing frequencies. In all cases, four strategies are compared where two out of the four strategies are the minimum variance and minimum surplus variance portfolios. The third and fourth strategies are the optimal portfolios in absence and presence of liabilities with a return requirement of 12%. The initial allocations for the four strategies are set in accordance with earlier analysis using assets and liabilities data from January 2003 to June 2008. The index level for the liability index is set as 100 in July 2008 and portfolio index values set accordingly with respect to funding ratio. The historical returns for the asset classes and liabilities from July 2008 to January 2012 are used to calculate and plot the portfolio and the liability indices in parallel for a historical performance comparison between the strategies and also to compare the performance of the strategies with the liabilities. New optimal allocations are found on an annual, semi-annual and monthly basis and the optimal portfolios are rebalanced accordingly. The comparison is made in terms of portfolio and liability index values, surplus associated with the strategies and average surplus for the strategies for the time period from July 2008 to January 2012.

The time series data used for this comparison include a time period with severely adverse market conditions, i.e. the 2008 market downturn. For the time period from July 2008 to January 2012, all comparative cases show that the strategies where liabilities were taken into account provide better hedge against increase in liabilities than their classic mean-variance counterparts, observed as higher surplus. The performance patterns for the four strategies are similar for all three funding ratios and all three rebalancing frequencies. In all cases, the minimum surplus variance portfolio performs the best, both in terms of asset performance and in assets net of liabilities. The classic minimum variance portfolio is the second best choice although the surplus optimal portfolio with a return requirement of 12%, a 2.7% in excess of the minimum variance portfolio return, manages to ascend above the minimum variance portfolio in terms of performance and surplus for some period in time. The traditional mean-variance optimal portfolio with a return requirement of 12% gains the least surplus in all cases. Interestingly, the portfolios taking liabilities into account seem to recover faster after the 2008 crisis and also, they grow a little faster on average than their asset-only counterparts in the post-crisis era. As a result, using this historical data, the surplus optimal strategies outperform the traditional mean-variance strategies in terms of generating surplus and provide better hedge against liabilities.

In stable market conditions, one might expect that the order of portfolios in terms of performance would be the other way around, i.e. that portfolios without a liability benchmark would have a superior performance to those who have such a benchmark. Also, in a stable environment, one might expect that portfolios with higher return requirement would perform better than minimum variance portfolios. The opposite performance observed in the comparison can be explained by the different objectives of the two models and as a consequence of the liability index characteristics. The surplus optimal portfolios aim to minimize surplus return volatility for a given level of expected surplus return. Since the liability index in this analysis had a stable growth with a low volatility, benchmarking portfolio returns with this liability index resulted in a low-risk investment strategy that suffered less from the 2008 market downswing than the traditional mean-variance strategies. This might suggest that for periods of market volatility, benchmarking portfolios with steady-growth and low-volatile indices could reduce the negative effects of market downturns.

In surplus optimization, the funding status determines the concentration in the surplus return hedge, i.e. how much risk the optimizer can afford to take in gaining surplus and how strong the hedge against surplus volatility should be in order to maintain surplus risk at acceptable levels. As the funding ratio decreases, the ability to take risk decreases and so should the willingness to take risk. Lower funding ratio increases the emphasis on hedging against liabilities by increasing the allocation in assets that provide similarly behaving returns as the liabilities do and give the highest potential in gaining acceptable surplus returns with as little volatility as possible via diversification.

Surplus optimization can easily be applied on problems outside the scope of managing pension assets. Index benchmarking is commonly applied practice in asset management where a portfolio is supposed to track and/or grow faster than a specified benchmark index. Instead of gaining surplus on a liability index, the surplus optimization model can be used to gain surplus on any preferred index in an efficient way via this benchmark approach. Also, by using the additional assumptions on benchmark returns from section 2.7, the surplus optimization approach allows for multiple index benchmarking as the hedge component can be decomposed into several components associated with each benchmark factor.

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Appendices

Appendices 1-26: Proofs of derivations in chapter 2.

Appendix 1:

Theorem 1: The minimum return variance portfolio.

The minimum return variance portfolio allocation vector is obtained by solving (2.2.5) *explicitly for* w_P^* *which results in*

$$w_{MVP} = \frac{\Sigma_A^{-1}\upsilon}{Q_{22}} \tag{2.2.6}$$

The expected return on the minimum return variance portfolio is

$$E\left[R_P\left(w_{MVP}\right)\right] = r_{MVP} = \frac{Q_{12}}{Q_{22}} \tag{2.2.7}$$

and the minimum return variance is

$$VAR\left[R_P\left(w_{MVP}\right)\right] = \sigma_{MVP}^2 = \frac{1}{Q_{22}} \tag{2.2.8}$$

Proof:

The minimum return variance portfolio w_{MVP} is obtained by omitting the constraint on portfolio return (2.2.2). This requires $\lambda_1 = 0$ and equation (2.2.4) becomes

$$w_P^* = \lambda_2 \Sigma_A^{-1}\upsilon \tag{A1.1}$$

Inserting (A1.1) into the constraint equation $w_P^T\upsilon = 1$ and isolating λ_2 gives

$$\lambda_2 = \frac{1}{\upsilon^T\Sigma_A^{-1}\upsilon} = \frac{1}{Q_{22}} \tag{A1.2}$$

This yields the minimum return variance portfolio

$$w_{MVP} = \frac{\Sigma_A^{-1}\upsilon}{Q_{22}} \tag{2.2.6}$$

The expected return on the minimum return variance portfolio (2.2.6) is

$$E\left[R_P\left(w_{MVP}\right)\right] = r_{MVP} = \mu_A^T w_{MVP} = \frac{Q_{12}}{Q_{22}} \tag{2.2.7}$$

and the minimum return variance is

$$VAR\left[R_P\left(w_{MVP}\right)\right] = \sigma_{MVP}^2 = w_{MVP}^T\Sigma_A w_{MVP} = \frac{\upsilon^T\Sigma_A^{-1}}{Q_{22}}\Sigma_A\frac{\Sigma_A^{-1}\upsilon}{Q_{22}} = \frac{Q_{22}}{Q_{22}^2} = \frac{1}{Q_{22}} \tag{2.2.8}$$

Appendix 2:

Theorem 2: The optimal return variance portfolio.

Given a preferred return requirement r_P*, the optimal return variance portfolio asset allocation vector is obtained by solving* (2.2.4) *for* w_P^* *which results in*

$$w_{rp} = \frac{\Sigma_A^{-1}\upsilon}{Q_{22}} + \frac{r_P Q_{22} - Q_{12}}{|Q_{\#1}|}\left[\Sigma_A^{-1}\mu_A - \frac{Q_{12}}{Q_{22}}\Sigma_A^{-1}\upsilon\right] \tag{2.2.9}$$

The optimal allocation vector (2.2.9) *is composed out of two separate portfolios*

$$w_{rp} = w_{MVP} + w_\eta \tag{2.2.10}$$

where

$$w_\eta = \frac{r_P Q_{22} - Q_{12}}{|Q_{\#1}|}\left[\Sigma_A^{-1}\mu_A - \frac{Q_{12}}{Q_{22}}\Sigma_A^{-1}\upsilon\right] \tag{2.2.11}$$

and $\upsilon^T w_\eta = 0$.

Proof:

The optimal solution in terms of Lagrange parameters was

$$w_P^* = \Sigma_A^{-1}\left[\mu_A \quad \upsilon\right]\left[\lambda_1 \quad \lambda_2\right]^T \tag{2.2.4}$$

The equations for the constraints, (2.2.2) and (2.2.3), can be rewritten as

$$\begin{bmatrix} \mu_A^T \\ \upsilon^T \end{bmatrix} w_P^* = \begin{bmatrix} r_P \\ 1 \end{bmatrix} \tag{A2.1}$$

Inserting equation (2.2.4) into equation (A2.1) yields

$$\begin{bmatrix} r_P \\ 1 \end{bmatrix} = \begin{bmatrix} \mu_A^T \\ \upsilon^T \end{bmatrix} \Sigma_A^{-1}\left[\mu_A \quad \upsilon\right]\begin{bmatrix} \lambda_1 \\ \lambda_2 \end{bmatrix} = \begin{bmatrix} Q_{11} & Q_{12} \\ Q_{12} & Q_{22} \end{bmatrix}\begin{bmatrix} \lambda_1 \\ \lambda_2 \end{bmatrix}$$

Matrix multiplication yields a system of two linear equations

$$\begin{bmatrix} r_P \\ 1 \end{bmatrix} = \begin{bmatrix} Q_{11}\lambda_1 + Q_{12}\lambda_2 \\ Q_{12}\lambda_1 + Q_{22}\lambda_2 \end{bmatrix} \tag{A2.2) – (A2.3}$$

Expanding (A2.1) gives

$$w_P^* = \lambda_1\Sigma_A^{-1}\mu_A + \lambda_2\Sigma_A^{-1}\upsilon \tag{A2.4}$$

Isolating λ_2 from (A2.3) gives

$$\lambda_2 = \frac{1 - Q_{12}\lambda_1}{Q_{22}} \tag{A2.5}$$

Inserting (A2.5) it into (A2.4) results in

$$w_P^* = \lambda_1\Sigma_A^{-1}\mu_A + \left(\frac{1 - Q_{12}\lambda_1}{Q_{22}}\right)\Sigma_A^{-1}\upsilon$$

or

$$w_{rp} = \frac{\Sigma_A^{-1}\upsilon}{Q_{22}} + \lambda_1 \left[\Sigma_A^{-1}\mu_A - \frac{Q_{12}}{Q_{22}} \Sigma_A^{-1}\upsilon \right] \tag{A2.6}$$

Equation (A2.6) is the optimal solution in terms of the Lagrangian return parameter λ_1. Inserting (A2.5) into (A2.2) gives

$$r_P = \lambda_1 \left[Q_{11} - \frac{Q_{12}^2}{Q_{22}} \right] + \frac{Q_{12}}{Q_{22}}$$

Isolating λ_1 gives

$$\lambda_1 = \left(r_P - \frac{Q_{12}}{Q_{22}} \right) \left(Q_{11} - \frac{Q_{12}^2}{Q_{22}} \right)^{-1}$$

The expression for λ_1 can be simplified by extending the fraction as

$$\lambda_1 = \left(r_P \frac{Q_{22}}{Q_{22}} - \frac{Q_{12}}{Q_{22}} \right) \left(Q_{11} \frac{Q_{22}}{Q_{22}} - \frac{Q_{12}^2}{Q_{22}} \right)^{-1}$$

which gives

$$\lambda_1 = \frac{r_P Q_{22} - Q_{12}}{Q_{11}Q_{22} - Q_{12}^2}$$

or

$$\lambda_1 = \frac{r_P Q_{22} - Q_{12}}{|Q_{\#1}|} \tag{A2.7}$$

Finally, inserting (A2.7) into (A2.6) yields

$$w_{rp} = \frac{\Sigma_A^{-1}\upsilon}{Q_{22}} + \frac{r_P Q_{22} - Q_{12}}{|Q_{\#1}|} \left[\Sigma_A^{-1}\mu_A - \frac{Q_{12}}{Q_{22}} \Sigma_A^{-1}\upsilon \right] \tag{2.2.9}$$

The optimal allocation vector (2.2.9) is composed out of two separate portfolios

$$w_{rp} = w_{MVP} + w_\eta \tag{2.2.10}$$

where the return generating component is

$$w_\eta = \frac{r_P Q_{22} - Q_{12}}{|Q_{\#1}|} \left[\Sigma_A^{-1}\mu_A - \frac{Q_{12}}{Q_{22}} \Sigma_A^{-1}\upsilon \right] \tag{2.2.11}$$

Multiplying w_η with υ^T from the left gives

$$\upsilon^T w_\eta = \frac{r_P Q_{22} - Q_{12}}{|Q_{\#1}|} \left[\upsilon^T \Sigma_A^{-1}\mu_A - \frac{Q_{12}}{Q_{22}} \upsilon^T \Sigma_A^{-1}\upsilon \right] = \frac{r_P Q_{22} - Q_{12}}{|Q_{\#1}|} \left[Q_{12} - \frac{Q_{12}}{Q_{22}} Q_{22} \right] = 0$$

Appendix 3:

Proposition 1: The expected return on the return generating component.

The expected return on the return generating component (2.2.11) *is*

$$E\left[R_P\left(w_\eta\right)\right] = r_P - \frac{Q_{12}}{Q_{22}} \tag{2.2.15}$$

Proof:

The expected return on the return generating component (2.2.11) is:

$$\begin{aligned}
E\left[R_P\left(w_\eta\right)\right] &= \mu_A^T w_\eta \\
&= \frac{r_P Q_{22} - Q_{12}}{\left|Q_{\#1}\right|}\left[\mu_A^T \Sigma_A^{-1} \mu_A - \frac{Q_{12}}{Q_{22}} \mu_A^T \Sigma_A^{-1} \upsilon\right] \\
&= \frac{r_P Q_{22} - Q_{12}}{\left|Q_{\#1}\right|}\left[Q_{11} - \frac{Q_{12}^2}{Q_{22}}\right] \\
&= \frac{r_P Q_{22} - Q_{12}}{\left|Q_{\#1}\right|} \frac{1}{Q_{22}}\left[Q_{11} Q_{22} - Q_{12}^2\right] \\
&= \frac{r_P Q_{22} - Q_{12}}{\left|Q_{\#1}\right|} \frac{\left|Q_{\#1}\right|}{Q_{22}} \\
&= r_P - \frac{Q_{12}}{Q_{22}}
\end{aligned} \tag{2.2.15}$$

Therefore, the expected return on the optimal return variance portfolio can be written as

$$\begin{aligned}
E\left[R_P\left(w_{rp}\right)\right] &= E\left[R_P\left(w_{MVP}\right)\right] + E\left[R_P\left(w_\eta\right)\right] \\
&= \frac{Q_{12}}{Q_{22}} + r_P - \frac{Q_{12}}{Q_{22}} \\
&= r_P
\end{aligned}$$

Appendix 4:

Proposition 2: The zero return covariance between the minimum return variance and the return generating components.

Any optimal return variance portfolio can be written as $w_{rp} = w_{MVP} + w_{\eta}$. *The return covariance between the minimum return variance component,* w_{MVP}, *and the return generating component,* w_{η} *is*

$$COV\left[R_P\left(w_{MVP}\right) \ , \ R_P\left(w_{\eta}\right)\right] = 0 \tag{2.2.16}$$

Proof:

The return covariance between the portfolio returns on the minimum variance component, w_{MVP}, and the return generating component, w_{η}, can be written as

$$COV\left[R_P\left(w_{MVP}\right) \ , \ R_P\left(w_{\eta}\right)\right] = w_{MVP}^T \Sigma_A w_{\eta} \tag{A4.1}$$

Inserting (2.2.6) transposed and (2.2.11) into (A4.1) results in

$$\begin{aligned} w_{MVP}^T \Sigma_A w_{\eta} &= \frac{\upsilon^T}{Q_{22}} \frac{r_P Q_{22} - Q_{12}}{\left|Q_{\#1}\right|}\left[\Sigma_A^{-1}\mu_A - \frac{Q_{12}}{Q_{22}} \Sigma_A^{-1}\upsilon \right] \\ &= \frac{r_P Q_{22} - Q_{12}}{\left|Q_{\#1}\right|}\left(\frac{Q_{12}}{Q_{22}} - \frac{Q_{12}}{Q_{22}} \frac{Q_{22}}{Q_{22}} \right) \\ &= \frac{r_P Q_{22} - Q_{12}}{\left|Q_{\#1}\right|}\left(\frac{Q_{12}}{Q_{22}} - \frac{Q_{12}}{Q_{22}} \right) = 0 \end{aligned}$$

Thus

$$COV\left[R_P\left(w_{MVP}\right) \ , \ R_P\left(w_{\eta}\right)\right] = 0 \tag{2.2.16}$$

Similarly, $COV\left[R_P\left(w_{MVP}\right) \ , \ R_P\left(w_{\eta}\right)\right] = w_{\eta}^T \Sigma_A w_{MVP} = 0$

Appendix 5:

Proposition 3: The return variance on the return generating component.

The return variance associated with the return generating component (2.2.11) *is*

$$VAR\left[R_P\left(w_\eta\right)\right]=\sigma_\eta^2=\frac{\left(r_P Q_{22}-Q_{12}\right)^2}{Q_{22}\left|Q_{\#1}\right|} \tag{2.2.17}$$

Proof:

The return variance associated with the return generating component can be found by

$$\begin{aligned}
\sigma_\eta^2 = w_\eta^T \Sigma_A w_\eta &= \left(\frac{r_P Q_{22}-Q_{12}}{\left|Q_{\#1}\right|}\right)^2 \left[\mu_A^T - \frac{Q_{12}}{Q_{22}}\upsilon^T\right]\left[\Sigma_A^{-1}\mu_A - \frac{Q_{12}}{Q_{22}}\Sigma_A^{-1}\upsilon\right] \\
&= \left(\frac{r_P Q_{22}-Q_{12}}{\left|Q_{\#1}\right|}\right)^2 \left(Q_{11} - 2\frac{Q_{12}}{Q_{22}}Q_{12} + \frac{Q_{12}}{Q_{22}}\frac{Q_{12}}{Q_{22}}Q_{22}\right) \\
&= \left(\frac{r_P Q_{22}-Q_{12}}{\left|Q_{\#1}\right|}\right)^2 \left(Q_{11} - \frac{Q_{12}^2}{Q_{22}}\right) \\
&= \left(\frac{r_P Q_{22}-Q_{12}}{\left|Q_{\#1}\right|}\right)^2 \frac{1}{Q_{22}}\left(Q_{11}Q_{22} - Q_{12}^2\right) \\
&= \left(\frac{r_P Q_{22}-Q_{12}}{\left|Q_{\#1}\right|}\right)^2 \frac{\left|Q_{\#1}\right|}{Q_{22}} \\
&= \frac{\left(r_P Q_{22}-Q_{12}\right)^2}{Q_{22}\left|Q_{\#1}\right|}
\end{aligned} \tag{2.2.17}$$

Appendix 6:

Theorem 3: The return variance on the optimal return variance portfolio.

Any optimal return variance portfolio can be written as $w_{rp} = w_{MVP} + w_\eta$. *As a result of theorem 1 and propositions 2 and 3, the return variance associated with the optimal return variance portfolio* (2.2.9) *is*

$$\sigma_{rp}^2 = \sigma_{MVP}^2 + \sigma_\eta^2 = \frac{1}{Q_{22}}\left(1+\frac{\left(r_P Q_{22}-Q_{12}\right)^2}{\left|Q_{\#1}\right|}\right)$$

Simplification of this expression results in

$$VAR\left[R_P\left(w_{rp}\right)\right]=\sigma_{rp}^2=\frac{r_P^2 Q_{22}-2r_P Q_{12}+Q_{11}}{\left|Q_{\#1}\right|} \tag{2.2.18}$$

Proof:

The return variance of the optimal return variance portfolio can be expressed as

$$VAR\left[R_P\left(w_{rp}\right)\right]=VAR\left[R_P\left(w_{MVP}\right)\right]+VAR\left[R_P\left(w_\eta\right)\right]+2COV\left[R_P\left(w_{MVP}\right),R_P\left(w_\eta\right)\right]$$

The results of proposition 2 yielded $COV\left[R_P\left(w_{MVP}\right),R_P\left(w_\eta\right)\right]=0$ and thus

$$VAR\left[R_P\left(w_{rp}\right)\right]=VAR\left[R_P\left(w_{MVP}\right)\right]+VAR\left[R_P\left(w_\eta\right)\right]$$

Therefore, the return variance of the optimal return variance portfolio can be expressed as the sum of the return variances of the minimum return variance component and the return generating component

$$\sigma_{rp}^2=\sigma_{MVP}^2+\sigma_\eta^2=\frac{1}{Q_{22}}+\frac{r_P^2Q_{22}-2r_PQ_{12}+Q_{12}^2/Q_{22}}{\left|Q_{\#1}\right|}$$

Simplifying this expression yields

$$\begin{aligned}VAR\left[R_P\left(w_{rp}\right)\right]=\sigma_{rp}^2&=\frac{1}{Q_{22}}\frac{\left|Q_{\#1}\right|}{\left|Q_{\#1}\right|}+\frac{r_P^2Q_{22}-2r_PQ_{12}+Q_{12}^2/Q_{22}}{\left|Q_{\#1}\right|}\\&=\frac{1}{Q_{22}}\frac{\left(Q_{11}Q_{22}-Q_{12}^2\right)}{\left|Q_{\#1}\right|}+\frac{r_P^2Q_{22}-2r_PQ_{12}+Q_{12}^2/Q_{22}}{\left|Q_{\#1}\right|}\\&=\frac{Q_{11}-Q_{12}^2/Q_{22}+r_P^2Q_{22}-2r_PQ_{12}+Q_{12}^2/Q_{22}}{\left|Q_{\#1}\right|}\\&=\frac{r_P^2Q_{22}-2r_PQ_{12}+Q_{11}}{\left|Q_{\#1}\right|}\end{aligned}\tag{2.2.18}$$

Appendix 7:

Theorem 4: The market portfolio.

Given the risk-free rate r_f, the risky assets market portfolio asset allocation vector is

$$w_{MKT} = \frac{\Sigma_A^{-1}\left(\mu_A - r_f \upsilon\right)}{\upsilon^T \Sigma_A^{-1}\left(\mu_A - r_f \upsilon\right)} \tag{2.3.3}$$

The expected return on the market portfolio is

$$E\left[R_P\left(w_{MKT}\right)\right] = r_{MKT} = \frac{\mu_A^T \Sigma_A^{-1}\left(\mu_A - r_f \upsilon\right)}{\upsilon^T \Sigma_A^{-1}\left(\mu_A - r_f \upsilon\right)} \tag{2.3.4}$$

and the market portfolio return variance is

$$VAR\left[R_P\left(w_{MKT}\right)\right] = \sigma_{MKT}^2 = \frac{\left(\mu_A - r_f \upsilon\right)^T \Sigma_A^{-1}\left(\mu_A - r_f \upsilon\right)}{\left(\upsilon^T \Sigma_A^{-1}\left(\mu_A - r_f \upsilon\right)\right)^2} \tag{2.3.5}$$

Proof:

The expected return on a portfolio can be decomposed into the risk-free rate, r_f, and the portfolio's risk premium on the risk-free rate, $w_P^T\left(\mu_A - r_f \upsilon\right)$. Formally, the decomposed expected return on any portfolio can be expressed as

$$E\left[R_P\left(w_P\right)\right] = w_P^T \mu_A = w_P^T\left(\mu_A - r_f \upsilon\right) + r_f = r_P \tag{2.3.2}$$

To solve for the market portfolio, (2.2.1) is still the objective, subject to the decomposed expected portfolio return constraint (2.3.2);

$$L(\bullet) = \frac{1}{2} w_P^T \Sigma_A w_P + \lambda\left(r_P - w_P^T\left(\mu_A - r_f \upsilon\right) - r_f\right)$$

The partial derivatives of L w.r.t. w_P yield necessary and sufficient first order conditions:

$$\Sigma_A w_P^* - \lambda_1 \mu_A - \lambda_2 \upsilon = 0, \quad \text{and} \quad w_P^T\left(\mu_A - r_f \upsilon\right) - r_f = r_P$$

A second order condition for a minimum is satisfied since the covariance matrix is positive definite. Isolating the optimal allocation vector w_P^* from the initial first order condition gives

$$w_P^* = \lambda \Sigma_A^{-1}\left(\mu_A - r_f \upsilon\right) \tag{A6.1}$$

The allocation vector is not unit normalized, so we consider the normalization constraint (2.2.3)

$$\upsilon^T w_P^* = 1 \tag{A6.2}$$

Inserting (A6.1) into (A6.2) yields

$$\lambda \upsilon^T \Sigma_A^{-1} \left(\mu_A - r_f \upsilon \right) = 1$$

Isolating λ gives

$$\lambda = \frac{1}{\upsilon^T \Sigma_A^{-1} \left(\mu_A - r_f \upsilon \right)}$$

Finally, inserting λ into (A6.1) results in the commonly known expression for the market portfolio:

$$w_{MKT} = \frac{\Sigma_A^{-1} \left(\mu_A - r_f \upsilon \right)}{\upsilon^T \Sigma_A^{-1} \left(\mu_A - r_f \upsilon \right)} \tag{2.3.3}$$

According to (2.3.3), the market portfolio can be considered as a risk premium portfolio that is linear in the risk premium vector $\left(\mu_A - r_f \upsilon \right)$; a unique portfolio, for any given value of the risk-free rate, r_f .

The expected return on the market portfolio is

$$E\left[R_P \left(w_{MKT} \right) \right] = r_{MKT} = \mu_A^T w_{MKT} = \frac{\mu_A^T \Sigma_A^{-1} \left(\mu_A - r_f \upsilon \right)}{\upsilon^T \Sigma_A^{-1} \left(\mu_A - r_f \upsilon \right)} \tag{2.3.4}$$

and the market portfolio return variance is

$$\begin{aligned} VAR\left[R_P \left(w_{MKT} \right) \right] &= \sigma_{MKT}^2 \\ &= w_{MKT}^T \Sigma_A w_{MKT} \\ &= \frac{\left(\mu_A - r_f \upsilon \right)^T}{\upsilon^T \Sigma_A^{-1} \left(\mu_A - r_f \upsilon \right)} \frac{\Sigma_A^{-1} \left(\mu_A - r_f \upsilon \right)}{\upsilon^T \Sigma_A^{-1} \left(\mu_A - r_f \upsilon \right)} \\ &= \frac{\left(\mu_A - r_f \upsilon \right)^T \Sigma_A^{-1} \left(\mu_A - r_f \upsilon \right)}{\left(\upsilon^T \Sigma_A^{-1} \left(\mu_A - r_f \upsilon \right) \right)^2} \end{aligned} \tag{2.3.5}$$

Appendix 8:

Theorem 5: The minimum surplus return variance portfolio.

The minimum surplus return variance portfolio (MSVP) allocation vector is obtained by solving (2.6.5) *explicitly for* $w^*_{P,S}$ *which results in*

$$w_{MSVP} = \frac{\Sigma_A^{-1}\upsilon}{Q_{22}} + \frac{\theta}{F}\left[\Sigma_A^{-1}\Sigma_{AL} - \frac{Q_{23}}{Q_{22}}\Sigma_A^{-1}\upsilon\right] \tag{2.6.6}$$

The optimal allocation vector (2.6.6) *is composed out of two separate portfolios*

$$w_{MSVP} = w_{MVP} + \theta\phi_{MSVP} \tag{2.6.7}$$

where w_{MVP} (2.2.6) *is in accordance with theorem 1, the minimum surplus variance correction component is*

$$\phi_{MSVP} = \frac{1}{F}\left[\Sigma_A^{-1}\Sigma_{AL} - \frac{Q_{23}}{Q_{22}}\Sigma_A^{-1}\upsilon\right] \tag{2.6.8}$$

and $\upsilon^T\phi_{MSVP} = 0$.

Proof:

The minimum surplus variance portfolio, w_{MSVP}, is obtained by omitting the constraint on portfolio return (2.2.2). This requires $\lambda_1 = 0$ and equation (2.6.4) becomes

$$w^*_{P,S} = \Sigma_A^{-1}\left[\upsilon \quad \Sigma_{AL}\right]\left[\lambda_2 \quad \theta F^{-1}\right]^T \tag{A7.1}$$

Expanding (A7.1) yields

$$w^*_{P,S} = \lambda_2\Sigma_A^{-1}\upsilon + \frac{\theta}{F}\Sigma_A^{-1}\Sigma_{AL} \tag{A7.2}$$

Considering the normalization constraint(2.6.3), multiplying (A7.2) from left with υ^T gives

$$\lambda_2\upsilon^T\Sigma_A^{-1}\upsilon + \frac{\theta}{F}\upsilon^T\Sigma_A^{-1}\Sigma_{AL} = 1 \tag{A7.3}$$

Isolating λ_2 from (A7.3) gives

$$\lambda_2 = \frac{1 - Q_{23}\theta F^{-1}}{Q_{22}} \tag{A7.4}$$

Inserting (A7.4) into (A7.2) gives

$$w^*_{P,S} = \left(\frac{1 - Q_{23}\theta F^{-1}}{Q_{22}}\right)\Sigma_A^{-1}\upsilon + \frac{\theta}{F}\Sigma_A^{-1}\Sigma_{AL}$$

that results in

$$w_{MSVP} = \frac{\Sigma_A^{-1}\upsilon}{Q_{22}} + \frac{\theta}{F}\left[\Sigma_A^{-1}\Sigma_{AL} - \frac{Q_{23}}{Q_{22}}\Sigma_A^{-1}\upsilon\right] \tag{2.6.6}$$

where

$$w_{MSVP} = w_{MVP} + \theta\phi_{MSVP} \tag{2.6.7}$$

with

$$\phi_{MSVP} = \frac{1}{F}\left[\Sigma_A^{-1}\Sigma_{AL} - \frac{Q_{23}}{Q_{22}}\Sigma_A^{-1}\upsilon\right] \tag{2.6.8}$$

Multiplying ϕ_{MSVP} with υ^T from the left gives

$$\upsilon^T\phi_{MSVP} = \frac{1}{F}\left[\upsilon^T\Sigma_A^{-1}\Sigma_{AL} - \frac{Q_{23}}{Q_{22}}\upsilon^T\Sigma_A^{-1}\upsilon\right] = \frac{1}{F}\left[Q_{23} - \frac{Q_{23}}{Q_{22}}Q_{22}\right] = 0\,.$$

Appendix 9:

Proposition 4: The expected return on the minimum surplus return variance portfolio.

The expected return on the minimum surplus return variance correction component (2.6.8) *is*

$$E\left[R_P\left(\phi_{MSVP}\right)\right] = -\frac{1}{F}\frac{\left|Q_{\#2}\right|}{Q_{22}} \tag{2.6.11}$$

Accordingly, the expected return on the minimum surplus return variance portfolio (2.6.6) *is*

$$\begin{aligned} E\left[R_P\left(w_{MSVP}\right)\right] &= E\left[R_P\left(w_{MVP}\right)\right] + \theta E\left[R_P\left(\phi_{MSVP}\right)\right] \\ &= \frac{1}{Q_{22}}\left(Q_{12} - \frac{\theta}{F}\left|Q_{\#2}\right|\right) \end{aligned} \tag{2.6.12}$$

Proof:

The expected return on the minimum surplus return variance correction component (2.6.8) is

$$\begin{aligned} E\left[R_P\left(\phi_{MSVP}\right)\right] = \mu_A^T\phi_{MSVP} &= \frac{1}{F}\left[\mu_A^T\Sigma_A^{-1}\Sigma_{AL} - \frac{Q_{23}}{Q_{22}}\mu_A^T\Sigma_A^{-1}\upsilon\right] \\ &= \frac{1}{F}\left[Q_{13} - \frac{Q_{23}}{Q_{22}}Q_{12}\right] = -\frac{1}{F}\frac{\left|Q_{\#2}\right|}{Q_{22}} \end{aligned} \tag{2.6.11}$$

From (2.2.7) and (2.6.11) it can readily be seen that the portfolio return on the MSVP (2.6.6) is

$$\begin{aligned} E\left[R_P\left(w_{MSVP}\right)\right] &= E\left[R_P\left(w_{MVP}\right)\right] + \theta E\left[R_P\left(\phi_{MSVP}\right)\right] \\ &= \frac{1}{Q_{22}}\left(Q_{12} - \frac{\theta}{F}\left|Q_{\#2}\right|\right) \end{aligned} \tag{2.6.12}$$

Appendix 10:

Proposition 5: The return variance on the minimum surplus return variance correction component.

The return variance associated with the minimum surplus return variance correction component (2.6.8) *is;*

$$VAR\left[R_P\left(\phi_{MSVP}\right)\right] = \sigma^2_{\phi MSVP} = \frac{1}{F^2}\frac{\left|Q_{\#4}\right|}{Q_{22}} \tag{2.6.13}$$

Proof:

The return variance associated with the minimum surplus return variance correction component (2.6.8) is;

$$\begin{aligned}
VAR\left[R_P\left(\phi_{MSVP}\right)\right] &= \sigma^2_{\phi MSVP} \\
&= \phi^T_{MSVP}\Sigma_A\phi_{MSVP} \\
&= \frac{1}{F}\left[\Sigma^T_{AL}\Sigma^{-1}_A - \frac{Q_{23}}{Q_{22}}\upsilon^T\Sigma^{-1}_A\right]\Sigma_A\frac{1}{F}\left[\Sigma^{-1}_A\Sigma_{AL} - \frac{Q_{23}}{Q_{22}}\Sigma^{-1}_A\upsilon\right] \\
&= \frac{1}{F^2}\left[\Sigma^T_{AL}\Sigma^{-1}_A\Sigma_{AL} - 2\frac{Q_{23}}{Q_{22}}\upsilon^T\Sigma^{-1}_A\Sigma_{AL} - \frac{Q^2_{23}}{Q^2_{22}}\upsilon^T\Sigma^{-1}_A\upsilon\right] \\
&= \frac{1}{F^2}\left[Q_{33} - 2\frac{Q^2_{23}}{Q_{22}} - \frac{Q^2_{23}}{Q^2_{22}}Q_{22}\right] \\
&= \frac{1}{F^2}\frac{1}{Q_{22}}\left[Q_{22}Q_{33} - Q^2_{23}\right] \\
&= \frac{1}{F^2}\frac{\left|Q_{\#4}\right|}{Q_{22}}
\end{aligned} \tag{2.6.13}$$

Appendix 11:

Proposition 6: The zero return covariance between the minimum return variance component and the minimum surplus return variance correction component.

The minimum surplus return variance portfolio is written as $w_{MSVP} = w_{MVP} + \theta\phi_{MSVP}$. *The return covariance between the minimum return variance component,* w_{MVP}, *and the minimum surplus return variance correction component,* ϕ_{MSVP}, *is*

$$COV\left[R_P\left(w_{MVP}\right)\ ,\ R_P\left(\phi_{MSVP}\right)\right] = 0 \tag{2.6.14}$$

Proof:

The return covariance between the portfolio returns on the minimum variance component, w_{MVP}, and the minimum surplus return variance correction component, ϕ_{MSVP}, can be written as

$$COV\left[R_P\left(w_{MVP}\right)\ ,\ R_P\left(\phi_{MSVP}\right)\right] = w_{MVP}^T \Sigma_A \phi_{MSVP} \tag{A10.1}$$

Inserting (2.2.6) transposed and (2.6.8) into (A10.1) results in

$$\begin{aligned} w_{MVP}^T \Sigma_A \phi_{MSVP} &= \left[\frac{\upsilon^T \Sigma_A^{-1}}{Q_{22}}\right] \Sigma_A \frac{1}{F}\left[\Sigma_A^{-1}\Sigma_{AL} - \frac{Q_{23}}{Q_{22}}\Sigma_A^{-1}\upsilon\right] \\ &= \frac{1}{F}\left(\frac{\upsilon^T \Sigma_A^{-1}\Sigma_{AL}}{Q_{22}} - \frac{\upsilon^T \Sigma_A^{-1}\upsilon}{Q_{22}}\frac{Q_{23}}{Q_{22}}\right) \\ &= \frac{1}{F}\left(\frac{Q_{23}}{Q_{22}} - \frac{Q_{22}}{Q_{22}}\frac{Q_{23}}{Q_{22}}\right) = \frac{1}{F}\left(\frac{Q_{23}}{Q_{22}} - \frac{Q_{23}}{Q_{22}}\right) = 0 \end{aligned}$$

Thus

$$COV\left[R_P\left(w_{MVP}\right)\ ,\ R_P\left(\phi_{MSVP}\right)\right] = 0 \tag{2.6.14}$$

Similarly, $COV\left[R_P\left(\phi_{MSVP}\right)\ ,\ R_P\left(w_{MVP}\right)\right] = \phi_{MSVP}^T \Sigma_A w_{MVP} = 0$

Appendix 12:

Theorem 6: The return variance on the minimum surplus return variance portfolio.

The minimum surplus return variance portfolio is written as $w_{MSVP} = w_{MVP} + \theta\phi_{MSVP}$. *The return variance associated with the minimum surplus return variance portfolio is*

$$\begin{aligned} VAR\left[R_P\left(w_{MSVP}\right)\right] &= \sigma^2_{MSVP} \\ &= \sigma^2_{MVP} + \theta^2\sigma^2_{\phi MSVP} \\ &= \frac{1}{Q_{22}}\left(1 + \frac{\theta^2}{F^2}\left|Q_{\#4}\right|\right) \end{aligned} \tag{2.6.15}$$

This result is in accordance with the results of theorem 1, and propositions 5 - 6.

Proof:

As the MSVP (2.6.6) is composed out of two separated portfolios, w_{MVP} (2.2.6) and ϕ_{MSVP} (2.6.8), the return variance associated with the MSVP can be expressed by

$$\begin{aligned} VAR\left[R_P\left(w_{MSVP}\right)\right] = VAR\left[R_P\left(w_{MVP}\right)\right] + VAR\left[R_P\left(\phi_{MSVP}\right)\right] \\ + 2COV\left[R_P\left(w_{MVP}\right)\ ,\ R_P\left(\phi_{MSVP}\right)\right] \end{aligned} \tag{A11.1}$$

The results of proposition 6 (2.6.14) showed that

$$COV\left[R_P\left(w_{MVP}\right)\ ,\ R_P\left(\phi_{MSVP}\right)\right] = COV\left[R_P\left(\phi_{MSVP}\right)\ ,\ R_P\left(w_{MVP}\right)\right] = 0$$

Therefore, the return variance associated with the MSVP simplifies to

$$VAR\left[R_P\left(w_{MSVP}\right)\right] = VAR\left[R_P\left(w_{MVP}\right)\right] + VAR\left[R_P\left(\phi_{MSVP}\right)\right]$$

Using (2.2.6) and (2.6.13) yields

$$\begin{aligned} VAR\left[R_P\left(w_{MSVP}\right)\right] &= \sigma^2_{MSVP} \\ &= \sigma^2_{MVP} + \theta^2\sigma^2_{\phi MSVP} \\ &= \frac{1}{Q_{22}}\left(1 + \frac{\theta^2}{F^2}\left|Q_{\#4}\right|\right) \end{aligned} \tag{2.6.15}$$

The return variance associated with the MSVP (2.6.6) can equivalently be achieved from

$$\sigma^2_{MSVP} = w^T_{MSVP}\Sigma_A w_{MSVP} \tag{A11.2}$$

Inserting (2.6.6) into (A11.2) gives

$$
\begin{aligned}
\sigma^2_{MSVP} &= w^T_{MSVP}\Sigma_A w^T_{MSVP} \\
&= \left[\frac{\upsilon^T\Sigma_A^{-1}}{Q_{22}} + \frac{\theta}{F}\left[\Sigma^T_{AL}\Sigma_A^{-1} - \frac{Q_{23}}{Q_{22}}\upsilon^T\Sigma_A^{-1}\right]\right]\Sigma_A\left[\frac{\Sigma_A^{-1}\upsilon}{Q_{22}} + \frac{\theta}{F}\left[\Sigma_A^{-1}\Sigma_{AL} - \frac{Q_{23}}{Q_{22}}\Sigma_A^{-1}\upsilon\right]\right] \\
&= \left[\frac{\upsilon^T}{Q_{22}} + \frac{\theta}{F}\left[\Sigma^T_{AL} - \frac{Q_{23}}{Q_{22}}\upsilon^T\right]\right]\left[\frac{\Sigma_A^{-1}\upsilon}{Q_{22}} + \frac{\theta}{F}\left[\Sigma_A^{-1}\Sigma_{AL} - \frac{Q_{23}}{Q_{22}}\Sigma_A^{-1}\upsilon\right]\right]
\end{aligned}
$$

or

$$
\begin{aligned}
\sigma^2_{MSVP} &= \frac{Q_{22}}{Q_{22}^2} + \frac{\theta}{F}\frac{Q_{23}}{Q_{22}} - \frac{\theta}{F}\frac{Q_{22}Q_{23}}{Q_{22}^2} \\
&\quad + \frac{\theta}{F}\frac{Q_{23}}{Q_{22}} + \frac{\theta^2}{F^2}Q_{33} - \frac{\theta^2}{F^2}\frac{Q_{23}^2}{Q_{22}} \\
&\quad - \frac{\theta}{F}\frac{Q_{22}Q_{23}}{Q_{22}^2} - \frac{\theta^2}{F^2}\frac{Q_{23}^2}{Q_{22}} + \frac{\theta^2}{F^2}\frac{Q_{23}^2Q_{22}}{Q_{22}^2}
\end{aligned}
$$

which simplifies to

$$
\begin{aligned}
\sigma^2_{MSVP} &= \frac{1}{Q_{22}} + \frac{\theta^2}{F^2}Q_{33} - \frac{\theta^2}{F^2}\frac{Q_{23}^2}{Q_{22}} \\
&= \frac{1}{Q_{22}} + \frac{\theta^2}{F^2}\left(Q_{33} - \frac{Q_{23}^2}{Q_{22}}\right) \\
&= \frac{1}{Q_{22}} + \frac{\theta^2}{F^2}\frac{1}{Q_{22}}\left(Q_{33}Q_{22} - Q_{23}^2\right) \\
&= \frac{1}{Q_{22}}\left(1 + \frac{\theta^2}{F^2}\left|Q_{\#4}\right|\right)
\end{aligned}
\tag{2.6.15}
$$

This proofs that the expressions (A11.1) and (A11.2) are equivalent and the return variance on the MSVP (2.6.6) can be expressed by (2.6.15).

Appendix 13:

Theorem 7: The optimal surplus return variance portfolio.

Given the preferred return requirement r_P*, the optimal surplus return portfolio asset allocation vector is obtained by solving* (2.6.4) *for* $w^*_{P,S}$ *which results in*

$$\begin{aligned} w_{rp,S} &= \frac{\Sigma_A^{-1}\upsilon}{Q_{22}} + \frac{\theta}{F}\left[\Sigma_A^{-1}\Sigma_{AL} - \frac{Q_{23}}{Q_{22}}\Sigma_A^{-1}\upsilon\right] \\ &\quad + \frac{r_P Q_{22} - Q_{12} + \theta F^{-1}\left|Q_{\#2}\right|}{\left|Q_{\#1}\right|}\left[\Sigma_A^{-1}\mu_A - \frac{Q_{12}}{Q_{22}}\Sigma_A^{-1}\upsilon\right] \end{aligned} \tag{2.6.16}$$

The optimal allocation vector is composed out of four separate portfolios

$$w_{rp,S} = w_{MVP} + \theta\phi_{MSVP} + w_\eta + \theta w_{\eta,S} \tag{2.6.17}$$

where w_{MVP}*,* w_η *and* ϕ_{MSVP} *are in accordance with theorems 1, 2 and 5, respectively. The return generating correction component is*

$$w_{\eta,S} = \frac{1}{F}\frac{\left|Q_{\#2}\right|}{\left|Q_{\#1}\right|}\left[\Sigma_A^{-1}\mu_A - \frac{Q_{12}}{Q_{22}}\Sigma_A^{-1}\upsilon\right] \tag{2.6.18}$$

and $\upsilon^T w_{\eta,S} = 0$.

Proof:

The optimal solution in terms of Lagrange parameters was

$$w^*_{P,S} = \Sigma_A^{-1}\left[\mu_A \quad \upsilon \quad \Sigma_{AL}\right]\left[\lambda_1 \quad \lambda_2 \quad \theta F^{-1}\right]^T \tag{2.6.4}$$

The equations for the constraints, (2.6.2) and (2.6.3), can be rewritten as

$$\begin{bmatrix} \mu_A^T \\ \upsilon^T \end{bmatrix} w^*_{P,S} = \begin{bmatrix} r_P \\ 1 \end{bmatrix} \tag{A12.1}$$

Inserting equation (2.6.4) into equation (A12.1) yields

$$\begin{bmatrix} r_P \\ 1 \end{bmatrix} = \begin{bmatrix} \mu_A^T \\ \upsilon^T \end{bmatrix}\Sigma_A^{-1}\left[\mu_A \quad \upsilon \quad \Sigma_{AL}\right]\begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \theta F^{-1} \end{bmatrix} = \begin{bmatrix} Q_{11} & Q_{12} & Q_{13} \\ Q_{21} & Q_{22} & Q_{23} \end{bmatrix}\begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \theta F^{-1} \end{bmatrix}$$

Matrix multiplication yields a system of two linear equations

$$\begin{bmatrix} r_P \\ 1 \end{bmatrix} = \begin{bmatrix} Q_{11}\lambda_1 + Q_{12}\lambda_2 + Q_{13}\theta F^{-1} \\ Q_{12}\lambda_1 + Q_{22}\lambda_2 + Q_{23}\theta F^{-1} \end{bmatrix} \tag{A12.2) – (A12.3}$$

Expanding (2.6.4) gives

$$w_{P,S}^{*} = \lambda_1 \Sigma_A^{-1} \mu_A + \lambda_2 \Sigma_A^{-1} \upsilon + \frac{\theta}{F} \Sigma_A^{-1} \Sigma_{AL} \tag{A12.4}$$

Isolating λ_2 from (A12.3) gives

$$\lambda_2 = \frac{1 - Q_{12}\lambda_1 - Q_{23}\theta F^{-1}}{Q_{22}} \tag{A12.5}$$

Inserting (A12.5) into (A12.4) results in

$$w_{P,S}^{*} = \lambda_1 \Sigma_A^{-1} \mu_A + \left(\frac{1 - Q_{12}\lambda_1 - Q_{23}\theta F^{-1}}{Q_{22}} \right) \Sigma_A^{-1} \upsilon + \frac{\theta}{F} \Sigma_A^{-1} \Sigma_{AL}$$

or

$$w_{rp,S} = \frac{\Sigma_A^{-1}\upsilon}{Q_{22}} + \frac{\theta}{F}\left[\Sigma_A^{-1}\Sigma_{AL} - \frac{Q_{23}}{Q_{22}} \Sigma_A^{-1}\upsilon \right] + \lambda_1 \left[\Sigma_A^{-1}\mu_A - \frac{Q_{12}}{Q_{22}} \Sigma_A^{-1}\upsilon \right] \tag{A12.6}$$

Equation (A12.6) is the optimal solution in terms of the Lagrangian return parameter λ_1. Inserting (A12.5) into (A12.2) gives

$$r_P = \lambda_1 \left[Q_{11} - \frac{Q_{12}^2}{Q_{22}} \right] + \frac{Q_{12} - Q_{12}Q_{23}\theta F^{-1}}{Q_{22}} + Q_{13}\theta F^{-1}$$

Isolating λ_1 gives

$$\lambda_1 = \left(r_P - \frac{Q_{12} - Q_{12}Q_{23}\theta F^{-1}}{Q_{22}} - Q_{13}\theta F^{-1} \right) \left(Q_{11} - \frac{Q_{12}^2}{Q_{22}} \right)^{-1}$$

The expression for λ_1 can be simplified by extending the fraction as

$$\lambda_1 = \left(r_P \frac{Q_{22}}{Q_{22}} - \frac{Q_{12} - Q_{12}Q_{23}\theta F^{-1}}{Q_{22}} - Q_{13} \frac{Q_{22}}{Q_{22}} \theta F^{-1} \right) \left(Q_{11} \frac{Q_{22}}{Q_{22}} - \frac{Q_{12}^2}{Q_{22}} \right)^{-1}$$

which gives

$$\lambda_1 = \frac{r_P Q_{22} - Q_{12} + \theta F^{-1}\left(Q_{12}Q_{23} - Q_{13}Q_{22} \right)}{Q_{11}Q_{22} - Q_{12}Q_{21}}$$

or

$$\lambda_1 = \frac{r_P Q_{22} - Q_{12} + \theta F^{-1} \det\left(Q_{\#2}\right)}{\det\left(Q_{\#1}\right)} \tag{A12.7}$$

Finally, inserting (A12.7) into (A12.6) yields

$$
\begin{aligned}
w_{rp,S} &= \frac{\Sigma_A^{-1}\upsilon}{Q_{22}} + \frac{\theta}{F}\left[\Sigma_A^{-1}\Sigma_{AL} - \frac{Q_{23}}{Q_{22}}\Sigma_A^{-1}\upsilon\right] \\
&\quad + \frac{r_P Q_{22} - Q_{12} + \theta F^{-1}\left|Q_{\#2}\right|}{\left|Q_{\#1}\right|}\left[\Sigma_A^{-1}\mu_A - \frac{Q_{12}}{Q_{22}}\Sigma_A^{-1}\upsilon\right]
\end{aligned}
\tag{2.6.16}
$$

The optimal allocation vector (2.6.16) is composed out of four separate portfolios

$$
w_{rp,S} = w_{MVP} + \theta\phi_{MSVP} + w_\eta + \theta w_{\eta,S} \tag{2.6.17}
$$

where the return generating correction component is

$$
w_{\eta,S} = \frac{1}{F}\frac{\left|Q_{\#2}\right|}{\left|Q_{\#1}\right|}\left[\Sigma_A^{-1}\mu_A - \frac{Q_{12}}{Q_{22}}\Sigma_A^{-1}\upsilon\right] \tag{2.6.18}
$$

and w_{MVP}, w_η and ϕ_{MSVP} are in accordance with theorems 1, 2 and 5, respectively.

Multiplying $w_{\eta,S}$ with υ^T from the left gives

$$
\begin{aligned}
\upsilon^T w_{\eta,S} &= \frac{1}{F}\frac{\left|Q_{\#2}\right|}{\left|Q_{\#1}\right|}\left[\upsilon^T\Sigma_A^{-1}\mu_A - \frac{Q_{12}}{Q_{22}}\upsilon^T\Sigma_A^{-1}\upsilon\right] \\
&= \frac{1}{F}\frac{\left|Q_{\#2}\right|}{\left|Q_{\#1}\right|}\left[Q_{12} - \frac{Q_{12}}{Q_{22}}Q_{22}\right] \\
&= 0
\end{aligned}
$$

Appendix 14:

Proposition 7: The expected return on the return generating correction component.

The expected return on the return generating correction component $w_{\eta,S}$ (2.6.18) *is;*

$$
E\left[R_P\left(w_{\eta,S}\right)\right] = \frac{1}{F}\frac{\left|Q_{\#2}\right|}{Q_{22}} \tag{2.6.23}
$$

From (2.6.11) *and* (2.6.23)*, it can be seen that*

$$
E\left[R_P\left(\phi_{MSVP}\right)\right] + E\left[R_P\left(w_{\eta,S}\right)\right] = 0 \tag{2.6.24}
$$

Proof:

The expected return on the return generating correction component (2.6.18) is;

$$
\begin{aligned}
E\left[R_P\left(w_{\eta,S}\right)\right] &= \mu_A^T w_{\eta,S} \\
&= \frac{1}{F}\frac{\left|Q_{\#2}\right|}{\left|Q_{\#1}\right|}\left[\mu_A^T\Sigma_A^{-1}\mu_A - \frac{Q_{12}}{Q_{22}}\mu_A^T\Sigma_A^{-1}\upsilon\right] \\
&= \frac{1}{F}\frac{\left|Q_{\#2}\right|}{\left|Q_{\#1}\right|}\left(Q_{11} - \frac{Q_{12}^2}{Q_{22}}\right) \\
&= \frac{1}{F}\frac{\left|Q_{\#2}\right|}{\left|Q_{\#1}\right|}\frac{1}{Q_{22}}\left(Q_{11}Q_{22} - Q_{12}^2\right) \\
&= \frac{1}{F}\frac{\left|Q_{\#2}\right|}{\left|Q_{\#1}\right|}\frac{\left|Q_{\#1}\right|}{Q_{22}} \\
&= \frac{1}{F}\frac{\left|Q_{\#2}\right|}{Q_{22}}
\end{aligned}
\tag{2.6.23}
$$

It can seen from (2.6.11) and (2.6.23) that

$$
E\left[R_P\left(\phi_{MSVP}\right)\right] + E\left[R_P\left(w_{\eta,S}\right)\right] = -\frac{1}{F}\frac{\left|Q_{\#2}\right|}{Q_{22}} + \frac{1}{F}\frac{\left|Q_{\#2}\right|}{Q_{22}} = 0
\tag{2.6.24}
$$

Therefore, the expected return on the surplus return variance portfolio can be written as

$$
\begin{aligned}
E\left[R_P\left(w_{rp,S}\right)\right] &= E\left[R_P\left(w_{MVP}\right)\right] + \theta E\left[R_P\left(\phi_{MSVP}\right)\right] + E\left[R_P\left(w_\eta\right)\right] + \theta E\left[R_P\left(w_{\eta,S}\right)\right] \\
&= \frac{Q_{12}}{Q_{22}} - \frac{\theta}{F}\frac{\left|Q_{\#2}\right|}{Q_{22}} + r_P - \frac{Q_{12}}{Q_{22}} + \frac{\theta}{F}\frac{\left|Q_{\#2}\right|}{Q_{22}}
\end{aligned}
$$

which turns out to be

$$
\begin{aligned}
E\left[R_P\left(w_{rp,S}\right)\right] &= E\left[R_P\left(w_{rp}\right)\right] \\
&= E\left[R_P\left(w_{MVP}\right)\right] + E\left[R_P\left(w_\eta\right)\right] \\
&= \frac{Q_{12}}{Q_{22}} + r_P - \frac{Q_{12}}{Q_{22}} = r_P
\end{aligned}
$$

Appendix 15:

Proposition 8: The return variance on the return generating correction component.

The return variance associated with the return generating correction component (2.6.18) *is;*

$$VAR\left[R_P\left(w_{\eta,S}\right)\right]=\sigma_{\eta,S}^2=\frac{1}{F^2}\frac{\left|Q_{\#2}\right|^2}{Q_{22}\left|Q_{\#1}\right|} \tag{2.6.25}$$

Proof:

The return variance associated with the return generating correction component (2.6.18) can be found by;

$$\begin{aligned}
VAR\left[R_P\left(w_{\eta,S}\right)\right] &= \sigma_{\eta,S}^2 \\
&= w_{\eta,S}^T \Sigma_A w_{\eta,S} \\
&= \frac{1}{F}\frac{\left|Q_{\#2}\right|}{\left|Q_{\#1}\right|}\left[\mu_A^T-\frac{Q_{12}}{Q_{22}}\upsilon^T\right]\frac{1}{F}\frac{\left|Q_{\#2}\right|}{\left|Q_{\#1}\right|}\left[\Sigma_A^{-1}\mu_A-\frac{Q_{12}}{Q_{22}}\Sigma_A^{-1}\upsilon\right] \\
&= \frac{1}{F^2}\left(\frac{\left|Q_{\#2}\right|}{\left|Q_{\#1}\right|}\right)^2\left(Q_{11}-\frac{Q_{12}^2}{Q_{22}}-\frac{Q_{12}^2}{Q_{22}}+\frac{Q_{12}}{Q_{22}}\frac{Q_{12}}{Q_{22}}Q_{22}\right) \\
&= \frac{1}{F^2}\left(\frac{\left|Q_{\#2}\right|}{\left|Q_{\#1}\right|}\right)^2\left(Q_{11}-\frac{Q_{12}^2}{Q_{22}}\right) \\
&= \frac{1}{F^2}\left(\frac{\left|Q_{\#2}\right|}{\left|Q_{\#1}\right|}\right)^2\frac{1}{Q_{22}}\left(Q_{11}Q_{22}-Q_{12}^2\right) \\
&= \frac{1}{F^2}\left(\frac{\left|Q_{\#2}\right|}{\left|Q_{\#1}\right|}\right)^2\frac{1}{Q_{22}}\left|Q_{\#1}\right| \\
&= \frac{1}{F^2}\frac{\left|Q_{\#2}\right|^2}{Q_{22}\left|Q_{\#1}\right|} \qquad (2.6.25)
\end{aligned}$$

Appendix 16:

Proposition 9: The zero return covariance between the minimum return variance and the return generating correction component.

The return covariance between the minimum return variance component, w_{MVP}*, and the return generating correction component,* $w_{\eta,S}$*, is*

$$COV\left[R_P\left(w_{MVP}\right)\ ,\ R_P\left(w_{\eta,S}\right)\right]=0 \tag{2.6.26}$$

Proof:

The return covariance between the portfolio returns on the minimum variance component, w_{MVP}, and the return generating correction component, $w_{\eta,S}$, can be written as

$$COV\left[R_P\left(w_{MVP}\right)\ ,\ R_P\left(w_{\eta,S}\right)\right]=w_{MVP}^T\Sigma_A w_{\eta,S} \tag{A16.1}$$

Inserting (2.2.6) transposed and (2.6.18) into (A16.1) results in

$$\begin{aligned}
w_{MVP}^T\Sigma_A w_{\eta,S} &= \frac{\upsilon^T}{Q_{22}}\frac{1}{F}\frac{\left|Q_{\#2}\right|}{\left|Q_{\#1}\right|}\left[\Sigma_A^{-1}\mu_A-\frac{Q_{12}}{Q_{22}}\Sigma_A^{-1}\upsilon\right] \\
&= \frac{1}{F}\frac{\left|Q_{\#2}\right|}{\left|Q_{\#1}\right|}\left(\frac{Q_{12}}{Q_{22}}-\frac{Q_{12}}{Q_{22}}\frac{Q_{22}}{Q_{22}}\right) \\
&= \frac{\theta}{F}\frac{\left|Q_{\#2}\right|}{\left|Q_{\#1}\right|}\left(\frac{Q_{12}}{Q_{22}}-\frac{Q_{12}}{Q_{22}}\right) \\
&= 0
\end{aligned}$$

Thus

$$COV\left[R_P\left(w_{MVP}\right)\ ,\ R_P\left(w_{\eta,S}\right)\right]=0 \tag{2.6.26}$$

Similarly, $COV\left[R_P\left(w_{\eta,S}\right)\ ,\ R_P\left(w_{MVP}\right)\right]=w_{\eta,S}^T\Sigma_A w_{MVP}=0$

Appendix 17:

Proposition 10: The return covariance between the minimum surplus return variance correction and the return generating components.

The return covariance between the minimum surplus return variance correction component, ϕ_{MSVP}*, and the return generating component,* w_η*, is*

$$COV\left[R_P\left(\phi_{MSVP}\right)\ ,\ R_P\left(w_\eta\right)\right] = -\frac{1}{F}\frac{\left|Q_{\#2}\right|}{\left|Q_{\#1}\right|}\left(r_P - \frac{Q_{12}}{Q_{22}}\right) \qquad (2.6.27)$$

Proof:

The return covariance between the portfolio returns on the minimum surplus return variance correction component, ϕ_{MSVP}, and the return generating component, w_η, can be written as

$$COV\left[R_P\left(\phi_{MSVP}\right)\ ,\ R_P\left(w_\eta\right)\right] = \phi_{MSVP}^T \Sigma_A w_\eta \qquad (A17.1)$$

Inserting (2.6.8) transposed and (2.2.11) into (A17.1) results in

$$\begin{aligned}
\phi_{MSVP}^T \Sigma_A w_\eta &= \frac{1}{F}\left[\Sigma_{AL}^T - \frac{Q_{23}}{Q_{22}}\upsilon^T\right]\frac{r_P Q_{22} - Q_{12}}{\left|Q_{\#1}\right|}\left[\Sigma_A^{-1}\mu_A - \frac{Q_{12}}{Q_{22}}\Sigma_A^{-1}\upsilon\right] \\
&= \frac{1}{F}\frac{r_P Q_{22} - Q_{12}}{\left|Q_{\#1}\right|}\left(Q_{13} - \frac{Q_{12}Q_{23}}{Q_{22}} - \frac{Q_{12}Q_{23}}{Q_{22}} + \frac{Q_{12}}{Q_{22}}\frac{Q_{23}}{Q_{22}}Q_{22}\right) \\
&= \frac{1}{F}\frac{r_P Q_{22} - Q_{12}}{\left|Q_{\#1}\right|}\left(Q_{13} - \frac{Q_{12}Q_{23}}{Q_{22}}\right) \\
&= \frac{1}{F}\frac{r_P Q_{22} - Q_{12}}{\left|Q_{\#1}\right|}\frac{1}{Q_{22}}\left(Q_{13}Q_{22} - Q_{12}Q_{23}\right) \\
&= -\frac{1}{F}\frac{r_P Q_{22} - Q_{12}}{\left|Q_{\#1}\right|}\frac{\left|Q_{\#2}\right|}{Q_{22}} \\
&= -\frac{1}{F}\frac{\left|Q_{\#2}\right|}{\left|Q_{\#1}\right|}\left(r_P - \frac{Q_{12}}{Q_{22}}\right)
\end{aligned}$$

or

$$COV\left[R_P\left(\phi_{MSVP}\right)\ ,\ R_P\left(w_\eta\right)\right] = -\frac{1}{F}\frac{\left|Q_{\#2}\right|}{\left|Q_{\#1}\right|}\left(r_P - \frac{Q_{12}}{Q_{22}}\right) \qquad (2.6.27)$$

Similarly, $COV\left[R_P\left(w_\eta\right)\ ,\ R_P\left(\phi_{MSVP}\right)\right] = w_\eta^T \Sigma_A \phi_{MSVP} = -\frac{1}{F}\frac{\left|Q_{\#2}\right|}{\left|Q_{\#1}\right|}\left(r_P - \frac{Q_{12}}{Q_{22}}\right)$

Appendix 18:

Proposition 11: The return covariance between the minimum surplus return variance correction and the return generating correction components.

The return covariance between the minimum surplus return variance correction component, ϕ_{MSVP} *, and the return generating correction component,* $w_{\eta,S}$ *, is*

$$COV\left[R_P\left(\phi_{MSVP}\right)\ ,\ R_P\left(w_{\eta,S}\right)\right]=-\frac{1}{F^2}\frac{\left|Q_{\#2}\right|^2}{Q_{22}\left|Q_{\#1}\right|} \tag{2.6.28}$$

Proof:

The return covariance between the portfolio returns on the minimum surplus return variance correction component, ϕ_{MSVP}, and the return generating correction component, $w_{\eta,S}$, can be written as

$$COV\left[R_P\left(\phi_{MSVP}\right)\ ,\ R_P\left(w_{\eta,S}\right)\right]=\phi_{MSVP}^T\Sigma_A w_{\eta,S} \tag{A18.1}$$

Inserting (2.6.8) transposed and (2.6.18) into (A18.1) results in

$$\begin{aligned}
\phi_{MSVP}^T\Sigma_A w_{\eta,S} &= \frac{1}{F}\left[\Sigma_{AL}^T-\frac{Q_{23}}{Q_{22}}\upsilon^T\right]\frac{1}{F}\frac{\left|Q_{\#2}\right|}{\left|Q_{\#1}\right|}\left[\Sigma_A^{-1}\mu_A-\frac{Q_{12}}{Q_{22}}\Sigma_A^{-1}\upsilon\right]\\
&=\frac{1}{F^2}\frac{\left|Q_{\#2}\right|}{\left|Q_{\#1}\right|}\left(Q_{13}-\frac{Q_{12}Q_{23}}{Q_{22}}-\frac{Q_{12}Q_{23}}{Q_{22}}+\frac{Q_{12}}{Q_{22}}\frac{Q_{23}}{Q_{22}}Q_{22}\right)\\
&=\frac{1}{F^2}\frac{\left|Q_{\#2}\right|}{\left|Q_{\#1}\right|}\left(Q_{13}-\frac{Q_{12}Q_{23}}{Q_{22}}\right)\\
&=\frac{1}{F^2}\frac{\left|Q_{\#2}\right|}{\left|Q_{\#1}\right|}\frac{1}{Q_{22}}\left(Q_{13}Q_{22}-Q_{12}Q_{23}\right)\\
&=-\frac{1}{F^2}\frac{\left|Q_{\#2}\right|^2}{Q_{22}\left|Q_{\#1}\right|}
\end{aligned}$$

or

$$COV\left[R_P\left(\phi_{MSVP}\right)\ ,\ R_P\left(w_{\eta,S}\right)\right]=-\frac{1}{F^2}\frac{\left|Q_{\#2}\right|^2}{Q_{22}\left|Q_{\#1}\right|} \tag{2.6.28}$$

Similarly, $COV\left[R_P\left(w_{\eta,S}\right)\ ,\ R_P\left(\phi_{MSVP}\right)\right]=w_{\eta,S}^T\Sigma_A\phi_{MSVP}=-\frac{1}{F^2}\frac{\left|Q_{\#2}\right|^2}{Q_{22}\left|Q_{\#1}\right|}$

Appendix 19:

Proposition 12: The return covariance between the return generating and the return generating correction components.

The return covariance between the return generating component, w_η *, and the return generating correction component,* $w_{\eta,S}$ *, is*

$$COV\left[R_P\left(w_\eta\right)\ ,\ R_P\left(w_{\eta,S}\right)\right]=\frac{1}{F}\frac{\left|Q_{\#2}\right|}{\left|Q_{\#1}\right|}\left(r_P-\frac{Q_{12}}{Q_{22}}\right) \qquad (2.6.29)$$

Proof:

The covariance between the portfolio returns on the return generating component, w_η , and the return generating correction component, $w_{\eta,S}$, can be written as

$$COV\left[R_P\left(w_\eta\right)\ ,\ R_P\left(w_{\eta,S}\right)\right]=w_\eta^T\Sigma_A w_{\eta,S} \qquad (A19.1)$$

Inserting (2.2.11) transposed and (2.6.18) into (A19.1) results in

$$\begin{aligned}
w_\eta^T\Sigma_A w_{\eta,S} &= \frac{r_PQ_{22}-Q_{12}}{\left|Q_{\#1}\right|}\left[\mu_A^T-\frac{Q_{12}}{Q_{22}}\upsilon^T\right]\frac{1}{F}\frac{\left|Q_{\#2}\right|}{\left|Q_{\#1}\right|}\left[\Sigma_A^{-1}\mu_A-\frac{Q_{12}}{Q_{22}}\Sigma_A^{-1}\upsilon\right]\\
&=\frac{1}{F}\frac{r_PQ_{22}-Q_{12}}{\left|Q_{\#1}\right|}\frac{\left|Q_{\#2}\right|}{\left|Q_{\#1}\right|}\left(Q_{11}-\frac{Q_{12}^2}{Q_{22}}-\frac{Q_{12}^2}{Q_{22}}+\frac{Q_{12}}{Q_{22}}\frac{Q_{12}}{Q_{22}}Q_{22}\right)\\
&=\frac{1}{F}\frac{r_PQ_{22}-Q_{12}}{\left|Q_{\#1}\right|}\frac{\left|Q_{\#2}\right|}{\left|Q_{\#1}\right|}\frac{1}{Q_{22}}\left(Q_{11}Q_{22}-Q_{12}^2\right)\\
&=\frac{1}{F}\frac{r_PQ_{22}-Q_{12}}{\left|Q_{\#1}\right|}\frac{\left|Q_{\#2}\right|}{\left|Q_{\#1}\right|}\frac{1}{Q_{22}}\left|Q_{\#1}\right|\\
&=\frac{1}{F}\frac{r_PQ_{22}-Q_{12}}{\left|Q_{\#1}\right|}\frac{\left|Q_{\#2}\right|}{Q_{22}}\\
&=\frac{1}{F}\frac{\left|Q_{\#2}\right|}{\left|Q_{\#1}\right|}\left(r_P-\frac{Q_{12}}{Q_{22}}\right)
\end{aligned}$$

or

$$COV\left[R_P\left(w_\eta\right)\ ,\ R_P\left(w_{\eta,S}\right)\right]=\frac{1}{F}\frac{\left|Q_{\#2}\right|}{\left|Q_{\#1}\right|}\left(r_P-\frac{Q_{12}}{Q_{22}}\right) \qquad (2.6.29)$$

Similarly, $COV\left[R_P\left(w_{\eta,S}\right)\ ,\ R_P\left(w_\eta\right)\right]=w_{\eta,S}^T\Sigma_A w_\eta=\frac{1}{F}\frac{\left|Q_{\#2}\right|}{\left|Q_{\#1}\right|}\left(r_P-\frac{Q_{12}}{Q_{22}}\right)$

Appendix 20:

Theorem 8: The return variance on the optimal surplus return variance portfolio.

The optimal surplus risk-return portfolio is expressed as $w_{rp,S} = w_{MVP} + \theta\phi_{MSVP} + w_{\eta} + \theta w_{\eta,S}$.

The return variance associated with the optimal surplus risk-return portfolio is

$$VAR\left[R_P\left(w_{rp.S}\right)\right] = \sigma^2_{rp,S} = \frac{r_P^2 Q_{22} - 2r_P Q_{12} + Q_{11} + \theta^2 F^{-2}\left|Q\right|}{\left|Q_{\#1}\right|} \qquad (2.6.30)$$

This result is in accordance with the results of theorems 1, 3, and 6, and propositions 2, 3, 5, 6 and 9 - 12.

Proof:

The optimal surplus return portfolio is composed out of four separate portfolios and expressed by (2.6.17) as;

$$w_{rp,S} = w_{MVP} + \theta\phi_{MSVP} + w_{\eta} + \theta w_{\eta,S}$$

The return variance associated with the optimal surplus return portfolio can be found via two methods, i.e.

I. $$VAR\left[R_P\left(w_{rp,S}\right)\right] = \sigma^2_{rp,S} = w^T_{rp,S}\Sigma_A w_{rp,S} \qquad (A20.1)$$

II. If (2.6.17) is expressed as $w_{rp,S} = \sum_{i=1}^{4} w_{P,i}$, then the return variance can be expressed as

$$VAR\left[R_P\left(w_{rp,S}\right)\right] = \sum_{x=1}^{4} VAR\left[R_P\left(w_x\right)\right] + \sum_{\substack{x,y=1\\x\neq y}}^{4} Cov\left[R_P\left(w_x\right)\ ,\ R_P\left(w_y\right)\right] \qquad (A20.2)$$

As all return variance and covariance terms have been derived in theorems 1, 3, and 6, and in propositions 2, 3, 5, 6, and 8 – 12, both methods are used for consistency.

<u>Method I:</u>

As method I suggests, the return variance can be written as

$$\begin{aligned}\sigma^2_{rp,S} &= w^T_{rp,S}\Sigma_A w_{rp,S}\\ &= \left[\frac{\upsilon^T}{Q_{22}} + \frac{\theta}{F}\left[\Sigma^T_{AL} - \frac{Q_{23}}{Q_{22}}\upsilon^T\right] + \frac{r_P Q_{22} - Q_{12} + \theta F^{-1}\left|Q_{\#2}\right|}{\left|Q_{\#1}\right|}\left[\mu^T_A - \frac{Q_{12}}{Q_{22}}\upsilon^T\right]\right]\\ &\bullet\left[\frac{\Sigma^{-1}_A\upsilon}{Q_{22}} + \frac{\theta}{F}\left[\Sigma^{-1}_A\Sigma_{AL} - \frac{Q_{23}}{Q_{22}}\Sigma^{-1}_A\upsilon\right] + \frac{r_P Q_{22} - Q_{12} + \theta F^{-1}\left|Q_{\#2}\right|}{\left|Q_{\#1}\right|}\left[\Sigma^{-1}_A\mu_A - \frac{Q_{12}}{Q_{22}}\Sigma^{-1}_A\upsilon\right]\right]\end{aligned}$$

Standard multiplication gives

$$\begin{aligned}
\sigma^2_{rp,S} =& \frac{Q_{22}}{Q_{22}^2} + \frac{\theta}{F}\frac{Q_{23}}{Q_{22}} - \frac{\theta}{F}\frac{Q_{23}}{Q_{22}}\frac{Q_{22}}{Q_{22}} + \frac{r_P Q_{22} - Q_{12} + \theta F^{-1}|Q_{\#2}|}{|Q_{\#1}|}\left(\frac{Q_{12}}{Q_{22}} - \frac{Q_{12}}{Q_{22}}\frac{Q_{22}}{Q_{22}}\right) \\
&+ \frac{\theta}{F}\frac{Q_{23}}{Q_{22}} + \frac{\theta^2}{F^2}Q_{33} - \frac{\theta^2}{F^2}\frac{Q_{23}^2}{Q_{22}} + \frac{\theta}{F}\frac{r_P Q_{22} - Q_{12} + \theta F^{-1}|Q_{\#2}|}{|Q_{\#1}|}\left(Q_{13} - \frac{Q_{12}Q_{23}}{Q_{22}}\right) \\
&- \frac{\theta}{F}\frac{Q_{23}}{Q_{22}}\frac{Q_{22}}{Q_{22}} - \frac{\theta^2}{F^2}\frac{Q_{23}^2}{Q_{22}} + \frac{\theta^2}{F^2}\frac{Q_{23}^2}{Q_{22}^2}Q_{22} \\
&+ \frac{\theta}{F}\frac{r_P Q_{22} - Q_{12} + \theta F^{-1}|Q_{\#2}|}{|Q_{\#1}|}\frac{Q_{23}}{Q_{22}}\left(-Q_{12} + \frac{Q_{12}}{Q_{22}}Q_{22}\right) \\
&+ \frac{r_P Q_{22} - Q_{12} + \theta F^{-1}|Q_{\#2}|}{|Q_{\#1}|}\left(\frac{Q_{12}}{Q_{22}} + \frac{\theta}{F}Q_{13} - \frac{\theta}{F}\frac{Q_{23}}{Q_{22}}Q_{12}\right) \\
&+ \left(\frac{r_P Q_{22} - Q_{12} + \theta F^{-1}|Q_{\#2}|}{|Q_{\#1}|}\right)^2\left(Q_{11} - \frac{Q_{12}^2}{Q_{22}}\right) \\
&+ \frac{r_P Q_{22} - Q_{12} + \theta F^{-1}|Q_{\#2}|}{|Q_{\#1}|}\frac{Q_{12}}{Q_{22}}\left(-\frac{Q_{22}}{Q_{22}} - \frac{\theta}{F}Q_{23} + \frac{\theta}{F}\frac{Q_{23}}{Q_{22}}Q_{22}\right) \\
&+ \left(\frac{r_P Q_{22} - Q_{12} + \theta F^{-1}|Q_{\#2}|}{|Q_{\#1}|}\right)^2\frac{Q_{12}}{Q_{22}}\left(-Q_{12} + \frac{Q_{12}}{Q_{22}}Q_{22}\right)
\end{aligned}$$

This simplifies to

$$\begin{aligned}
\sigma^2_{rp,S} =& \frac{1}{Q_{22}} + \frac{\theta}{F}\frac{Q_{23}}{Q_{22}}(1-1) + \frac{r_P Q_{22} - Q_{12} + \theta F^{-1}|Q_{\#2}|}{|Q_{\#1}|}\frac{Q_{12}}{Q_{22}}(1-1) \\
&+ \frac{\theta}{F}\frac{Q_{23}}{Q_{22}} + \frac{\theta^2}{F^2}\frac{\left(Q_{33}Q_{22} - Q_{23}^2\right)}{Q_{22}} + \frac{\theta}{F}\frac{r_P Q_{22} - Q_{12} + \theta F^{-1}|Q_{\#2}|}{|Q_{\#1}|}\frac{\left(Q_{13}Q_{22} - Q_{12}Q_{23}\right)}{Q_{22}} \\
&- \frac{\theta}{F}\frac{Q_{23}}{Q_{22}} + \frac{\theta^2}{F^2}\frac{Q_{23}^2}{Q_{22}}(-1+1) \\
&+ \frac{\theta}{F}\frac{r_P Q_{22} - Q_{12} + \theta F^{-1}|Q_{\#2}|}{|Q_{\#1}|}\frac{Q_{12}Q_{23}}{Q_{22}}(-1+1) \\
&+ \frac{r_P Q_{22} - Q_{12} + \theta F^{-1}|Q_{\#2}|}{|Q_{\#1}|}\frac{Q_{12}}{Q_{22}} + \frac{\theta}{F}\frac{r_P Q_{22} - Q_{12} + \theta F^{-1}|Q_{\#2}|}{|Q_{\#1}|}\frac{\left(Q_{13}Q_{22} - Q_{12}Q_{23}\right)}{Q_{22}} \\
&+ \left(\frac{r_P Q_{22} - Q_{12} + \theta F^{-1}|Q_{\#2}|}{|Q_{\#1}|}\right)^2\frac{\left(Q_{11}Q_{22} - Q_{12}^2\right)}{Q_{22}} \\
&- \frac{r_P Q_{22} - Q_{12} + \theta F^{-1}|Q_{\#2}|}{|Q_{\#1}|}\frac{Q_{12}}{Q_{22}} + \frac{\theta}{F}\frac{r_P Q_{22} - Q_{12} + \theta F^{-1}|Q_{\#2}|}{|Q_{\#1}|}\frac{Q_{12}Q_{23}}{Q_{22}}(-1+1) \\
&+ \left(\frac{r_P Q_{22} - Q_{12} + \theta F^{-1}|Q_{\#2}|}{|Q_{\#1}|}\right)^2\frac{Q_{12}^2}{Q_{22}}(-1+1)
\end{aligned}$$

Further simplification gives

$$\sigma^2_{rp,S} = \frac{1}{Q_{22}} + \frac{\theta}{F}\frac{Q_{23}}{Q_{22}}(1-1) + \frac{\theta^2}{F^2}\frac{\left(Q_{33}Q_{22} - Q_{23}^2\right)}{Q_{22}}$$
$$+\frac{2\theta}{F}\frac{r_P Q_{22} - Q_{12} + \theta F^{-1}\left|Q_{\#2}\right|}{\left|Q_{\#1}\right|}\frac{\left(Q_{13}Q_{22} - Q_{12}Q_{23}\right)}{Q_{22}}$$
$$+\frac{r_P Q_{22} - Q_{12} + \theta F^{-1}\left|Q_{\#2}\right|}{\left|Q_{\#1}\right|}\frac{Q_{12}}{Q_{22}}(1-1)$$
$$+\left(\frac{r_P Q_{22} - Q_{12} + \theta F^{-1}\left|Q_{\#2}\right|}{\left|Q_{\#1}\right|}\right)^2\frac{\left(Q_{11}Q_{22} - Q_{12}^2\right)}{Q_{22}}$$

or

$$\sigma^2_{rp,S} = \frac{1}{Q_{22}} + \frac{\theta^2}{F^2}\frac{\left|Q_{\#4}\right|}{Q_{22}}$$
$$-\frac{2\theta}{F}\frac{r_P Q_{22} - Q_{12} + \theta F^{-1}\left|Q_{\#2}\right|}{\left|Q_{\#1}\right|}\frac{\left|Q_{\#2}\right|}{Q_{22}}$$
$$+\left(\frac{r_P Q_{22} - Q_{12} + \theta F^{-1}\left|Q_{\#2}\right|}{\left|Q_{\#1}\right|}\right)^2\frac{\left|Q_{\#1}\right|}{Q_{22}}$$

Extending fractions so as to get a common numerator and expanding the second power term gives

$$\sigma^2_{rp,S} = \frac{1}{Q_{22}}\frac{\left|Q_{\#1}\right|}{\left|Q_{\#1}\right|} + \frac{\theta^2}{F^2}\frac{\left|Q_{\#4}\right|}{Q_{22}}\frac{\left|Q_{\#1}\right|}{\left|Q_{\#1}\right|}$$
$$-\frac{2\theta}{F}\frac{r_P\left|Q_{\#2}\right|}{\left|Q_{\#1}\right|} + \frac{2\theta}{F}\frac{Q_{12}\left|Q_{\#2}\right|}{Q_{22}\left|Q_{\#1}\right|} - \frac{2\theta^2}{F^2}\frac{\left|Q_{\#2}\right|^2}{Q_{22}\left|Q_{\#1}\right|}$$
$$+\frac{r_P^2 Q_{22}^2 - 2r_P Q_{12}Q_{22} + 2r_P Q_{22}\theta F^{-1}\left|Q_{\#2}\right| + Q_{12}^2 - 2Q_{12}\theta F^{-1}\left|Q_{\#2}\right| + \theta^2 F^{-2}\left|Q_{\#2}\right|^2}{Q_{22}\left|Q_{\#1}\right|}$$

Simplification along with changing expression for determinants gives

$$\sigma^2_{rp,S} = \frac{\left(Q_{11}Q_{22} - Q_{12}^2\right)}{Q_{22}\left|Q_{\#1}\right|} + \frac{\theta^2}{F^2}\frac{\left(Q_{22}Q_{33} - Q_{23}^2\right)}{Q_{22}}\frac{\left|Q_{\#1}\right|}{\left|Q_{\#1}\right|}$$
$$-\frac{2\theta}{F}\frac{r_P\left|Q_{\#2}\right|}{\left|Q_{\#1}\right|} + \frac{2\theta}{F}\frac{Q_{12}\left|Q_{\#2}\right|}{Q_{22}\left|Q_{\#1}\right|} - \frac{2\theta^2}{F^2}\frac{\left|Q_{\#2}\right|^2}{Q_{22}\left|Q_{\#1}\right|}$$
$$+\frac{r_P^2 Q_{22} - 2r_P Q_{12} + Q_{12}^2/Q_{22}}{\left|Q_{\#1}\right|} + \frac{2\theta}{F}\frac{r_P\left|Q_{\#2}\right|}{\left|Q_{\#1}\right|} - \frac{2\theta}{F}\frac{Q_{12}\left|Q_{\#2}\right|}{Q_{22}\left|Q_{\#1}\right|} + \frac{\theta^2}{F^2}\frac{\left|Q_{\#2}\right|^2}{Q_{22}\left|Q_{\#1}\right|}$$

which simplifies to

$$\sigma_{rp,S}^2 = \frac{Q_{11} - Q_{12}^2 / Q_{22}}{|Q_{\#1}|} + \frac{\theta^2}{F^2} \frac{\left(Q_{33} - Q_{23}^2 / Q_{22}\right)|Q_{\#1}|}{|Q_{\#1}|} + \frac{r_P^2 Q_{22} - 2 r_P Q_{12} + Q_{12}^2 / Q_{22}}{|Q_{\#1}|}$$
$$+ \frac{2\theta}{F} \frac{r_P |Q_{\#2}|}{|Q_{\#1}|}(-1+1) + \frac{2\theta}{F} \frac{Q_{12} |Q_{\#2}|}{Q_{22} |Q_{\#1}|}(1-1) + \frac{\theta^2}{F^2} \frac{|Q_{\#2}|^2}{Q_{22} |Q_{\#1}|}(-2+1)$$

Further simplification results in

$$\sigma_{rp,S}^2 = \frac{r_P^2 Q_{22} - 2 r_P Q_{12} + Q_{11} + Q_{12}^2 / Q_{22} (-1+1)}{|Q_{\#1}|}$$
$$+ \frac{\theta^2}{F^2} \left(\frac{\left(Q_{33} - Q_{23}^2 / Q_{22}\right)|Q_{\#1}|}{|Q_{\#1}|} - \frac{|Q_{\#2}|^2}{Q_{22} |Q_{\#1}|} \right)$$

Again, changing expressions for determinants gives

$$\sigma_{rp,S}^2 = \frac{r_P^2 Q_{22} - 2 r_P Q_{12} + Q_{11}}{|Q_{\#1}|}$$
$$+ \frac{\theta^2}{F^2} \left(\frac{\left(Q_{33} - Q_{23}^2 / Q_{22}\right)\left(Q_{11} Q_{22} - Q_{12}^2\right)}{|Q_{\#1}|} - \frac{\left(Q_{12} Q_{23} - Q_{13} Q_{22}\right)^2}{Q_{22} |Q_{\#1}|} \right)$$

and multiplication yields

$$\sigma_{rp,S}^2 = \frac{r_P^2 Q_{22} - 2 r_P Q_{12} + Q_{11}}{|Q_{\#1}|}$$
$$+ \frac{\theta^2}{F^2} \left(\frac{Q_{11} Q_{22} Q_{33} - Q_{12}^2 Q_{33} - Q_{11} Q_{23}^2 + Q_{12}^2 Q_{23}^2 / Q_{22}}{|Q_{\#1}|} - \frac{Q_{12}^2 Q_{23}^2 / Q_{22} - 2 Q_{12} Q_{13} Q_{23} + Q_{13}^2 Q_{22}}{|Q_{\#1}|} \right)$$

which simplifies to

$$\sigma_{rp,S}^2 = \frac{r_P^2 Q_{22} - 2 r_P Q_{12} + Q_{11}}{|Q_{\#1}|}$$
$$+ \frac{\theta^2}{F^2} \left(\frac{Q_{11} Q_{22} Q_{33} - Q_{11} Q_{23}^2 - Q_{12}^2 Q_{33} + Q_{12} Q_{13} Q_{23} + Q_{13} Q_{12} Q_{23} - Q_{13}^2 Q_{22}}{|Q_{\#1}|} \right)$$

Now, factorizing results in

$$\sigma_{rp,S}^2 = \frac{r_P^2 Q_{22} - 2 r_P Q_{12} + Q_{11}}{|Q_{\#1}|}$$
$$+ \frac{\theta^2}{F^2} \left(\frac{Q_{11}\left(Q_{22} Q_{33} - Q_{23}^2\right) - Q_{12}\left(Q_{12} Q_{33} - Q_{13} Q_{23}\right) + Q_{13}\left(Q_{12} Q_{23} - Q_{13} Q_{22}\right)}{|Q_{\#1}|} \right)$$

which finally gives the desired result

$$\sigma_{rp,S}^2 = \frac{r_P^2 Q_{22} - 2 r_P Q_{12} + Q_{11}}{|Q_{\#1}|} + \frac{\theta^2}{F^2} \frac{|Q|}{|Q_{\#1}|}$$

or more precisely:

$$VAR\left[R_P\left(w_{rp,S}\right)\right]=\sigma_{rp,S}^2=\frac{r_P^2Q_{22}-2r_PQ_{12}+Q_{11}+\theta^2F^{-2}\left|Q\right|}{\left|Q_{\#1}\right|} \tag{2.6.30}$$

<u>Method II:</u>

The return variance of the optimal surplus return portfolio can equivalently be expressed as

$$VAR\left[R_P\left(w_{rp,S}\right)\right]=\sum_{x=1}^{4}VAR\left[R_P\left(w_x\right)\right]+\sum_{\substack{x,y=1\\x\neq y}}^{4}Cov\left[R_P\left(w_x\right)\ ,\ R_P\left(w_y\right)\right] \tag{A20.2}$$

The return variances for the four components from theorem 1, and propositions 3, 5, and 8 are expressed as, respectively:

$$VAR\left[R_P\left(w_{MVP}\right)\right]=\frac{1}{Q_{22}}\ ,\quad VAR\left[R_P\left(w_\eta\right)\right]=\frac{\left(r_PQ_{22}-Q_{12}\right)^2}{Q_{22}\left|Q_{\#1}\right|}\ , \qquad \text{(2.2.8) - (2.2.17)}$$

$$VAR\left[R_P\left(\phi_{MSVP}\right)\right]=\frac{1}{F^2}\frac{\left|Q_{\#4}\right|}{Q_{22}}\ \text{ and }\ VAR\left[R_P\left(w_{\eta,S}\right)\right]=\frac{1}{F^2}\frac{\left|Q_{\#2}\right|^2}{Q_{22}\left|Q_{\#1}\right|} \qquad \text{(2.6.13) - (2.6.25)}$$

The covariances of returns between the four components are given by propositions 2, 6, and 9 – 12, respectively as:

$$COV\left[R_P\left(w_{MVP}\right)\ ,\ R_P\left(w_\eta\right)\right]=COV\left[R_P\left(w_\eta\right)\ ,\ R_P\left(w_{MVP}\right)\right]=0 \qquad \text{(Prop. 2)}$$

$$COV\left[R_P\left(w_{MVP}\right)\ ,\ R_P\left(\phi_{MSVP}\right)\right]=COV\left[R_P\left(\phi_{MSVP}\right)\ ,\ R_P\left(w_{MVP}\right)\right]=0 \qquad \text{(Prop. 6)}$$

$$COV\left[R_P\left(w_{MVP}\right)\ ,\ R_P\left(w_{\eta,S}\right)\right]=COV\left[R_P\left(w_{\eta,S}\right)\ ,\ R_P\left(w_{MVP}\right)\right]=0 \qquad \text{(Prop. 9)}$$

$$\begin{aligned}COV\left[R_P\left(\phi_{MSVP}\right)\ ,\ R_P\left(w_\eta\right)\right]&=COV\left[R_P\left(w_\eta\right)\ ,\ R_P\left(\phi_{MSVP}\right)\right]\\&=-\frac{1}{F}\frac{\left|Q_{\#2}\right|}{\left|Q_{\#1}\right|}\left(r_P-\frac{Q_{12}}{Q_{22}}\right)\end{aligned} \qquad \text{(Prop. 10)}$$

$$\begin{aligned}COV\left[R_P\left(\phi_{MSVP}\right)\ ,\ R_P\left(w_{\eta,S}\right)\right]&=COV\left[R_P\left(w_{\eta,S}\right)\ ,\ R_P\left(\phi_{MSVP}\right)\right]\\&=-\frac{1}{F^2}\frac{\left|Q_{\#2}\right|^2}{Q_{22}\left|Q_{\#1}\right|}\end{aligned} \qquad \text{(Prop. 11)}$$

$$\begin{aligned}COV\left[R_P\left(w_\eta\right)\ ,\ R_P\left(w_{\eta,S}\right)\right]&=COV\left[R_P\left(w_{\eta,S}\right)\ ,\ R_P\left(w_\eta\right)\right]\\&=\frac{1}{F}\frac{\left|Q_{\#2}\right|}{\left|Q_{\#1}\right|}\left(r_P-\frac{Q_{12}}{Q_{22}}\right)\end{aligned} \qquad \text{(Prop. 12)}$$

From propositions 10 and 12, it can be readily seen that

$$COV\left[R_P\left(\phi_{MSVP}\right)\ ,\ R_P\left(w_\eta\right)\right]+COV\left[R_P\left(w_\eta\right)\ ,\ R_P\left(w_{\eta,S}\right)\right]=0$$

As all these results are combined into (A20.2), the return variance associated with the optimal surplus return variance portfolio can be expressed as:

$$\begin{aligned}VAR\left[R_P\left(w_{rp,S}\right)\right]&=VAR\left[R_P\left(w_{MVP}\right)\right]+\theta^2 VAR\left[R_P\left(\phi_{MSVP}\right)\right]\\&+VAR\left[R_P\left(w_\eta\right)\right]+\theta^2 VAR\left[R_P\left(w_{\eta,S}\right)\right]\\&+2\theta COV\left[R_P\left(\phi_{MSVP}\right)\ ,\ R_P\left(w_{\eta,S}\right)\right]\end{aligned}$$

Inserting (2.2.8), (2.2.17), (2.6.13), (2.6.25) and (2.6.28) into (A20.2) gives

$$\begin{aligned}VAR\left[R_P\left(w_{rp,S}\right)\right]&=\frac{1}{Q_{22}}+\frac{\theta^2}{F^2}\frac{\left|Q_{\#4}\right|}{Q_{22}}+\frac{\left(r_P Q_{22}-Q_{12}\right)^2}{Q_{22}\left|Q_{\#1}\right|}\\&+\frac{\theta^2}{F^2}\frac{\left|Q_{\#2}\right|^2}{Q_{22}\left|Q_{\#1}\right|}-\frac{2\theta^2}{F^2}\frac{\left|Q_{\#2}\right|^2}{Q_{22}\left|Q_{\#1}\right|}\end{aligned}$$

This simplifies to

$$VAR\left[R_P\left(w_{rp,S}\right)\right]=\frac{1}{Q_{22}}+\frac{\theta^2}{F^2}\frac{\left|Q_{\#4}\right|}{Q_{22}}+\frac{\left(r_P Q_{22}-Q_{12}\right)^2}{Q_{22}\left|Q_{\#1}\right|}-\frac{\theta^2}{F^2}\frac{\left|Q_{\#2}\right|^2}{Q_{22}\left|Q_{\#1}\right|}$$

Extending fractions so as to get a common numerator gives

$$VAR\left[R_P\left(w_{rp,S}\right)\right]=\frac{\left|Q_{\#1}\right|}{Q_{22}\left|Q_{\#1}\right|}+\frac{\theta^2}{F^2}\frac{\left|Q_{\#4}\right|\left|Q_{\#1}\right|}{Q_{22}\left|Q_{\#1}\right|}+\frac{\left(r_P Q_{22}-Q_{12}\right)^2}{Q_{22}\left|Q_{\#1}\right|}-\frac{\theta^2}{F^2}\frac{\left|Q_{\#2}\right|^2}{Q_{22}\left|Q_{\#1}\right|}$$

Rearranging terms gives

$$VAR\left[R_P\left(w_{rp,S}\right)\right]=\frac{\left(r_P Q_{22}-Q_{12}\right)^2+\left|Q_{\#1}\right|}{Q_{22}\left|Q_{\#1}\right|}+\frac{\theta^2}{F^2}\frac{\left|Q_{\#4}\right|\left|Q_{\#1}\right|-\left|Q_{\#2}\right|^2}{Q_{22}\left|Q_{\#1}\right|}$$

Expanding the second power term and changing expression for determinants yields

$$\begin{aligned}VAR\left[R_P\left(w_{rp,S}\right)\right]&=\frac{r_P^2 Q_{22}^2-2r_P Q_{12}Q_{22}+Q_{12}^2+Q_{11}Q_{22}-Q_{12}^2}{Q_{22}\left|Q_{\#1}\right|}\\&+\frac{\theta^2}{F^2}\frac{\left(Q_{22}Q_{33}-Q_{23}^2\right)\left(Q_{11}Q_{22}-Q_{12}^2\right)-\left(Q_{12}Q_{23}-Q_{13}Q_{22}\right)^2}{Q_{22}\left|Q_{\#1}\right|}\end{aligned}$$

Simplifying the former fraction and multiplying in the second fraction gives

$$VAR\left[R_P\left(w_{rp,S}\right)\right] = \frac{r_P^2 Q_{22} - 2r_P Q_{12} + Q_{11}}{|Q_{\#1}|} + \frac{\theta^2}{F^2}\frac{Q_{11}Q_{22}Q_{33} - Q_{12}^2 Q_{33} - Q_{11}Q_{23}^2 + 2Q_{12}Q_{13}Q_{23} - Q_{13}^2 Q_{22}}{|Q_{\#1}|} + \frac{\theta^2}{F^2}\frac{Q_{12}^2 Q_{23}^2 / Q_{22}\left(1-1\right)}{|Q_{\#1}|}$$

which simplifies to

$$VAR\left[R_P\left(w_{rp,S}\right)\right] = \frac{r_P^2 Q_{22} - 2r_P Q_{12} + Q_{11}}{|Q_{\#1}|} + \frac{\theta^2}{F^2}\frac{Q_{11}Q_{22}Q_{33} - Q_{11}Q_{23}^2 - Q_{12}^2 Q_{33} + Q_{12}Q_{13}Q_{23} + Q_{13}Q_{12}Q_{23} - Q_{13}^2 Q_{22}}{|Q_{\#1}|}$$

Now, factorizing results in

$$VAR\left[R_P\left(w_{rp,S}\right)\right] = \frac{r_P^2 Q_{22} - 2r_P Q_{12} + Q_{11}}{|Q_{\#1}|} + \frac{\theta^2}{F^2}\frac{Q_{11}\left(Q_{22}Q_{33} - Q_{23}^2\right) - Q_{12}\left(Q_{12}Q_{33} - Q_{13}Q_{23}\right) + Q_{13}\left(Q_{12}Q_{23} - Q_{13}Q_{22}\right)}{|Q_{\#1}|}$$

which finally gives the desired result

$$VAR\left[R_P\left(w_{rp,S}\right)\right] = \frac{r_P^2 Q_{22} - 2r_P Q_{12} + Q_{11}}{|Q_{\#1}|} + \frac{\theta^2}{F^2}\frac{|Q|}{|Q_{\#1}|}$$

or

$$VAR\left[R_P\left(w_{rp,S}\right)\right] = \sigma_{rp,S}^2 = \frac{r_P^2 Q_{22} - 2r_P Q_{12} + Q_{11} + \theta^2 F^{-2}|Q|}{|Q_{\#1}|} \tag{2.6.30}$$

This proofs the expression for the optimal surplus portfolio return variance and that the two methods for finding the variance are equivalent.

Appendix 21:

Proof of equation (2.8.4).

The expression for the return variance on the optimal portfolio (2.8.4) was

$$VAR\left[R_P\left(w_{rp}\right)\right]=VAR\left[R_P\left(w_{MVP}\right)\right]+\left(\frac{r_PQ_{22}-Q_{12}}{\left|Q_{\#1}\right|}\right)^2 VAR\left[R_P\left(w_{\alpha}\right)\right]$$

The return variance on the MVP (2.2.8) was

$$VAR\left[R_P\left(w_{MVP}\right)\right]=\frac{1}{Q_{22}}$$

The return variance on w_α is

$$\begin{aligned}VAR\left[R_P\left(w_{\alpha}\right)\right]&=w_a^T\Sigma_A w_\alpha\\&=\left[\mu_A^T-\frac{Q_{12}}{Q_{22}}\upsilon^T\right]\left[\Sigma_A^{-1}\mu_A-\frac{Q_{12}}{Q_{22}}\Sigma_A^{-1}\upsilon\right]\\&=Q_{11}-\frac{Q_{12}^2}{Q_{22}} &&\text{(A21.1)}\\&=\frac{1}{Q_{22}}\left(Q_{11}Q_{22}-Q_{12}^2\right)\\&=\frac{\left|Q_{\#1}\right|}{Q_{22}}\end{aligned}$$

Inserting (2.2.8) and (A21.1) into (2.8.4) gives

$$\begin{aligned}VAR\left[R_P\left(w_{rp}\right)\right]&=VAR\left[R_P\left(w_{MVP}\right)\right]+\left(\frac{r_PQ_{22}-Q_{12}}{\left|Q_{\#1}\right|}\right)^2 VAR\left[R_P\left(w_{\alpha}\right)\right]\\&=\frac{1}{Q_{22}}+\left(\frac{r_PQ_{22}-Q_{12}}{\left|Q_{\#1}\right|}\right)^2\frac{\left|Q_{\#1}\right|}{Q_{22}}\\&=\frac{1}{Q_{22}}\frac{\left|Q_{\#1}\right|}{\left|Q_{\#1}\right|}+\frac{r_P^2Q_{22}^2-2r_PQ_{12}Q_{22}+Q_{12}^2}{\left|Q_{\#1}\right|^2}\frac{\left|Q_{\#1}\right|}{Q_{22}} &&\text{(A21.2)}\\&=\frac{Q_{11}Q_{22}-Q_{12}^2+r_P^2Q_{22}^2-2r_PQ_{12}Q_{22}+Q_{12}^2}{Q_{22}\left|Q_{\#1}\right|}\\&=\frac{r_P^2Q_{22}-2r_PQ_{12}+Q_{11}}{\left|Q_{\#1}\right|}\end{aligned}$$

Equation (A21.2) shows that (2.2.18) and (2.8.4) are equivalent expressions for the return variance on the optimal set in absence of liabilities.

Appendix 22:

Proof of equation (2.8.6)

The expression for the return variance on the optimal portfolio (2.8.6) was

$$
\begin{aligned}
VAR\left[R_P\left(w_{rp,S}\right)\right] &= VAR\left[R_P\left(w_{MSVP}\right)\right] \\
&\quad + \left(\frac{r_P Q_{22} - Q_{12} + \theta F^{-1}\left|Q_{\#2}\right|}{\left|Q_{\#1}\right|}\right)^2 VAR\left[R_P\left(w_\alpha\right)\right] \\
&\quad + 2\frac{r_P Q_{22} - Q_{12} + \theta F^{-1}\left|Q_{\#2}\right|}{\left|Q_{\#1}\right|} COV\left[R_P\left(w_{MSVP}\right)\ ,\ R_P\left(w_\alpha\right)\right]
\end{aligned}
$$

The return variance on the MSVP (2.6.15) was

$$
VAR\left[R_P\left(w_{MSVP}\right)\right] = \frac{1}{Q_{22}}\left(1 + \frac{\theta^2}{F^2}\left|Q_{\#4}\right|\right)
$$

Also

$$
\begin{aligned}
&\left(\frac{r_P Q_{22} - Q_{12} + \theta F^{-1}\left|Q_{\#2}\right|}{\left|Q_{\#1}\right|}\right)^2 \\
&\qquad = \frac{r_P^2 Q_{22}^2 - 2r_P Q_{12} Q_{22} + 2r_P Q_{22}\theta F^{-1}\left|Q_{\#2}\right| + Q_{12}^2 - 2Q_{12}\theta F^{-1}\left|Q_{\#2}\right| + \theta^2 F^{-2}\left|Q_{\#2}\right|^2}{\left|Q_{\#1}\right|^2}
\end{aligned}
$$

Equation (A21.1) was

$$
VAR\left[R_P\left(w_\alpha\right)\right] = \frac{\left|Q_{\#1}\right|}{Q_{22}}
$$

The covariance term in (2.8.6) is

$$
\begin{aligned}
COV\left[R_P\left(w_{MSVP}\right)\ ,\ R_P\left(w_\alpha\right)\right] &= w_{MSVP}^T \Sigma_A^{-1} w_\alpha \\
&= \left[\frac{\upsilon^T}{Q_{22}} + \frac{\theta}{F}\left[\Sigma_{AL}^T - \frac{Q_{23}}{Q_{22}}\upsilon^T\right]\right]\left[\Sigma_A^{-1}\mu_A - \frac{Q_{12}}{Q_{22}}\Sigma_A^{-1}\upsilon\right] \\
&= \frac{Q_{12}}{Q_{22}} - \frac{Q_{12}}{Q_{22}}\frac{Q_{22}}{Q_{22}} + \frac{\theta}{F}\left(Q_{13} - \frac{Q_{12}}{Q_{22}}Q_{23} - \frac{Q_{23}}{Q_{22}}Q_{12} + \frac{Q_{23}}{Q_{22}}\frac{Q_{12}}{Q_{22}}Q_{22}\right) \\
&= \frac{\theta}{F}\left(Q_{13} - \frac{Q_{12}Q_{23}}{Q_{22}}\right) \\
&= \frac{\theta}{F}\frac{\left(Q_{13}Q_{22} - Q_{12}Q_{23}\right)}{Q_{22}} \\
&= -\frac{\theta}{F}\frac{\left|Q_{\#2}\right|}{Q_{22}}
\end{aligned}
$$

Inserting these expressions into (2.8.6) results in

$$
\begin{aligned}
VAR\left[R_P\left(w_{rp,S}\right)\right] = {} & \frac{1}{Q_{22}} + \frac{\theta^2}{F^2}\frac{\left|Q_{\#4}\right|}{Q_{22}} \\
& - \frac{2\theta}{F}\frac{r_P Q_{22} - Q_{12} + \theta F^{-1}\left|Q_{\#2}\right|}{\left|Q_{\#1}\right|}\frac{\left|Q_{\#2}\right|}{Q_{22}} \\
& + \left(\frac{r_P Q_{22} - Q_{12} + \theta F^{-1}\left|Q_{\#2}\right|}{\left|Q_{\#1}\right|}\right)^2 \frac{\left|Q_{\#1}\right|}{Q_{22}}
\end{aligned}
\tag{A22.1}
$$

Equation (A22.1) can be found in the derivation of $VAR\left[R_P\left(w_{rp,S}\right)\right]$ in theorem 8, method I. This shows that (2.8.6) and (2.6.30) are equivalent expressions for the return variance on the optimal surplus return portfolio.

Appendix 23:

Proposition 13: The zero surplus return covariance between the minimum surplus return variance and the return generating components.

All portfolios in the frontier set, W_S*, can be written as* $w_{rp,S} = w_{MSVP} + \alpha w_\alpha$*. The surplus return covariance between the minimum variance component,* w_{MSVP}*, and the return generating component,* w_α*, is*

$$COV\left[R_P\left(w_\alpha\right) \;,\; R_P\left(w_{MSVP}\right) - \frac{\theta}{F} R_L \right] = 0 \tag{2.9.1}$$

Proof:

The surplus return covariance between the w_{MSVP}, and the total return generating component w_α, can be written as

$$COV\left[R_P\left(w_\alpha\right) \;,\; R_P\left(w_{MSVP}\right) - \frac{\theta}{F} R_L \right] = w_\alpha^T \Sigma_A w_{MSVP} - \frac{\theta}{F} w_\alpha^T \Sigma_{AL} \tag{A23.1}$$

Inserting (2.2.14) transposed and (2.6.6) into (A23.1) results in

$$\begin{aligned} w_\alpha^T \Sigma_A w_{MSVP} - \frac{\theta}{F} w_\alpha^T \Sigma_{AL} &= \left[\mu_A^T - \frac{Q_{12}}{Q_{22}} \upsilon^T \right] \left(\frac{\Sigma_A^{-1} \upsilon}{Q_{22}} + \frac{\theta}{F} \left[\Sigma_A^{-1} \Sigma_{AL} - \frac{Q_{23}}{Q_{22}} \Sigma_A^{-1} \upsilon \right] \right) \\ &\quad - \frac{\theta}{F} \left[\mu_A^T \Sigma_A^{-1} - \frac{Q_{12}}{Q_{22}} \upsilon^T \Sigma_A^{-1} \right] \Sigma_{AL} \\ &= \frac{Q_{12}}{Q_{22}} - \frac{Q_{12}}{Q_{22}} \frac{Q_{22}}{Q_{22}} \\ &\quad + \frac{\theta}{F} \left(Q_{13} - \frac{Q_{23}}{Q_{22}} Q_{12} - \frac{Q_{12}}{Q_{22}} Q_{23} + \frac{Q_{12}}{Q_{22}} \frac{Q_{23}}{Q_{22}} Q_{22} - Q_{13} + \frac{Q_{12}}{Q_{22}} Q_{23} \right) \\ &= 0 \end{aligned}$$

Thus

$$COV\left[R_P\left(w_\alpha\right) \;,\; R_P\left(w_{MSVP}\right) - \frac{\theta}{F} R_L \right] = 0 \tag{2.9.1}$$

Appendix 24:

Proof of equation (2.9.3)

Equation (2.9.3) was

$$VAR\left[R_P\left(w_{rp,S}\right)-\frac{\theta}{F}R_L\right]=VAR\left[R_P\left(w_{MSVP}\right)-\frac{\theta}{F}R_L\right]+\alpha^2 VAR\left[R_P\left(w_\alpha\right)\right]$$

Equation (2.9.3) states that the surplus return variance for any portfolio on the frontier can be composed out of the minimum surplus return variance and additional surplus return variance, resulting from $r_P \neq r_{MSVP}$.

The surplus return variance for any portfolio on the frontier can also be achieved directly from

$$\begin{aligned}VAR\left[R_p\left(w_{rp,S}\right)-\frac{\theta}{F}R_L\right]=VAR\left[R_p\left(w_{rp,S}\right)\right]+\frac{\theta^2}{F^2}VAR\left[R_L\right]\\ -\frac{2\theta}{F}COV\left[R_p\left(w_{rp,S}\right)\ ,\ R_L\right]\end{aligned} \tag{A24.1}$$

Expressions (2.9.3) and (A24.1) must be equal for (2.9.3) to be true. From the proof of theorem 8, the return variance of the optimal surplus return portfolio. (2.6.30) was written before simplification as

$$VAR\left[R_P\left(w_{rp,S}\right)\right]=\frac{1}{Q_{22}}+\frac{\theta^2}{F^2}\frac{\left|Q_{\#4}\right|}{Q_{22}}+\frac{\left(r_P Q_{22}-Q_{12}\right)^2}{Q_{22}\left|Q_{\#1}\right|}-\frac{\theta^2}{F^2}\frac{\left|Q_{\#2}\right|^2}{Q_{22}\left|Q_{\#1}\right|}$$

Using (2.6.16) and that $COV\left[R_P\left(w_{rp,S}\right)\ ,\ R_L\right]=\Sigma_{AL}^T w_{rp,S}$, then

$$COV\left[R_P\left(w_{rp,S}\right)\ ,\ R_L\right]=\frac{Q_{23}}{Q_{22}}+\frac{\theta}{F}\frac{\left|Q_{\#4}\right|}{Q_{22}}-\frac{r_P Q_{22}-Q_{12}+\theta F^{-1}\left|Q_{\#2}\right|}{\left|Q_{\#1}\right|}\frac{\left|Q_{\#2}\right|}{Q_{22}}$$

Then (A24.1) becomes

$$\begin{aligned}VAR\left[R_P\left(w_{rp,S}\right)-\frac{\theta}{F}R_L\right]&=VAR\left[R_P\left(w_{rp,S}\right)\right]+\frac{\theta^2}{F^2}VAR\left[R_L\right]-\frac{2\theta}{F}COV\left[R_P\left(w_{rp,S}\right)\ ,\ R_L\right]\\ &=\frac{1}{Q_{22}}+\frac{\theta^2}{F^2}\frac{\left|Q_{\#4}\right|}{Q_{22}}+\frac{\left(r_P Q_{22}-Q_{12}\right)^2}{Q_{22}\left|Q_{\#1}\right|}-\frac{\theta^2}{F^2}\frac{\left|Q_{\#2}\right|^2}{Q_{22}\left|Q_{\#1}\right|}+\frac{\theta^2}{F^2}VAR\left[R_L\right]\\ &\quad-\frac{2\theta}{F}\frac{Q_{23}}{Q_{22}}-\frac{2\theta^2}{F^2}\frac{\left|Q_{\#4}\right|}{Q_{22}}+\frac{2\theta}{F}\frac{r_P Q_{22}-Q_{12}+\theta F^{-1}\left|Q_{\#2}\right|}{\left|Q_{\#1}\right|}\frac{\left|Q_{\#2}\right|}{Q_{22}}\end{aligned}$$

which simplifies to

$$\begin{aligned}VAR\left[R_P\left(w_{rp,S}\right)-\frac{\theta}{F}R_L\right]&=\frac{1}{Q_{22}}-\frac{2\theta}{F}\frac{Q_{23}}{Q_{22}}-\frac{\theta^2}{F^2}\frac{\left|Q_{\#4}\right|}{Q_{22}}+\frac{\theta^2}{F^2}\frac{\left|Q_{\#2}\right|^2}{Q_{22}\left|Q_{\#1}\right|}+\frac{\theta^2}{F^2}VAR\left[R_L\right]\\ &\quad+\frac{r_P^2 Q_{22}^2-2r_P Q_{12}Q_{22}+Q_{12}^2}{Q_{22}\left|Q_{\#1}\right|}+\frac{2\theta}{F}\frac{r_P Q_{22}-Q_{12}}{\left|Q_{\#1}\right|}\frac{\left|Q_{\#2}\right|}{Q_{22}}\end{aligned}$$

or

$$VAR\left[R_P\left(w_{rp,S}\right)-\frac{\theta}{F}R_L\right]=\frac{1}{Q_{22}}-\frac{2\theta}{F}\frac{Q_{23}}{Q_{22}}-\frac{\theta^2}{F^2}\frac{\left|Q_{\#4}\right|}{Q_{22}}+\frac{\theta^2}{F^2}\frac{\left|Q_{\#2}\right|^2}{Q_{22}\left|Q_{\#1}\right|}+\frac{\theta^2}{F^2}VAR\left[R_L\right] \\ +\frac{r_P^2Q_{22}^2-2r_PQ_{12}Q_{22}+Q_{12}^2+2r_PQ_{22}\theta F^{-1}\left|Q_{\#2}\right|-2Q_{12}\theta F^{-1}\left|Q_{\#1}\right|}{Q_{22}\left|Q_{\#1}\right|} \tag{A24.2}$$

For analyzing (2.9.3), the minimum surplus return variance from (A24.1) is

$$\begin{aligned}VAR\left[R_P\left(w_{MSVP}\right)-\frac{\theta}{F}R_L\right]&=VAR\left[R_P\left(w_{MSVP}\right)\right]+\frac{\theta^2}{F^2}VAR\left[R_L\right]-\frac{2\theta}{F}\Sigma_{AL}^T w_{MSVP}\\&=\frac{1}{Q_{22}}\left(1+\frac{\theta^2}{F^2}\left|Q_{\#4}\right|\right)+\frac{\theta^2}{F^2}VAR\left[R_L\right]-\frac{2\theta}{F}\left(\frac{Q_{23}}{Q_{22}}+\frac{\theta}{F}\left(Q_{33}-\frac{Q_{23}}{Q_{22}}Q_{23}\right)\right)\\&=\frac{1}{Q_{22}}\left(1+\frac{\theta^2}{F^2}\left|Q_{\#4}\right|\right)+\frac{\theta^2}{F^2}VAR\left[R_L\right]-\frac{2\theta}{F}\frac{Q_{23}}{Q_{22}}-\frac{2\theta^2}{F^2Q_{22}}\left|Q_{\#4}\right|\\&=\frac{1}{Q_{22}}-\frac{2\theta}{F}\frac{Q_{23}}{Q_{22}}-\frac{\theta^2}{F^2}\frac{\left|Q_{\#4}\right|}{Q_{22}}+\frac{\theta^2}{F^2}VAR\left[R_L\right]\end{aligned}$$

and the variance on w_α is

$$\begin{aligned}VAR\left[R_P\left(w_\alpha\right)\right]&=\left[\mu_A^T-\frac{Q_{12}}{Q_{22}}\upsilon^T\right]\left[\Sigma_A^{-1}\mu_A-\frac{Q_{12}}{Q_{22}}\Sigma_A^{-1}\upsilon\right]\\&=Q_{11}-\frac{Q_{12}^2}{Q_{22}}\\&=\frac{\left|Q_{\#1}\right|}{Q_{22}}\end{aligned}$$

Then (2.9.3) becomes

$$\begin{aligned}VAR\left[R_P\left(w_{rp,S}\right)-\frac{\theta}{F}R_L\right]&=VAR\left[R_P\left(w_{MSVP}\right)-\frac{\theta}{F}R_L\right]\\&\quad+\left(\frac{r_PQ_{22}-Q_{12}+\theta F^{-1}\left|Q_{\#2}\right|}{\left|Q_{\#1}\right|}\right)^2 VAR\left[R_P\left(w_\alpha\right)\right]\\&=\frac{1}{Q_{22}}-\frac{2\theta}{F}\frac{Q_{23}}{Q_{22}}-\frac{\theta^2}{F^2}\frac{\left|Q_{\#4}\right|}{Q_{22}}\\&\quad+\frac{\left(r_PQ_{22}-Q_{12}+\theta F^{-1}\left|Q_{\#2}\right|\right)^2}{Q_{22}\left|Q_{\#1}\right|}+\frac{\theta^2}{F^2}VAR\left[R_L\right]\end{aligned}$$

or

$$VAR\left[R_P\left(w_{rp,S}\right)-\frac{\theta}{F}R_L\right]=\frac{1}{Q_{22}}-\frac{2\theta}{F}\frac{Q_{23}}{Q_{22}}-\frac{\theta^2}{F^2}\frac{|Q_{\#4}|}{Q_{22}}+\frac{\theta^2}{F^2}\frac{|Q_{\#2}|^2}{Q_{22}|Q_{\#1}|}+\frac{\theta^2}{F^2}VAR[R_L] + \frac{r_P^2Q_{22}^2-2r_PQ_{12}Q_{22}+Q_{12}^2+2r_PQ_{22}\theta F^{-1}|Q_{\#2}|-2Q_{12}\theta F^{-1}|Q_{\#1}|}{Q_{22}|Q_{\#1}|} \tag{A24.3}$$

By comparing (A24.2) and (A24.3), the expressions (2.9.3) and (A24.1) are equal and thus (2.9.3) is true.

Appendix 25:

Proposition 14: The funding ratio for absolute minimum surplus return variance and the absolute minimum surplus return variance.

Given any reasonable data used for surplus optimization, the absolute minimum surplus return variance for this data can be found. The funding ratio that minimizes the surplus return variance is

$$F_{MSV}=\frac{\theta\left(Q_{22}\sigma_L^2-|Q_{\#4}|\right)}{Q_{23}} \tag{2.11.3}$$

This funding ratio exists if the condition

$$\frac{3\theta}{F}\left(Q_{22}\sigma_L^2-|Q_{\#4}|\right)-2Q_{23}>0 \tag{2.11.4}$$

is satisfied for $F_{\min}\le F\le F_{\max}$.

The absolute minimum surplus variance that can be achieved for any feasible surplus optimal portfolio is

$$\begin{aligned} VAR\left[R_P\left(w_{MSV}\right)-\frac{\theta}{F_{MSV}}R_L\right] &= \sigma_{S,\min}^2 \\ &= \sigma_{S,MSVP}^2\Big|_{F=F_{MSV}} \\ &= \frac{1}{Q_{22}}\left(1-\frac{Q_{23}^2}{Q_{22}\sigma_L^2-|Q_{\#4}|}\right) \end{aligned} \tag{2.11.5}$$

Proof:

The minimum surplus variance portfolio was defined according to (2.6.6)

$$w_{MSVP} = \frac{\Sigma_A^{-1}\upsilon}{Q_{22}} + \frac{\theta}{F}\left[\Sigma_A^{-1}\Sigma_{AL} - \frac{Q_{23}}{Q_{22}}\Sigma_A^{-1}\upsilon\right]$$

and the return variance of (2.6.6) with (2.6.15)

$$\sigma_{MSVP}^2 = \frac{1}{Q_{22}}\left(1 + \frac{\theta^2}{F^2}\left|Q_{\#4}\right|\right)$$

The return covariance between the MSVP and liabilities is

$$\Sigma_{AL}^T w_{MSVP} = \frac{Q_{23}}{Q_{22}} + \frac{\theta}{F}\left(Q_{33} - \frac{Q_{23}}{Q_{22}}Q_{23}\right) = \frac{Q_{23}}{Q_{22}} + \frac{\theta}{F}\frac{\left|Q_{\#4}\right|}{Q_{22}}$$

The surplus return variance of the MSVP is written as

$$\begin{aligned}\sigma_{S,MSVP}^2 &= \sigma_{MSVP}^2 + \frac{\theta^2}{F^2}\sigma_L^2 - \frac{2\theta}{F}\Sigma_{AL}^T w_{MSVP} \\ &= \frac{1}{Q_{22}} + \frac{\theta^2}{F^2}\frac{\left|Q_{\#4}\right|}{Q_{22}} + \frac{\theta^2}{F^2}\sigma_L^2 - \frac{2\theta}{F}\frac{Q_{23}}{Q_{22}} - \frac{2\theta^2}{F^2}\frac{\left|Q_{\#4}\right|}{Q_{22}} \\ &= \frac{1}{Q_{22}} - \frac{\theta^2}{F^2}\frac{\left|Q_{\#4}\right|}{Q_{22}} + \frac{\theta^2}{F^2}\sigma_L^2 - \frac{2\theta}{F}\frac{Q_{23}}{Q_{22}} \\ &= \frac{1}{Q_{22}} + \frac{\theta^2}{F^2}\left(\sigma_L^2 - \frac{\left|Q_{\#4}\right|}{Q_{22}}\right) - \frac{2\theta}{F}\frac{Q_{23}}{Q_{22}}\end{aligned} \qquad (2.11.2)$$

The minimum is found by

$$\frac{\partial}{\partial F}\sigma_{S,MSVP}^2 = 0 \quad \text{and} \quad \frac{\partial^2}{\partial F^2}\sigma_S^2 > 0$$

Then

$$\frac{\partial}{\partial F}\sigma_{S,MSVP}^2 = \frac{2\theta^2}{F^3}\frac{\left|Q_{\#4}\right|}{Q_{22}} - \frac{2\theta^2\sigma_L^2}{F^3} + \frac{2\theta}{F^2}\frac{Q_{23}}{Q_{22}} = 0$$

or

$$\frac{\theta}{F^2}\frac{Q_{23}}{Q_{22}} = \frac{\theta^2\sigma_L^2}{F^3} - \frac{\theta^2}{F^3}\frac{\left|Q_{\#4}\right|}{Q_{22}}$$

Multiplying with F^3/θ gives

$$F\frac{Q_{23}}{Q_{22}} = \theta\sigma_L^2 - \frac{\theta\left|Q_{\#4}\right|}{Q_{22}}$$

from which the minimum surplus return variance funding ratio can be found:

$$F_{MSV} = \frac{\theta\left(Q_{22}\sigma_L^2 - |Q_{\#4}|\right)}{Q_{23}} \tag{2.11.3}$$

The second order condition for a minimum is

$$\frac{\partial^2}{\partial F^2}\sigma_{S,MSVP}^2 = \frac{6\theta^2\sigma_L^2}{F^4} - \frac{6\theta^2}{F^4}\frac{|Q_{\#4}|}{Q_{22}} - \frac{4\theta}{F^3}\frac{Q_{23}}{Q_{22}} > 0$$

or

$$\frac{3\theta^2}{F^4}\left(\sigma_L^2 - \frac{|Q_{\#4}|}{Q_{22}}\right) - \frac{2\theta}{F^3}\frac{Q_{23}}{Q_{22}} > 0$$

Multiplying with $Q_{22}F^3/\theta$ gives

$$\frac{3\theta}{F}\left(Q_{22}\sigma_L^2 - |Q_{\#4}|\right) - 2Q_{23} > 0 \tag{2.11.4}$$

For $F = F_{MSV}$, the second order condition yields $(3-2)Q_{23} = Q_{23} > 0$.

The minimum surplus variance is found by inserting (2.11.3) into (2.11.2)

$$\begin{aligned}\sigma_{S,\min}^2(w_{MSVP}) &= \frac{1}{Q_{22}} - \frac{\theta^2}{F^2}\frac{|Q_{\#4}|}{Q_{22}} + \frac{\theta^2}{F^2}\sigma_L^2 - \frac{2\theta}{F}\frac{Q_{23}}{Q_{22}} \\ &= \frac{1}{Q_{22}} - \frac{Q_{23}^2}{\left(Q_{22}\sigma_L^2 - |Q_{\#4}|\right)^2}\frac{|Q_{\#4}|}{Q_{22}} + \frac{Q_{23}^2\sigma_L^2}{\left(Q_{22}\sigma_L^2 - |Q_{\#4}|\right)^2} - \frac{2Q_{23}}{\left(Q_{22}\sigma_L^2 - |Q_{\#4}|\right)}\frac{Q_{23}}{Q_{22}}\end{aligned}$$

Writing this expression with a common numerator yields

$$\begin{aligned}\sigma_{S,\min}^2(w_{MSVP}) &= \frac{1}{Q_{22}} - \frac{Q_{23}^2|Q_{\#4}|}{Q_{22}\left(Q_{22}\sigma_L^2 - |Q_{\#4}|\right)^2} + \frac{Q_{22}Q_{23}^2\sigma_L^2}{Q_{22}\left(Q_{22}\sigma_L^2 - |Q_{\#4}|\right)^2} - \frac{2Q_{23}^2\left(Q_{22}\sigma_L^2 - |Q_{\#4}|\right)}{Q_{22}\left(Q_{22}\sigma_L^2 - |Q_{\#4}|\right)^2} \\ &= \frac{1}{Q_{22}} - \frac{-Q_{23}^2|Q_{\#4}| + Q_{22}Q_{23}^2\sigma_L^2 - 2Q_{22}Q_{23}^2\sigma_L^2 + 2Q_{23}^2|Q_{\#4}|}{Q_{22}\left(Q_{22}\sigma_L^2 - |Q_{\#4}|\right)^2}\end{aligned}$$

This simplifies to the expression for the absolute minimum surplus variance

$$\begin{aligned}\sigma_{S,\min}^2(w_{MSVP}) &= \frac{1}{Q_{22}} + \frac{Q_{23}^2|Q_{\#4}| - Q_{22}Q_{23}^2\sigma_L^2}{Q_{22}\left(Q_{22}\sigma_L^2 - |Q_{\#4}|\right)^2} \\ &= \frac{1}{Q_{22}} - \frac{Q_{23}^2\left(Q_{22}\sigma_L^2 - |Q_{\#4}|\right)}{Q_{22}\left(Q_{22}\sigma_L^2 - |Q_{\#4}|\right)^2} \\ &= \frac{1}{Q_{22}} - \frac{Q_{23}^2}{Q_{22}\left(Q_{22}\sigma_L^2 - |Q_{\#4}|\right)} \\ &= \frac{1}{Q_{22}}\left(1 - \frac{Q_{23}^2}{Q_{22}\sigma_L^2 - |Q_{\#4}|}\right)\end{aligned} \tag{2.11.5}$$

Appendix 26:

Theorem 9: The market portfolio in presence of liabilities.

The risky asset market portfolio, given in theorem 4, is expressed as (2.3.3)

$$w_{MKT} = \frac{\Sigma_A^{-1}\left(\mu_A - r_f \upsilon\right)}{\upsilon^T \Sigma_A^{-1}\left(\mu_A - r_f \upsilon\right)}$$

The allocation vector for the risky assets market portfolio in presence of liabilities is

$$w_{MKT,S} = \frac{\Sigma_A^{-1}\left(\mu_A - r_f \upsilon\right)}{\upsilon^T \Sigma_A^{-1}\left(\mu_A - r_f \upsilon\right)} + \frac{\theta}{F}\left[\Sigma_A^{-1}\Sigma_{AL} - \frac{\upsilon^T \Sigma_A^{-1}\Sigma_{AL}}{\upsilon^T \Sigma_A^{-1}\left(\mu_A - r_f \upsilon\right)}\Sigma_A^{-1}\left(\mu_A - r_f \upsilon\right)\right] \tag{2.12.1}$$

The portfolio (2.12.1) *is composed out of two portfolios;*

$$w_{MKT,S} = w_{MKT} + \theta\phi_{MKT} \tag{2.12.2}$$

where w_{MKT} *is in accordance with theorem 4, the market portfolio correction component is*

$$\phi_{MKT} = \frac{1}{F}\left[\Sigma_A^{-1}\Sigma_{AL} - \frac{\upsilon^T \Sigma_A^{-1}\Sigma_{AL}}{\upsilon^T \Sigma_A^{-1}\left(\mu_A - r_f \upsilon\right)}\Sigma_A^{-1}\left(\mu_A - r_f \upsilon\right)\right] \tag{2.12.3}$$

and $\upsilon^T \phi_{MKT} = 0$.

Proof:

The expected return on a portfolio can be decomposed into the risk-free rate, r_f, and the portfolio's risk premium on the risk-free rate, $w_{P,S}^T\left(\mu_A - r_f \upsilon\right)$. Formally, the decomposed expected return on any portfolio can be expressed as (2.3.2)

$$E\left[R_{P,S}\left(w_{P,S}\right)\right] = w_{P,S}^T \mu_A = w_{P,S}^T\left(\mu_A - r_f \upsilon\right) + r_f = r_{P,S}$$

To solve for the market portfolio in presence of liabilities, (2.12.1) is used subject to the decomposed expected portfolio return constraint (2.3.2);

$$L(\bullet) = \frac{1}{2} w_{P,S}^T \Sigma_A w_{P,S} - \frac{\theta}{F} w_{P,S}^T \Sigma_{AL} + \lambda\left(r_P - w_{P,S}^T\left(\mu_A - r_f \upsilon\right) - r_f\right)$$

The partial derivatives of L w.r.t. w_P yield necessary first order conditions for stationarity and primal feasibility:

$$\Sigma_A w_{P,S}^* - \frac{\theta}{F}\Sigma_{AL} - \lambda\left(\mu_A - r_f \upsilon\right) = 0, \quad \text{and} \quad w_{P,S}^T\left(\mu_A - r_f \upsilon\right) - r_f = r_{P,S}$$

A second order sufficient condition for a minimum is sufficient since the covariance matrix is positive definite. Isolating the optimal allocation vector $w_{P,S}^{*}$ from the initial first order condition gives

$$w_{P,S}^{*} = \frac{\theta}{F}\Sigma_A^{-1}\Sigma_{AL} + \lambda\Sigma_A^{-1}\left(\mu_A - r_f\upsilon\right) \tag{A26.1}$$

The allocation vector is not unit normalized, so we consider the normalization constraint

$$\upsilon^T w_{P,S}^{*} = 1 \tag{A26.2}$$

Inserting (A26.1) into (A26.2) yields

$$\frac{\theta}{F}\upsilon^T\Sigma_A^{-1}\Sigma_{AL} + \lambda\upsilon^T\Sigma_A^{-1}\left(\mu_A - r_f\upsilon\right) = 1$$

Isolating λ gives

$$\lambda_1 = \frac{1 - \frac{\theta}{F}\upsilon^T\Sigma_A^{-1}\Sigma_{AL}}{\upsilon^T\Sigma_A^{-1}\left(\mu_A - r_f\upsilon\right)}$$

Inserting λ into (A26.1) gives

$$w_{P,S}^{*} = \frac{\theta}{F}\Sigma_A^{-1}\Sigma_{AL} + \left(\frac{1 - \frac{\theta}{F}\upsilon^T\Sigma_A^{-1}\Sigma_{AL}}{\upsilon^T\Sigma_A^{-1}\left(\mu_A - r_f\upsilon\right)}\right)\Sigma_A^{-1}\left(\mu_A - r_f\upsilon\right)$$

that finally results in expression for the market portfolio with the liability hedge:

$$w_{MKT,S} = \frac{\Sigma_A^{-1}\left(\mu_A - r_f\upsilon\right)}{\upsilon^T\Sigma_A^{-1}\left(\mu_A - r_f\upsilon\right)} + \frac{\theta}{F}\left[\Sigma_A^{-1}\Sigma_{AL} - \frac{\upsilon^T\Sigma_A^{-1}\Sigma_{AL}}{\upsilon^T\Sigma_A^{-1}\left(\mu_A - r_f\upsilon\right)}\Sigma_A^{-1}\left(\mu_A - r_f\upsilon\right)\right] \tag{2.12.1}$$

This portfolio is composed out of two components:

$$w_{MKT,S} = w_{MKT} + \theta\phi_{MKT} \tag{2.12.2}$$

where the market portfolio, w_{MKT}, is in accordance with theorem 6 and the market portfolio correction component is

$$\phi_{MKT} = \frac{1}{F}\left[\Sigma_A^{-1}\Sigma_{AL} - \frac{\upsilon^T\Sigma_A^{-1}\Sigma_{AL}}{\upsilon^T\Sigma_A^{-1}\left(\mu_A - r_f\upsilon\right)}\Sigma_A^{-1}\left(\mu_A - r_f\upsilon\right)\right] \tag{2.12.3}$$

Multiplying ϕ_{MKT} with υ^T from the left results in

$$\begin{aligned}\upsilon^T\phi_{MKT} &= \frac{1}{F}\left[\upsilon^T\Sigma_A^{-1}\Sigma_{AL} - \frac{\upsilon^T\Sigma_A^{-1}\Sigma_{AL}}{\upsilon^T\Sigma_A^{-1}\left(\mu_A - r_f\upsilon\right)}\upsilon^T\Sigma_A^{-1}\left(\mu_A - r_f\upsilon\right)\right] \\ &= \frac{1}{F}\upsilon^T\Sigma_A^{-1}\Sigma_{AL}\left(1-1\right) \\ &= 0\end{aligned}$$

Appendix 27: The normality assumption of returns and the Jarque-Bera normality test

In financial modelling, logarithmic returns or continuously compounded returns are often assumed to be normally distributed, based on the roughly symmetric character of stock return distributions. The assumption on normality has strong empirical support. In general, normality tests of daily portfolio return do not reject the normality assumption, suggesting that the assumption of normality for stock portfolio returns may be acceptable even if individual stock returns are not normally distributed. By lengthening the observation horizon, the assumption on normality appears to become more acceptable for individual stock returns. Another reason for the normality assumption is the central limit theorem, which states that, under mild conditions, the mean of a large number of random variables independently drawn from the same distribution is distributed approximately normally, irrespective of the form of the original distribution.

The first two moments of a random variable *X*, mean and variance, uniquely determine a normal distribution. For a normal distribution, the skewness is zero and kurtosis, K_x, is three. Excess kurtosis is defined as $\mathrm{K}_x - 3$ and is therefore zero for a normal distribution.

A distribution with positive excess kurtosis has relatively more of its probability mass in its tails than normal distribution does, i.e. it contains more extreme values. A distribution with positive (negative) excess kurtosis is said to be leptocurtic (platykurtic).

Several tests for hypothesis testing on whether logarithmic returns are normally distributed are used. The Jarque–Bera test is a goodness-of-fit test of whether sample data have the skewness and kurtosis matching a normal distribution.

Definition. The test statistic *JB* is defined as

$$JB_x = \frac{N}{6}\left(\hat{\Psi}_x^2 + \frac{1}{4}\left(\hat{\mathrm{K}}_x - 3\right)^2\right) \tag{A27.1}$$

where N is the number of observations or degrees of freedom for the random variable X, the sample skewness and the sample kurtosis are defined as $\hat{\Psi}_x$ and $\hat{\mathrm{K}}_x$, respectively.

If the data come from a normal distribution, the *JB* statistic asymptotically has a chi-squared distribution with two degrees of freedom, so the statistic can be used to test the hypothesis that the data on logarithmic returns are from a normal distribution. The null hypothesis for a given p - value is a joint hypothesis of the skewness being zero and the excess kurtosis being zero. As mentioned before, samples from a normal distribution have an expected skewness and an expected excess kurtosis $(K_x - 3)$ of zero. As the definition of *JB* shows, any deviation from this increases the *JB* statistic.

Appendix 28: Non-linear Optimization

Appendices 28 and 29 on non-linear optimization and optimality conditions are based on Boyd & Vandenbergher (2004) and Rockafellar (1970).

An optimization problem involves finding the maximum or minimum of a function f, with or without subject to some constraint functions $g_1,...,g_m$ and $h_1,...,h_l$ that are defined on some domain $\Omega \subset \mathbb{R}^n$. Let the goal be to find a minimum defined by f, g_i and h_j. Then for $x \in \mathbb{R}^n$, the optimization model has the form

$$\min_{\mathbf{x}\in\Omega} \; f(\mathbf{x}) \tag{A28.1}$$

s.t.

$$g_i(x) \le 0 \quad \forall i \in I, \quad I = \{1,...,m\} \tag{A28.2}$$

$$h_j(x) = 0 \quad \forall j \in J, \quad J = \{1,...,l\} \tag{A28.3}$$

Let the cost function/objective $f : \mathbb{R}^n \to \mathbb{R}$ and the constraint functions $g_i : \mathbb{R}^n \to \mathbb{R}$, $h_j : \mathbb{R}^n \to \mathbb{R}$ all be continuously differentiable on the domain Ω. Then $\mathbf{x}^*$ is a local minimum of $f(\mathbf{x}^*)$ if and only if the Karush-Kuhn-Tucker (KKT) first order conditions provide necessary conditions for a solution to be optimal. Under convexity, the KKT conditions are also sufficient. If some of the functions g_i, h_j are non-differentiable, subdifferential versions of KKT conditions are available. The KKT approach to non-linear programming generalizes the method of Lagrange multipliers that is valid for equality constraints only.

If $\mathbf{x}^*$ is a local minimum that satisfies the KKT conditions, then there exist constants; KKT multipliers μ_i, λ_j, that provide necessary conditions for:

Stationarity;

$$-\nabla_x f(\mathbf{x}^*) = \sum_{i=1}^{m} \mu_i^* \nabla_x g_i(\mathbf{x}^*) + \sum_{j=1}^{l} \lambda_j^* \nabla_x h_j(\mathbf{x}^*) \tag{A28.4}$$

Primal feasibility;

$$g_i(\mathbf{x}^*) \le 0 \quad \forall i \in I \tag{A28.5}$$

$$h_j(\mathbf{x}^*) = 0 \quad \forall j \in J \tag{A28.6}$$

Dual feasibility;

$$\mu_i^* \ge 0 \quad \forall i \in I \tag{A28.7}$$

Complementary slackness;

$$\mu_i^* g_i\left(\mathbf{x}^*\right) \leq 0 \quad \forall i \in I \tag{A28.8}$$

In the particular case where no inequality constraints are present, i.e. $i = 0$, the KKT first order conditions turn into Lagrange first order conditions and the KKT multipliers are referred to as Lagrange multipliers; $\lambda_j \quad \forall j \in J$.

In some cases, the necessary conditions are also sufficient for optimality but in general, the necessary conditions are not sufficient for optimality and additional information is necessary, such as the second order sufficient conditions.

For solving the problem (A28.1) - (A28.3), it is convenient to define the constraint functions as

$$\mathbf{g}(\mathbf{x}) = \left[g_i(\mathbf{x})\right]_{i=1,\dots,m}^T, \quad \mathbf{h}(\mathbf{x}) = \left[h_j(\mathbf{x})\right]_{j=1,\dots,l}^T$$

and the KKT multipliers as

$$\boldsymbol{\mu} = \left[\mu_i\right]_{i=1,\dots,m}^T \quad \text{and} \quad \boldsymbol{\lambda} = \left[\lambda_j\right]_{j=1,\dots,l}^T$$

The Lagrangian is defined as

$$L(\mathbf{x}, \boldsymbol{\mu}, \boldsymbol{\lambda}) = f(\mathbf{x}) + \boldsymbol{\mu}^T \mathbf{g}(\mathbf{x}) + \boldsymbol{\lambda}^T \mathbf{h}(\mathbf{x}) \tag{A28.9}$$

If there exist unique μ_i^* satisfying the KKT conditions, then $\mathbf{x}^*$ is a local minimum on f . The KKT conditions (A28.4) – (A28.8) are found by the following:

The partial derivative of the Lagrangian w.r.t. $\mathbf{x}$ is set to zero for finding a stationary point:

$$\nabla_x L\left(\mathbf{x}^*, \boldsymbol{\mu}^*, \boldsymbol{\lambda}^*\right) = \mathbf{0}$$

Here, a constrained local optimum occurs at $\mathbf{x}^*$ when $\nabla_x f\left(\mathbf{x}^*\right)$ and $\nabla_x \mathbf{h}\left(\mathbf{x}^*\right)$ are parallel, i.e.

$$\nabla_x f\left(\mathbf{x}^*\right) + \sum_{i=1}^{m} \mu_i^* \nabla_x g_i\left(\mathbf{x}^*\right) + \sum_{j=1}^{l} \lambda_j^l \nabla_x h_j\left(\mathbf{x}^*\right) = 0 \tag{A28.10}$$

The direction of the normal is arbitrary as the constraints can be imposed as either $\mathbf{h}\left(\mathbf{x}^*\right) = 0$ or $-\mathbf{h}\left(\mathbf{x}^*\right) = 0$ and the same applies to the inequality constraints. Equation (A28.10) gives the formal stationarity condition:

$$-\nabla_x f\left(\mathbf{x}^*\right) = \sum_{i=1}^{m} \mu_i^* \nabla_x g_i\left(\mathbf{x}^*\right) + \sum_{j=1}^{l} \lambda_j^* \nabla_x h_j\left(\mathbf{x}^*\right) \quad \text{(A28.4)}$$

Primal feasibility is found via the derivative of the Lagrangian w.r.t. each μ_i and λ_j ;

$\nabla_{\mu i} L\left(\mathbf{x}^*, \boldsymbol{\mu}^*, \boldsymbol{\lambda}^*\right)$ yields

$$g_i\left(\mathbf{x}^*\right) \le 0 \quad \forall i \in I \quad \text{(A28.5)}$$

and

$\nabla_{\lambda j} L\left(\mathbf{x}^*, \boldsymbol{\mu}^*, \boldsymbol{\lambda}^*\right)$ yields

$$h_j\left(\mathbf{x}^*\right) \le 0 \quad \forall j \in J \quad \text{(A28.6)}$$

As a result of the primal feasibility condition (A28.6), dual feasibility condition requires

$$\mu_i^* \ge 0 \quad \forall i \in I \quad \text{(A28.7)}$$

Since the gradient (A28.10) is zero for $\mathbf{x} = \mathbf{x}^*$, then $\sum_{i=1}^{m} \mu_i^* g_i\left(\mathbf{x}^*\right) = 0$. Since every term in that sum is non-positive, each term is zero:

$$\mu_i^* g_i\left(\mathbf{x}^*\right) \le 0 \quad \forall i \in I \quad \text{(A28.8)}$$

Equation (A28.8) refers to as the complimentary slackness conditions. They imply that if the i- th constraint is strictly satisfied, then the corresponding dual variable, μ_i, is zero, Conversely, if $\mu_i > 0$ then $g_i\left(\mathbf{x}^*\right) = 0$.

The KKT conditions above are necessary first order conditions. In most cases, a sufficient second order condition is needed to ensure minimum/maximum, i.e. positive/negative semi-definite Hessian. In the minimization problem (A28.1) – (A28.3), the Hessian at $\mathbf{x}^*$ must be positive and the sufficient condition is

$$\mathbf{v}^T \left(\nabla_{xx} L\left(\mathbf{x}^*, \mu^*\right)\right)\mathbf{v} \ge 0, \quad \forall \mathbf{v} \in \mathbb{R}^n \quad \text{(A28.11)}$$

Where a minimization problem is only subject to equality constraints, the problem can be stated as:

$$\min_{x \in \Omega} \; f\left(x\right)$$

s.t.

$$h_j\left(x\right) = 0 \quad \forall j$$

The Lagrangian is defined as

$$L\left(\mathbf{x}, \mu, \lambda\right) = f\left(\mathbf{x}\right) + \lambda^T \mathbf{h}\left(\mathbf{x}\right)$$

As no inequality constraints are present, the KKT conditions turn into Lagrange conditions and as:

$\mathbf{x}^*$ is a local minimum $\Leftrightarrow$ there exist unique λ_j^* s.t.

1. $\nabla_x L\left(\mathbf{x}^*, \lambda^*\right) = \mathbf{0}$

   This condition encodes the stationarity condition (A28.4); $-\nabla_x f\left(\mathbf{x}^*\right) = \sum_{j=1}^{l} \lambda_j^* \nabla_x h_j\left(\mathbf{x}^*\right)$

   as $\nabla_x L\left(\mathbf{x}^*, \lambda^*\right) = \nabla_x f\left(\mathbf{x}^*\right) + \sum_{j=1}^{l} \lambda_j h_j\left(\mathbf{x}\right) = 0$

2. $\nabla_\lambda L\left(\mathbf{x}^*, \lambda^*\right) = \mathbf{0}$

   This condition encodes the primal feasibility condition (A28.6); $\mathbf{h}\left(\mathbf{x}^*\right) = \mathbf{0}$

   as $\nabla_\lambda L\left(\mathbf{x}^*, \lambda^*\right) = \mathbf{h}\left(\mathbf{x}^*\right) = \mathbf{0}$

The conditions above are necessary first order conditions. In most cases, a sufficient second order condition is needed to ensure minimum/maximum, i.e. positive/negative semi-definite Hessian.

For the minimization problem (A28.1) and (A28.3), the Hessian at $\mathbf{x}^*$ must be positive and the sufficient condition is

3. $\mathbf{v}^T \left(\nabla_{xx} L\left(\mathbf{x}^*\right)\right)\mathbf{v} \geq 0, \quad \forall \mathbf{v} \quad \text{s.t.} \quad \nabla_x \mathbf{h}\left(\mathbf{x}^*\right)^T \mathbf{v} = 0$

   This condition ensures minimum for $f\left(\mathbf{x}^*\right)$.

Note that for $\mathbf{x}^*$, $L\left(\mathbf{x}^*, \lambda^*\right) = f\left(\mathbf{x}^*\right)$ as $\mathbf{h}\left(\mathbf{x}^*\right) = \mathbf{0}$.

As the optimization problems in this study do not contain inequality constraints, a step-by-step explanation on the optimization process in the absence of inequality constraints is provided in appendix 29.

Appendix 29: Fundamental issues in equality constrained non-linear optimization

The minimization problem is stated as:

$$\min_{x \in \Omega} \; f(x) \tag{A29.1}$$

s.t.

$$h_j(x) = 0 \quad \forall j \in J, \quad J = \{1, ..., l\} \tag{A29.2}$$

The constraints can be rewritten for convenience using vector notation as

$$\mathbf{h}(\mathbf{x}) = \left[h_j(\mathbf{x})\right]^T_{j=1,...,l} = \mathbf{0} \tag{A29.3}$$

As the constraints imply, the feasible region is bounded by $\mathbf{h}(\mathbf{x}) = \mathbf{0}$. Let $\mathbf{x}_F$ denote any feasible point in Ω, i.e. the point satisfies $\mathbf{h}(\mathbf{x}_F) = \mathbf{0}$. To find a feasible point that decreases the cost function $f(\mathbf{x})$, a change $\delta\mathbf{x}$ has to be found subject to

$$\mathbf{h}(\mathbf{x}_F + \alpha\delta\mathbf{x}) = 0$$

and

$$f(\mathbf{x}_F + \alpha\delta\mathbf{x}) < f(\mathbf{x}_F)$$

for an appropriate α.

At any point $\tilde{\mathbf{x}}$, the direction of steepest decent of the cost function $f(\mathbf{x})$ is given by $-\nabla_x f(\tilde{\mathbf{x}})$. To move $\delta\mathbf{x}$ from $\mathbf{x}_F$ such that $f(\mathbf{x}_F + \alpha\delta\mathbf{x}) < f(\mathbf{x}_F)$, the move must result in $\delta\mathbf{x}(-\nabla_x f(\mathbf{x})) > 0$. Normals, perpendicular to the constraint surface are given by $\nabla_x \mathbf{h}(\mathbf{x}_F)$. To move a small $\delta\mathbf{x}$ from $\mathbf{x}_F$ and remain on the constraint surface, the move has to be in a direction orthogonal to $\nabla_x \mathbf{h}(\mathbf{x}_F)$. If $\mathbf{x}_F$ lies on the constraint surface, then:

a) Setting $\delta\mathbf{x}$ orthogonal to $\nabla_x \mathbf{h}(\mathbf{x}_F)$ ensures $\mathbf{h}(\mathbf{x}_F + \alpha\delta\mathbf{x}) = 0$ and
b) $f(\mathbf{x}_F + \alpha\delta\mathbf{x}) < f(\mathbf{x}_F)$, only if $\delta\mathbf{x}\left(-\nabla_x f(\mathbf{x}^*)\right) > 0$

As this search continues in the direction of steepest decent of the cost function $f(\mathbf{x})$, this should finally lead to the point when $\nabla_x f(\mathbf{x}_F)$ and $\nabla_x \mathbf{h}(\mathbf{x}_F)$ are parallel, i.e.

$$-\nabla_x f(\mathbf{x}^*) = \sum_{j=1}^{l} \lambda_j^* \nabla_x h_j(\mathbf{x}^*) \tag{A29.4}$$

where $\lambda_j \quad \forall j \in J$ are scalar.

When this occurs, if $\delta\mathbf{x}$ is orthogonal to $\nabla_x \mathbf{h}(\mathbf{x}_F)$, then

$$\delta\mathbf{x}\left(-\nabla_x f\left(\mathbf{x}^*\right)\right) = -\delta\mathbf{x}\left(\lambda^*\nabla_x\mathbf{h}\left(\mathbf{x}^*\right)\right) = 0$$

A decrease in the cost function, $-\nabla_x f\left(\mathbf{x}^*\right)$, cannot be achieved by remaining on the constraint surface and making a move from $\mathbf{x}^*$ and thus $\mathbf{x}^*$ corresponds to local optimum (minimum). In other words, constrained local optimum occurs at $\mathbf{x}^*$ when $\nabla_x f\left(\mathbf{x}^*\right)$ and $\nabla_x\mathbf{h}\left(\mathbf{x}^*\right)$ are parallel, i.e.

$$-\nabla_x f\left(\mathbf{x}^*\right) = \sum_{j=1}^{l}\lambda_j^*\nabla_x h_j\left(\mathbf{x}^*\right)$$

for unique scalars $\lambda_j \quad \forall j \in J$.

From the above, the necessary and sufficient conditions for a solution to be optimal can be expressed in more concrete way as in appendix A28. As the optimization problem was defined according to (A29.1) – (A29.2), the Lagrangian is defined as

$$L\left(\mathbf{x},\lambda\right) = f\left(\mathbf{x}\right) + \sum_{j=1}^{l}\lambda_j h_j\left(\mathbf{x}\right)$$

Let $\mathbf{x}^*$ be a local optimum (minimum) ⇔ there exist unique $\lambda_j^* \quad \forall j \in J$ s.t.

1. $\nabla_x L\left(\mathbf{x}^*,\lambda^*\right) = \mathbf{0}$

   This condition encodes (A29.4) for stationarity; $-\nabla_x f\left(\mathbf{x}^*\right) = \sum_{j=1}^{l}\lambda_j^*\nabla_x h_j\left(\mathbf{x}^*\right)$

   as $\nabla_x L\left(\mathbf{x}^*,\lambda^*\right) = \nabla_x f\left(\mathbf{x}^*\right) + \sum_{j=1}^{l}\lambda_j h_j\left(\mathbf{x}\right) = 0$

2. $\nabla_\lambda L\left(\mathbf{x}^*,\lambda^*\right) = \mathbf{0}$

   This condition encodes the equality constraints (A29.3) for primal feasibility; $\mathbf{h}\left(\mathbf{x}^*\right) = \mathbf{0}$

   as $\nabla_\lambda L\left(\mathbf{x}^*,\lambda^*\right) = \mathbf{h}\left(\mathbf{x}^*\right) = \mathbf{0}$

The conditions above are necessary first order conditions. In most cases, a sufficient second order condition is needed to ensure minimum/maximum, i.e. positive/negative semi-definite Hessian. In the case of (A29.1) – (A29.2), the Hessian at $\mathbf{x}^*$ must be positive and the sufficient condition is

3. $\mathbf{v}^T\left(\nabla_{xx}L\left(\mathbf{x}^*\right)\right)\mathbf{v} \geq 0, \quad \forall\mathbf{v} \quad \text{s.t.} \quad \nabla_x\mathbf{h}\left(\mathbf{x}^*\right)^T\mathbf{v} = 0$

   This condition ensures minimum for $f\left(\mathbf{x}^*\right)$.

Note that for $\mathbf{x}^*$, $L\left(\mathbf{x}^*,\lambda^*\right) = f\left(\mathbf{x}^*\right)$ as $\mathbf{h}\left(\mathbf{x}^*\right) = \mathbf{0}$