Group Actions on Infinite Digraphs and the Suborbit Function

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Abstract

This thesis is a study of infinite directed graphs, and how we can use tools from the theory of group actions to investigate them. For a group G acting on a set Ω, we define a group homomorphism from G to the multiplicative group of positive rational numbers, using the suborbits of the group action. This homomorphism will be called the suborbit function and we will see that it is equal to a well known function, defined on locally compact topological groups, called the modular function. There are a few objectives, and all main results are proved using the suborbit function. The first objective is to generalize a result of Cheryl E. Praeger from 1991 about homomorphic images of infinite directed graphs with certain additional properties. The second objective is to find a condition on edge transitive digraphs making them highly arc transitive. Next, we define Cayley–Abels digraphs of groups and use the suborbit function to give a lower bound on their valency. Then we consider the growth of graphs, showing that all infinite digraphs with the same additional properties as in Praeger's result, have exponential growth. Finally, the last chapter is dedicated to constructing examples using Cartesian products of digraphs.

Útdráttur

Þessi ritgerð fjallar um óendanleg stefnd net og aðferðir til þess að nota grúpuverkanir til að rannsaka þau. Látum G vera grúpu sem verkar á mengi Ω. Við notum hlutbrautir þessarar verkunar til þess að skilgreina grúpumótun frá G yfir í margföldunargrúpu jákvæðra ræðra talna. Við köllum þessa mótun hlutbrautafallið, og munum sjá að það er jafngilt vel þekktu falli sem kallast mátfallið, og er skilgreint á staðþjöppuðum granngrúpum. Markmið ritgerðarinnar eru nokkur, og allar helstu niðurstöður eru sannaðar með hjálp hlutbrautafallsins. Við útvíkkum niðurstöðu Cheryl E. Praeger frá 1991 um mótanamyndir óendanlegra stefndra neta með ákveðna eiginleika. Við gefum skilyrði á leggjagegnvirk, stefnd net sem tryggir að þau séu háörvavegagegnvirk. Við skilgreinum Cayley–Abels-net grúpna og notum hlutbrautafallið til þess að gefa neðra mark á stig slíkra neta. Að lokum skoðum við vöxt neta og sýnum að öll óendanleg stefnd net, með sömu eiginleika og í niðurstöðu Praeger, vaxa með veldisvísishraða. Í síðasta kaflanum notum við svo kartesk margfeldi til þess að búa til ýmis dæmi um stefnd net.

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1 Introduction

Let G be a group and Ω a set. A group action of G on Ω is a map:

$$\Omega \times G \to \Omega, \quad (\alpha, g) \mapsto \alpha^g$$

such that $(\alpha^g)^h = \alpha^{gh}$ and $\alpha^e = \alpha$ for all $\alpha \in \Omega$ and all $g, h \in G$.

We know many natural group actions from algebra. Every group acts on itself with conjugation and on the coset space of any subgroup with right multiplication. A permutation group of a set Ω acts naturally on Ω with a group action. An automorphism group of a structure, say a field or a graph, acts on this structure with a group action and so on and so forth. In this thesis we mostly concentrate on the connections between group actions and graphs.

Another way to connect these two is to consider a given group action and use it to construct a graph. By doing this we are building a bridge between different fields of mathematics. Connections like that can prove very useful, because they allow us to transfer results from one area of mathematics to another. Indeed we will do this here, applying results from the theory of group actions to the theory of infinite digraphs and vice versa.

In Chapter 2 we give some preliminaries and notation. In Chapter 3 we define a function that we will call the suborbit function, using the suborbits of a group action. This function is in fact a group homomorphism, and it was introduced by Cheryl E. Praeger in 1991 [12]. The suborbit function is essentially our most important tool, as we use it to prove all our main results.

Chapter 4 focuses on a link to group topology. If a group G acts on a set Ω, we can use this action to define a topology on G called the permutation topology. We will see that our suborbit function gains a whole new dimension if we do this. The purpose of this chapter is not to provide new results, but simply to connect what we are working with to already known results. Most of the conclusions in this chapter can be found in an article by Rögnvaldur G. Möller from 2010 [11].

Chapters 5 and 6 contain the main results of the thesis. The first section of Chapter 5 starts with a result of Cheryl E Praeger from 1991. Praeger showed that infinite, connected digraphs that are locally finite, vertex- and edge transitive and have unequal out-valency and in-valency, can be mapped with a graph homomorphism onto $\widetilde{\mathbb{Z}} := (\mathbb{Z}, \{(i, i+1) : i \in \mathbb{Z}\})$ and she constructed this homomorphism using the suborbit function. We then proceed to generalize this result by omitting the condition of edge transitivity. The first instinct was that we would get a similar graph homomorphism onto $\widetilde{\mathbb{Z}}^n$, the naturally directed graph on $\mathbb{Z}^n$, if we have $n < \infty$ orbits on edges. This is not true in general. However, if we construct a map in the same way as before, using the suborbit function, we get a graph homomorphism onto a certain Cayley digraph of the additive group $\mathbb{Z}^k$, for some $k \leq n$. Furthermore, adding some conditions on the automorphism group of the digraph, we can guarantee that this Cayley digraph is in fact $\widetilde{\mathbb{Z}}^k$. The conditions however are quite extensive, which raises the question of whether there actually exist such digraphs. We come back to this question in Chapter 6 where we give examples of digraphs satisfying these conditions.

Section 5.2 focuses on highly arc transitive graphs, starting with some examples. Our main result here is that infinite, connected, vertex- and edge transitive digraphs with relatively prime in-valency and out-valency are highly arc transitive. We prove this in two different ways, first using basic group theory and then using the suborbit function. In Section 5.3 we define Cayley–Abels digraphs of topological groups and use the image of the suborbit function to give a lower bound on their valency. This is a partial answer to a question of George A. Willis from 2014. In Section 5.4 we consider growth of graphs in relation to the suborbit function. The objective here is to show that every infinite, connected, vertex- and edge transitive digraph with finite, unequal in-valency and out-valency has exponential growth.

The principal goal of the last chapter is to construct some more examples. We do this by using products of graphs, mainly focusing on the Cartesian product. In the first section we define three different products and give some examples of them. The next three sections concentrate on Cartesian products, and how we can identify properties of a graph based on properties of its factors. Most of these results have been proved for finite, undirected graphs [8, 5], but we verify that they also hold true for infinite digraphs. In 6.4 we have collected the tools to construct an infinite family of digraphs that satisfy the conditions of the generalization of Praeger's result.

2 Preliminaries

2.1 Group actions

Let G be a group acting on a set Ω. For $g \in G$ and $\alpha \in \Omega$ we denote the image of α under the action of g by α^g. We say that G acts *transitively* on Ω, or that G is *transitive* on Ω, if for any two elements, α and β in Ω, there exists an element $g \in G$ such that $\alpha^g = \beta$.

We can think of each element in G as giving a permutation of the set Ω. Therefore we have a natural map from G to $\mathrm{Sym}(\Omega)$, taking $g \in G$ to the corresponding permutation. We say that the action of G on Ω is *faithful* if this map is injective. In this case we can think of G as a permutation group on Ω.

The *orbit* of an element, $\alpha \in \Omega$, is the set $\alpha^G := \{\alpha^g : g \in G\}$. It is clear that G is transitive if and only if every element of Ω lies in the same orbit, that is if for any $\alpha \in \Omega$ we have $\alpha^G = \Omega$. We define a relation on Ω with $\alpha \sim \beta$ if $\beta \in \alpha^G$. This is an equivalence relation and its equivalence classes are called the *orbits* of G.

The *point stabilizer* or simply the *stabilizer* of α is denoted by G_α and defined as the subset of G that fixes α, that is $G_\alpha := \{g \in G : \alpha^g = \alpha\}$. For a subset Δ of Ω we define the *setwise stabilizer* of Δ in G as

$$G_{\{\Delta\}} := \{g \in G : \Delta^g = \Delta\}$$

and the *pointwise stabilizer* of Δ in G as

$$G_{(\Delta)} := \{g \in G : \delta^g = \delta \text{ for all } \delta \in \Delta\}.$$

It is left to the reader to verify that $G_\alpha, G_{\{\Delta\}}$ and $G_{(\Delta)}$ are subgroups of G. When the set Δ is finite and we have $\Delta = \{\alpha_0, \alpha_1, \ldots, \alpha_n\}$, we often denote the pointwise stabilizer by $G_{\alpha_0\alpha_1\cdots\alpha_n}$ instead of $G_{(\Delta)}$.

If G acts transitively on a set Ω, it also has a natural action on the coset space, G/G_α, for any $\alpha \in \Omega$. In fact Ω looks exactly like this coset space in the sense that there exists a bijective function, $\theta : \Omega \to G/G_\alpha$ such that $\theta(\omega^g) = (\theta(\omega))^g$ for all $\omega \in \Omega$. This is further explained in [2, p. 22].

Let G be a group that is transitive on a set Ω and let $\alpha \in \Omega$. The subgroup G_α also acts on Ω and the orbits of this action are called the *suborbits of G on Ω.* The suborbit

$\alpha^{G_\alpha} = \{\alpha\}$ is called the *trivial suborbit*. We also have a natural action of G on the set $\Omega^2 = \Omega \times \Omega$ defined by $(\alpha, \beta)^g := (\alpha^g, \beta^g)$. The orbits of G on Ω^2 are called the *orbitals* of G and the orbital $\{(\alpha, \alpha) : \alpha \in \Omega\}$ is called the *diagonal orbital*. There is a one-to-one correspondence between suborbits and orbitals of G, given by $\beta^{G_\alpha} \leftrightarrow (\alpha, \beta)^G$, with the trivial suborbit corresponding to the diagonal orbital [2, Theorem 5.2].

2.2 Graphs

A *graph* Γ is an ordered pair of sets, $(V(\Gamma), E(\Gamma))$, where $V(\Gamma)$ is called the *vertex set* of Γ and $E(\Gamma) \subseteq \{\{x, y\} : x, y \in V(\Gamma), x \neq y\}$ is called the *edge set* of Γ. The elements of these sets are called *vertices* and *edges*, respectively. The *trivial graph* is the graph with one vertex and no edges. Two vertices, α and β in Γ, are *adjacent* or *neighbors* if there is an edge connecting them, that is if $\{\alpha, \beta\} \in E(\Gamma)$. The *valency* of a vertex α is the number of its neighbors, $|\{\beta \in V(\Gamma) : \{\alpha, \beta\} \in E(\Gamma)\}|$. If every vertex of Γ has finite valency, we say that Γ is *locally finite*.

A *directed graph* or *digraph*, Γ, is defined similarly, but with the edge set consisting of ordered pairs of elements in $V(\Gamma)$, that is $E(\Gamma) \subseteq V(\Gamma)^2$. We generally assume that our digraphs are without loops, that is $(\alpha, \alpha) \notin E(\Gamma)$ for any $\alpha \in V(\Gamma)$, except in Definition 5.1.8 of Cayley digraphs. The *trivial digraph* is defined just like the trivial graph. Two vertices, α and β, in a digraph are *adjacent* or *neighbors* if either (α, β) or (β, α) is an edge. The valency of a vertex is again defined as the number of its neighbors. Let $e := (\alpha, \beta) \in E(\Gamma)$. Then α is called the *initial vertex* of the edge e, and β its *terminal vertex*. We define the *out-valency* of a vertex α, as the number of edges with α as an initial vertex and the *in-valency* of α as the number of edges with α as a terminal vertex. It is clear that the valency of a vertex is the sum of the out-valency and the in-valency.

Let Γ_1 and Γ_2 be graphs. A *graph homomorphism* between Γ_1 and Γ_2 is a map, $\varphi : V(\Gamma_1) \to V(\Gamma_2)$ such that if $\{\alpha, \beta\} \in E(\Gamma_1)$ then $\{\varphi(\alpha), \varphi(\beta)\} \in E(\Gamma_2)$. A graph homomorphism between digraphs is defined similarly, taking edges to edges. A *graph epimorphism* is a surjective graph homomorphism, a *graph isomorphism* is a bijective graph homomorphism, and a *graph automorphism* is a graph isomorphism from a graph to itself. The set of all graph automorphisms on a graph Γ (directed or undirected) forms a group under composition of maps. This group is called the *automorphism group* of Γ and is denoted by $\mathrm{Aut}(\Gamma)$.

Let Γ be a graph, directed or undirected, and $G := \mathrm{Aut}(\Gamma)$. Then every $g \in G$ is a bijective map on both $V(\Gamma)$ and $E(\Gamma)$, so G acts on both sets in a natural way (in fact these actions are clearly faithful). We say that Γ is *vertex transitive* (resp. *edge transitive*) if this action is transitive on $V(\Gamma)$ (resp. on $E(\Gamma)$). If Γ is a vertex transitive digraph, every vertex must have the same out-valency, u and the same in-valency, v. In this case u and v are called the out-valency and in-valency of Γ, respectively. We define the valency of a vertex transitive graph (directed or undirected) similarly.

Let Γ be a vertex transitive digraph with $G := \mathrm{Aut}(\Gamma)$. Then $E(\Gamma)$ is a union of some orbitals of the action of G on $V(\Gamma)$. On the other hand, if a group G acts on a set Ω we can construct a digraph $\Gamma := (\Omega, \Delta_1 \cup \cdots \cup \Delta_n)$ where the Δ_i are some orbitals of the action, excluding the diagonal orbital. This digraph is called the *orbital digraph* of G with respect to the orbitals $\Delta_1, \ldots, \Delta_n$.

3 The suborbit function

We start by defining a group homomorphism from a group G acting on a set, to the multiplicative group of the positive rational numbers. We will call this homomorphism the suborbit function and denote it by ψ. In a paper [12] from 1991, Cheryl E. Praeger defines the same function in order to show that infinite digraphs with certain properties can be mapped homomorphically onto the naturally directed graph on $\mathbb{Z}$. We will see this result later on, as well as many other applications of this function.

3.1 Definition

Let G be a group acting transitively on a set Ω such that all suborbits of G are finite. Fix a reference point, $\alpha \in \Omega$, and define the *suborbit function* as follows:

$$\psi : G \to \mathbb{Q}_+, \quad g \mapsto \frac{\left|\beta^{G_\alpha}\right|}{\left|\alpha^{G_\beta}\right|}, \quad \text{where } \beta = \alpha^g.$$

This function is well defined since the suborbits of G are finite, but the fact that it is a group homomorphism is not obvious.

Proposition 3.1.1. *The suborbit function is a group homomorphism from G to the group of positive rational numbers. Moreover, it does not depend on the reference point, α.*

Proof. We note first that for every $\beta, \gamma \in \Omega$ we have $\left|\beta^{G_\gamma}\right| = |G_\gamma : G_{\gamma\beta}|$, and for every $x \in G$

$$x^{-1} G_\beta x = G_{\beta^x}$$

and therefore

$$\left|\beta^{G_{\beta^x}}\right| = \left|\beta^{x^{-1} G_\beta x}\right| = \left|(\beta^{x^{-1}})^{G_\beta}\right|.$$

Let $g, h \in G$ and set $\beta := \alpha^g$ and $\gamma := \alpha^{h^{-1}}$. Then $\left|\gamma^{G_\alpha}\right| = \left|\alpha^{G_{\alpha^h}}\right|$ and similarly $\left|\alpha^{G_\gamma}\right| = \left|\alpha^{G_{\alpha^{h^{-1}}}}\right| = \left|\alpha^{hG_\alpha}\right|$, so

$$\psi(h) = \frac{\left|\alpha^{hG_\alpha}\right|}{\left|\alpha^{G_{\alpha^h}}\right|} = \frac{\left|\alpha^{G_\gamma}\right|}{\left|\gamma^{G_\alpha}\right|} \quad \text{and} \quad \psi(gh) = \frac{\left|\alpha^{ghG_\alpha}\right|}{\left|\alpha^{G_{\alpha^{gh}}}\right|} = \frac{\left|\beta^{G_\gamma}\right|}{\left|\gamma^{G_\beta}\right|}.$$

Furthermore

$$1 = \frac{|G_\alpha : G_{\alpha\beta\gamma}|}{|G_\alpha : G_{\alpha\beta\gamma}|} = \frac{|G_\alpha : G_{\alpha\gamma}|\,|G_{\alpha\gamma} : G_{\alpha\beta\gamma}|}{|G_\alpha : G_{\alpha\beta}|\,|G_{\alpha\beta} : G_{\alpha\beta\gamma}|}.$$

Therefore

$$\begin{aligned}
\psi(g)\psi(h) &= \frac{|\beta^{G_\alpha}|}{|\alpha^{G_\beta}|}\cdot\frac{|\alpha^{G_\gamma}|}{|\gamma^{G_\alpha}|}\\
&= \frac{|G_\alpha : G_{\alpha\beta}|\,|G_\gamma : G_{\alpha\gamma}|}{|G_\beta : G_{\alpha\beta}|\,|G_\alpha : G_{\alpha\gamma}|}\cdot\frac{|G_\alpha : G_{\alpha\gamma}|\,|G_{\alpha\gamma} : G_{\alpha\beta\gamma}|}{|G_\alpha : G_{\alpha\beta}|\,|G_{\alpha\beta} : G_{\alpha\beta\gamma}|}\\
&= \frac{|G_\gamma : G_{\alpha\gamma}|\,|G_{\alpha\gamma} : G_{\alpha\beta\gamma}|}{|G_\beta : G_{\alpha\beta}|\,|G_{\alpha\beta} : G_{\alpha\beta\gamma}|} = \frac{|G_\gamma : G_{\alpha\beta\gamma}|}{|G_\beta : G_{\alpha\beta\gamma}|}\\
&= \frac{|G_\gamma : G_{\beta\gamma}|\,|G_{\beta\gamma} : G_{\alpha\beta\gamma}|}{|G_\beta : G_{\beta\gamma}|\,|G_{\beta\gamma} : G_{\alpha\beta\gamma}|}\\
&= \frac{|G_\gamma : G_{\beta\gamma}|}{|G_\beta : G_{\beta\gamma}|} = \frac{|\beta^{G_\gamma}|}{|\gamma^{G_\beta}|} = \psi(gh)
\end{aligned}$$

so ψ is a homomorphism from G to $\mathbb{Q}_+$. We will now show that it is independent of our reference point, α. Let ω be another point in Ω and define a homomorphism

$$\psi_\omega : G \to \mathbb{Q}_+, \quad g \mapsto \frac{|\beta^{G_\omega}|}{|\omega^{G_\beta}|} \quad \text{where } \beta = \omega^g.$$

Let $h \in G$ such that $\alpha = \omega^h$. Then $\beta^h = \alpha^{h^{-1}gh}$ and we get

$$\begin{aligned}
\psi_\omega(g) &= \frac{|\beta^{G_\omega}|}{|\omega^{G_\beta}|} = \frac{|\beta^{G_{\alpha^{h^{-1}}}}|}{|\alpha^{h^{-1}G_\beta}|} = \frac{|\beta^{hG_\alpha}|}{|\alpha^{G_{\beta^h}}|} = \frac{|\alpha^{h^{-1}ghG_\alpha}|}{|\alpha^{G_{\alpha^{h^{-1}gh}}}|}\\
&= \psi(h^{-1}gh) = \psi(h)^{-1}\psi(g)\psi(h) = \psi(g).
\end{aligned}$$

□

This last property of ψ allows us to define the function without fixing the point, α. The elements α and β in the definition are then simply any two elements in Ω such that $\alpha^g = \beta$. This becomes useful in determining some properties of ψ.

3.2 Basic properties

We investigate some properties of the kernel of the suborbit function before introducing a condition on the action of G making it trivial. It is worth noting that the kernel of ψ is often "large" in some sense, as it contains certain subgroups of G that can not be trivial if ψ is non-trivial.

Theorem 3.2.1.

(i) $G_\alpha \leq \ker\psi$ *for all* $\alpha \in \Omega$.

(ii) *$G' \leq \ker\psi$, where G' it the commutator subgroup of G.*

(iii) *If $g \in G$ is such that $\langle g \rangle$ has a finite orbit on Ω, then $g \in \ker\psi$.*

Proof.

(i) We have $\alpha^g = \alpha$ for every $g \in G_\alpha$ so

$$\psi(g) = \frac{|\alpha^{gG_\alpha}|}{|\alpha^{G_{\alpha^g}}|} = \frac{|\alpha^{G_\alpha}|}{|\alpha^{G_\alpha}|} = 1$$

(ii) This is obvious since the image of ψ is an abelian group.

(iii) Let $g \in G$, and suppose $H := \langle g \rangle$ has a finite orbit, α^H. Then $|\alpha^H| < \infty$, so there exists $n \in \mathbb{N}$ such that $\alpha^{g^n} = \alpha$. Then $g^n \in G_\alpha \subset \ker\psi$ (by (i)) so we have

$$\psi(g^n) = 1 = \psi(g)^n \in \mathbb{Q}_+$$

and therefore $\psi(g) = 1$ and $g \in \ker\psi$.

$\square$

Remark 3.2.1.1. If $g, h \in G$ and there exists a point $\alpha \in \Omega$ such that $\alpha^g = \alpha^h$ then $gh^{-1} \in G_\alpha$, so by (i) we have $\psi(gh^{-1}) = 1$, thus $\psi(g) = \psi(h)$.

Definition 3.2.2. Let G be a group acting faithfully on a set Ω. We say that the action is *quasi-primitive* if every non-trivial normal subgroup of G acts transitively on Ω.

As the name implies, quasi-primitivity is a generalization of another property of a group action called primitivity. To define it we first need to define *blocks*. A subset, $\Delta \subseteq \Omega$ is a *block* if for every $g \in G$ either $\Delta = \Delta^g$ or $\Delta \cap \Delta^g = \emptyset$. A group action is *primitive* if every block, $\Delta \subseteq \Omega$ is either trivial, that is $|\Delta| = 1$, or improper, that is $\Delta = \Omega$. We can see that primitivity implies quasi-primitivity because α^N is a block for any normal subgroup N of G. The following theorem is proved in [11, Corollary 2.6] for primitive group actions.

Theorem 3.2.3. *Let G be a permutation group acting transitively on a set Ω and assume that all suborbits of G are finite. If G is quasi-primitive, then the suborbit function is trivial.*

Proof. Let $K := \ker\psi$. Then K is a normal subgroup of G. Since G is quasi-primitive, we have that K is either trivial or transitive. Suppose $K = \{e\}$. Then by (i) in Theorem 3.2.1 we have that G_α is trivial for every $\alpha \in \Omega$, and therefore

$$\frac{|\beta^{G_\alpha}|}{|\alpha^{G_\beta}|} = \frac{|\{\beta\}|}{|\{\alpha\}|} = 1$$

for all $\alpha, \beta \in \Omega$. Thus ψ is trivial.

Now suppose K is transitive. Let $g \in G$ and set $\alpha, \beta \in \Omega$ such that $\beta = \alpha^g$. By the transitivity of K, there exists $k \in K$ such that $\beta = \alpha^k$. But then, by Remark 3.2.1.1, $\psi(g) = \psi(k) = 1$, thus ψ is trivial. $\square$

4 A little topology

Before we go on to the main topic of this thesis, we will introduce a connection to topology. The links between permutation groups and topological groups are many and diverse, however we will only touch on a few here.

4.1 Topological groups

We start by defining topological groups and looking at some of their properties. We will exclude basic definitions from topology, but note that we define neighborhoods to be open.

Definition 4.1.1. A *topological group* G is a topological Hausdorff space that is also a group, such that the functions

$$(g,h) \mapsto gh \quad \text{and} \quad g \mapsto g^{-1}$$

are continuous.

Remark. In this case, the function $g \mapsto g^{-1}$ is a homeomorphism, because it is its own inverse.

Having a lot of structure, topological groups have many convenient properties.

Proposition 4.1.2. *Let $x \in G$. The functions $G \to G$, given by*

$$g \mapsto gx, \quad g \mapsto xg \quad \textit{and} \quad g \mapsto x^{-1}gx$$

are homeomorphisms.

Proof. It is clear that the functions are all bijective. We will only show that the function $f : g \mapsto gx$ is continuous (the other proofs are similar). Let $g \in G$ and let $V \subset G$ be a neighborhood of $f(g) = gx$. Since the function $(g_1, g_2) \mapsto g_1g_2$ is continuous, there exist neighborhoods U_1 and U_2 of g and x respectively such that $U_1U_2 \subset V$. In particular $f(U_1) = U_1x \subset V$ so f is continuous. □

An obvious consequence of this proposition is that for every open set U in a topological group G and every $g \in G$, the sets Ug, gU and $g^{-1}Ug$ are open. In fact, every neighborhood of $g \in G$ is of the form Ug with U a neighborhood of the identity because if V is a neighborhood of g, then $U := Vg^{-1}$ is a neighborhood of the identity and $V = Ug$. It follows that every open set in G is of the form Ug with $g \in G$ and U a neighborhood of the identity. To define a topology on a group, it is therefore sufficient to give a neighborhood basis of the identity.

Proposition 4.1.3. *Let G and H be topological groups. A group homomorphism $\varphi : G \to H$ is continuous if and only if it is continuous at the identity, e_G.*

Proof. Obviously φ is continuous at the identity if it is continuous. Suppose it is continuous at e_G. Let $g \in G$, and let $V = W\varphi(g)$ a neighborhood of $\varphi(g)$ with W a neighborhood of $e_H = \varphi(e_G)$. There exists a neighborhood U of e_G such that $\varphi(U) \subset W$. But then $\varphi(Ug) = \varphi(U)\varphi(g) \subset W\varphi(g) = V$, so φ is continuous at g. $\square$

Definition 4.1.4. Let G be a locally compact group, Σ the σ-algebra generated by the open sets of G and μ a measure on Σ. We say that μ is *regular* if it satisfies the following:

(i) $\mu(K) < \infty$ for all compact sets $K \subset G$

(ii) For $A \in \Sigma$ we have $\mu(A) = \inf\{\mu(U) : A \subset U, U \text{ open}\}$.

(iii) For U an open subset of G we have $\mu(U) = \sup\{\mu(K) : K \subset U, K \text{ compact}\}$.

Definition 4.1.5. Let G be a locally compact topological group. A *right Haar measure* on G is a nontrivial, regular measure μ on G such that $\mu(Ag) = \mu(A)$ for all measurable subsets A of G and all elements $g \in G$.

The following theorem is on the existence and uniqueness of a right Haar measure. For proof see [4, Section 9.2].

Theorem 4.1.6. *Let G be a locally compact group. Then there exists a right Haar measure on G. Furthermore, if μ and μ' are two Haar measures on G, then there exists a number $a \in \mathbb{R}_+$ such that $\mu' = a\mu$.*

4.2 The permutation topology

In this section we will show that given a group G acting faithfully on a set, we can define a topology on G making it a topological group. We will also introduce two functions, the modular function and the scale function, and discover their connection to the suborbit function.

Definition 4.2.1. Let G be a group acting on a set Ω. We define a topology on G by taking as a neighborhood basis of the identity element the family of subgroups

$$\{G_{(\Delta)} : \Delta \text{ is a finite subset of } \Omega\},$$

that is, the pointwise stabilizers of finite sets. This topology is called the *permutation topology.*

Proposition 4.2.2. *A group G acting faithfully on a set Ω is a topological group with respect to the permutation topology.*

Proof. We start by showing that G is Hausdorff. Let $g, h \in G$. Since the action is faithful, there exists $\alpha \in \Omega$ such that $\alpha^g \neq \alpha^h$. Then $G_\alpha g$ and $G_\alpha h$ are disjoint neighborhoods of g and h respectively, thus G is Hausdorff.

It remains to show that the functions

$$(g, h) \mapsto gh \quad \text{and} \quad g \mapsto g^{-1}$$

are continuous. By Proposition 4.1.3 it suffices to show continuity at the identity, e_G. Let $V \subset G$ be a neighborhood of e_G and let Δ be a finite subset of Ω such that $G_{(\Delta)} \subset V$. Then $G_{(\Delta)} \times G_{(\Delta)}$ is a neighborhood of $e_{G\times G}$ and $gh \in V$ whenever $(g, h) \in G_{(\Delta)} \times G_{(\Delta)}$. Also $G_{(\Delta)}^{-1} = G_{(\Delta)}$ is a neighborhood of e_G, and $g^{-1} \in V$ for $g \in G_{(\Delta)}$, so both functions are continuous. □

Remark. Note that G is totally disconnected with respect to the permutation topology, if and only if the action if faithful.

If G is a topological group and U an open subgroup of G, then G acts transitively on the coset space G/U. The point stabilizers of the action are conjugates of U, thus open subgroups of G. Furthermore, pointwise stabilizers of finite sets are simply finite intersections of point stabilizers, so they are open as well. Therefore, the permutation topology on G, given by this action, is contained in the original topology on G. Furthermore, if U is also a compact subgroup of G then all pointwise stabilizers of finite sets are compact.

Let G be a group with the permutation topology, acting on a set Ω. We say that G is *closed* if its image in $\mathrm{Sym}(\Omega)$ is a closed subgroup of $\mathrm{Sym}(\Omega)$. If the action on Ω is faithful, we can regard G as a permutation group on Ω. In this case, G is closed if and only if it is a full automorphism group for some first-order structures on Ω (see [3, Section 2.4]).

Let μ be a right Haar measure on a locally compact group G and define for all $x \in G$ the measure μ_x by $\mu_x(A) := \mu(xA)$, for any measurable set $A \subseteq G$. Since the map $g \mapsto xg$ is a homeomorphism, μ_x is regular for any $x \in G$. Also, for $g \in G$ we have $\mu_x(Ag) = \mu(xAg) = \mu(xA) = \mu_x(A)$ so μ_x is a right Haar measure on G. Then, by Theorem 4.1.6, there exists a number $\Delta(x) \in \mathbb{R}_+$ such that $\mu_x = \Delta(x)\mu$. We can therefore define the following function.

Definition 4.2.3. Suppose G is a locally compact group and let μ be a right Haar measure on G. The *modular function* is a function Δ that satisfies $\mu(xA) = \Delta(x)\mu(A)$ for any measurable set A and any $x \in G$.

Remark. It is easy to see that Δ is a group homomorphism from G to $\mathbb{R}$.

The following identity was first proved in a paper by Günter Schlichting in 1979, [13, Lemma 1]. It also appears independently in a paper by V. I. Trofimov from 1985, in the proof of Theorem 1 [16].

Theorem 4.2.4. *Let G be a group acting transitively on Ω such that G is closed with the permutation topology. Suppose furthermore that all suborbits of the action are finite and let ψ be the suborbit function defined by this action. Then $\Delta = \psi$.*

Proof. Let $g \in G$ and $\alpha, \beta \in \Omega$ such that $\alpha^g = \beta$, and let μ be a right Haar measure on G. We note first that G_α, G_β and $G_{\alpha\beta}$ are compact sets (see [11, Lemma 2.2]), and so $\mu(G_\alpha), \mu(G_\beta), \mu(G_{\alpha\beta}) < \infty$. Also, G_α is a disjoint union of $k := |G_\alpha : G_{\alpha\beta}|$ cosets, $G_{\alpha\beta}g_1, \ldots, G_{\alpha\beta}g_k$ so we have

$$\mu(G_\alpha) = \mu\left(\bigcup_{i=1}^{k} G_{\alpha\beta}g_i\right) = \sum_{i=1}^{k} \mu(G_{\alpha\beta}g_i) = |G_\alpha : G_{\alpha\beta}|\, \mu(G_{\alpha\beta}) \tag{4.1}$$

Now we easily get:

$$\begin{aligned}\psi(g) = \frac{\left|\beta^{G_\alpha}\right|}{\left|\alpha^{G_\beta}\right|} &= \frac{|G_\alpha : G_{\alpha\beta}|}{|G_\beta : G_{\alpha\beta}|} = \frac{\mu(G_\alpha)/\mu(G_{\alpha\beta})}{\mu(G_\beta)/\mu(G_{\alpha\beta})}\\ &= \frac{\mu(G_\alpha)}{\mu(G_\beta)} = \frac{\mu(gG_\beta g^{-1})}{\mu(G_\beta)} = \frac{\Delta(g)\mu(G_\beta)}{\mu(G_\beta)} = \Delta(g)\end{aligned}$$

□

Remark 4.2.4.1. If G is a totally disconnected, locally compact group, then it contains a compact open subgroup U, and acts transitively on G/U. The pointwise stabilizers of this action are compact so all suborbits are finite by Equation (4.1). Furthermore, the topology on G contains the permutation topology, so the modular function stays the same. Theorem 4.2.4 therefore still holds, and we see that the suborbit function is independent of the subgroup U, as long as U is compact and open.

Definition 4.2.5. Let G be a totally disconnected, locally compact group. The *scale function* on G is defined as

$$\mathbf{s}(g) := \min\{|U : U \cap g^{-1}Ug| : U \text{ a compact open subgroup of } G\}.$$

A compact open subgroup U of G is said to be *tidy for g* if $\mathbf{s}(g) = |U : U \cap g^{-1}Ug|$.

In a paper from 2001, George A. Willis defines tidy subgroups in a fairly untidy way. In the same paper he shows however, that the definition is equivalent to the one above [18, Definition 2.1, Theorem 3.1].

Theorem 4.2.6. [17, Corollary 1] *Let G be a locally compact, totally disconnected group. For $g \in G$ we have*

$$\Delta(g) = \frac{\mathbf{s}(g)}{\mathbf{s}(g^{-1})}.$$

Proof. (A similar proof can be found in [10, Theorem 5.2]). Let $g \in G$ and choose compact open subgroups, U_1 and U_2 such that

$$|U_1 : U_1 \cap g^{-1}U_1g| = \mathbf{s}(g) \quad \text{and} \quad |U_2 : U_2 \cap gU_2g^{-1}| = \mathbf{s}(g^{-1}).$$

Since $\mathbf{s}(g) \leq |U_2 : U_2 \cap g^{-1}U_2g|$ and $\mathbf{s}(g^{-1}) \leq |U_1 : U_1 \cap gU_1g^{-1}|$, we get:

$$\frac{|U_1 : U_1 \cap g^{-1}U_1g|}{|U_1 : U_1 \cap gU_1g^{-1}|} \leq \frac{\mathbf{s}(g)}{\mathbf{s}(g^{-1})} \leq \frac{|U_2 : U_2 \cap g^{-1}U_2g|}{|U_2 : U_2 \cap gU_2g^{-1}|}.$$

Let G act on the space $\Omega := G/U_1$ and let $\alpha \in \Omega$ such that $U_1 = G_\alpha$. We use the fact that compact open subgroups in the permutation topology of this action are compact open subgroups in the topology on G. Also, because of the identity $\mu(G_\gamma) = |G_\gamma : G_{\gamma\delta}|\mu(G_{\gamma\delta})$ for a right Haar measure μ, and $\gamma, \delta \in \Omega$, we know that all suborbits of the action are finite. Now we have:

$$\frac{|U_1 : U_1 \cap g^{-1}U_1g|}{|U_1 : U_1 \cap gU_1g^{-1}|} = \frac{|G_\alpha : G_{\alpha\alpha^g}|}{|G_\alpha : G_{\alpha\alpha^{g^{-1}}}|} = \frac{|G_\alpha : G_{\alpha\alpha^g}|}{|G_{\alpha^g} : G_{\alpha^g\alpha}|} = \Delta(g)$$

by the proof of Theorem 4.2.4. By defining a similar action of G on G/U_2 we also get

$$\Delta(g) = \frac{|U_2 : U_2 \cap g^{-1}U_2g|}{|U_2 : U_2 \cap gU_2g^{-1}|}$$

and we have shown that $\mathbf{s}(g)/\mathbf{s}(g^{-1}) = \Delta(g)$. □

We note that a closed permutation group G that acts transitively on Ω with all its suborbits finite is always locally compact because stabilizers of points are compact. This is proved by Woess in [19, Lemma 1].

Corollary 4.2.7. *Let G be a group acting transitively on a set Ω. Suppose G is closed with the permutation topology and that all suborbits of the action are finite. Then*

$$\psi(g) = \frac{\mathbf{s}(g)}{\mathbf{s}(g^{-1})}$$

for all $g \in G$.

The following two theorems are proved in [10].

Theorem 4.2.8. [10, Theorem 7.7] *Let G be a totally disconnected, locally compact group and let $g \in G$. For any compact, open subgroup V of G,*

$$\mathbf{s}(g) = \lim_{n\to\infty} |V : V \cap g^{-n}Vg^n|^{1/n}.$$

Theorem 4.2.9. [10, Corollary 7.8] *Let G be a group acting transitively on a set Ω such that all suborbits of the action are finite. Let $\alpha \in \Omega$ and $g \in G$ and set $\alpha_n := \alpha^{g^n}$ for $n \in \mathbb{N}$. Then*

$$\mathbf{s}(g) = \lim_{n\to\infty} |\alpha_n^{G_\alpha}|^{1/n}.$$

5 Applications to graphs

In this chapter we consider graphs (mostly directed) and their automorphism groups. If a digraph is locally finite and vertex transitive, we can define the suborbit function by the action of its automorphism group on its vertex set. We will see that doing this, we can use the suborbit function to identify some properties of the digraph. Conversely, given these properties of the digraph, we can use them to describe the suborbit function.

5.1 Homomorphic images

By mapping a graph with a graph homomorphism onto a simpler graph, we can observe some of its features while excluding others. This can be convenient if the original graph is complicated. In this section we will see how we can use the suborbit function to build such graph homomorphisms for certain types of digraphs, starting with the result of Praeger that we mentioned in the beginning of Chapter 3, and then generalizing it to digraphs with less structure.

Definition 5.1.1. A *walk* in a graph (directed or undirected) from a vertex α to a vertex β is a sequence of vertices, $\alpha = \alpha_0, \alpha_1, \ldots, \alpha_n = \beta$ such that α_{i-1} and α_i are neighbors for every $i = 1, \ldots, n$. A *path* is a walk in which every two vertices are distinct. A graph, Γ is *connected* if for any two vertices, α and β there exists a walk from α to β.

Definition 5.1.2. A *cycle*, more specifically an *n-cycle*, is a path $\alpha_1, \ldots, \alpha_n$ where α_1 and α_n are adjacent. A 3-cycle is called a *triangle* and a 4-cycle is called a *square*.

Definition 5.1.3. A *tree* is a graph (directed or undirected) in which for any two vertices, α and β there is exactly one path from α to β.

Remark. A tree is always connected and without cycles.

Definition 5.1.4. A graph (directed or undirected) is said to be *bipartite* if its vertex set can be partitioned into two parts, such that every two adjacent vertices lie in different parts.

Let Γ be a connected digraph and G a subgroup of $\mathrm{Aut}(\Gamma)$ that acts transitively on vertices of Γ. Assume furthermore that all suborbits of G are finite and let ψ be the suborbit function defined by the action of G on $V(\Gamma)$. We label every directed edge, $e = (\alpha, \beta)$, with a number:

$$\psi_e := \frac{|\beta^{G_\alpha}|}{|\alpha^{G_\beta}|}$$

and note that if $g, h \in G$ are such that $\alpha^g = \beta$ and $\beta^h = \alpha$ then we have $\psi_{(\alpha,\beta)} = \psi(g)$ and $\psi(h) = \left(\psi_{(\alpha,\beta)}\right)^{-1}$. This way of labeling the edges of the digraph can be found in an article by Hyman Bass and Ravi Kulkarni from 1990 [1, Section 1].

Let $g \in G$ and α and β be vertices of Γ such that $\alpha^g = \beta$. Suppose furthermore that there exists a vertex $\gamma \in V(\Gamma)$ such that (α, γ) and (γ, β) are edges. Let $g_1, g_2 \in G$ such that $\alpha^{g_1} = \gamma$ and $\gamma^{g_2} = \beta$. Then $\alpha^g = \alpha^{g_1 g_2}$ so we have

$$\psi(g) = \psi(g_1 g_2) = \psi(g_1)\psi(g_2) = \psi_{(\alpha,\gamma)}\psi_{(\gamma,\beta)}.$$

Since Γ is connected, there exists a walk (not necessarily directed) between any two vertices α and β. For such a walk, we enumerate the edges, $e_1, \ldots, e_k$. Then if $\alpha^g = \beta$ we can extend the above to get:

$$\psi(g) = (\psi_{e_1})^{\varepsilon_1} \cdots (\psi_{e_k})^{\varepsilon_k}$$

where $\varepsilon_i \in \{\pm 1\}$. Thus the labeled graph describes the suborbit function completely.

Definition 5.1.5. We define a naturally directed graph on the integers as

$$\widetilde{\mathbb{Z}} := (\mathbb{Z}, E(\widetilde{\mathbb{Z}}))$$

where $E(\widetilde{\mathbb{Z}}) = \{(n, n+1) : n \in \mathbb{Z}\}$. More generally, for any $k \in \mathbb{N}^*$ we define a naturally directed graph on $\mathbb{Z}^k$ as

$$\widetilde{\mathbb{Z}}^k := (\mathbb{Z}^k, E(\widetilde{\mathbb{Z}}^k))$$

where $(\mathbf{n}, \mathbf{m}) \in E(\widetilde{\mathbb{Z}}^k)$, with $\mathbf{n} = (n_1, \ldots, n_k)$ and $\mathbf{m} = (m_1, \ldots, m_k)$, if and only if there exists a unique number, $l \in \{1, \ldots, k\}$ such that $m_l = n_l + 1$ and $m_i = n_i$ for all $i \in \{1, \ldots, k\} \backslash \{l\}$.

The following theorem was first proved by Cheryl E. Praeger in 1991 [12].

Theorem 5.1.6. *Let Γ be an infinite, connected, vertex transitive, edge transitive digraph with finite but unequal out-valency and in-valency. Then there exists a graph epimorphism, $\varphi : \Gamma \to \widetilde{\mathbb{Z}}$, s.t. the inverse image, $\varphi^{-1}(n)$, is infinite for any $n \in \mathbb{Z}$.*

Proof. Set $G := \mathrm{Aut}(\Gamma)$. We denote the out-valency of Γ with u and the in-valency with v. Let α and β be vertices of Γ such that (α, β) is an edge. Since G acts transitively on the edges of Γ we see that $\beta^{G_\alpha} = \{\gamma \in V(\Gamma) : (\alpha, \gamma) \in E(\Gamma)\}$ and it is obvious that the cardinality of this set is u. In the same way we get $\alpha^{G_\beta} = \{\gamma \in V(\Gamma) : (\gamma, \beta) \in E(\Gamma)\}$, and the cardinality of this set is v. Thus for an edge (α, β) we have

$$\psi_{(\alpha,\beta)} = \frac{u}{v}.$$

Now, let α and β be arbitrary vertices of Γ and $g \in G$ such that $\alpha^g = \beta$. Let $\alpha = \alpha_0, \alpha_1, \ldots, \alpha_k = \beta$ a walk between α and β. We find $g_1, \ldots g_k$ such that $\alpha_{i-1}^{g_i} = \alpha_i$ and we denote the directed edges of this walk by $e_1, \ldots, e_k$, so that we have:

$$e_i = (\alpha_{i-1}, \alpha_i) \quad \text{or} \quad e_i = (\alpha_i, \alpha_{i-1}).$$

Then $\psi(g) = \psi(g_1) \cdots \psi(g_k)$ and by the note above we can write:

$$\psi(g_i) = \begin{cases} \psi_{e_i} & \text{if} \quad e_i = (\alpha_{i-1}, \alpha_i) \\ (\psi_{e_i})^{-1} & \text{if} \quad e_i = (\alpha_i, \alpha_{i-1}) \end{cases}$$

where e_i is an edge, so for each i we have $\psi(g_i) = u/v$ or $\psi(g_i) = v/u$. It follows that for any $g \in G$ we can write

$$\psi(g) = \left(\frac{u}{v}\right)^n,$$

for some $n \in \mathbb{Z}$. Furthermore, we get that

$$\operatorname{Im}(\psi) = \left\{ \left(\frac{u}{v}\right)^n : n \in \mathbb{Z} \right\} \simeq \mathbb{Z}$$

because $u/v \neq 1$.

We can now define a function

$$\varphi : V(\Gamma) \to \mathbb{Z}, \quad \beta \mapsto n \qquad \text{if} \qquad \beta = \alpha^g \quad \text{and} \quad \psi(g) = \left(\frac{u}{v}\right)^n.$$

By Remark 3.2.1.1 this map is well defined. Also, it is easy to see that if (β, γ) is an edge, then $\varphi(\gamma) = \varphi(\beta) + 1$, thus φ is a graph homomorphism from Γ to $\widetilde{\mathbb{Z}}$, and it is surjective since the image of ψ is a subgroup of $\mathbb{Q}_+$ spanned by u/v.

Suppose that $\varphi^{-1}(n)$ is finite for some $n \in \mathbb{Z}$. We note that the fibers of φ are simply the orbits of the kernel of ψ, and thus they all have the same cardinality, say k. Then the number of edges with initial vertex in $\varphi^{-1}(n)$ is $u \cdot k$ and the number of edges with terminal vertex in $\varphi^{-1}(n+1)$ is $v \cdot k$, and these must be equal, so we get $uk = vk$ but this contradicts the hypothesis $u \neq v$. □

We can see that Theorem 5.1.6 fails if we omit the hypothesis of edge transitivity, which raises the question of whether we can build a similar epimorphism onto $\widetilde{\mathbb{Z}}^n$ if $\operatorname{Aut}(\Gamma)$ has n orbits on edges (note that $n < \infty$ since the graph is locally finite and vertex transitive). Generally the answer is no, and we give a counter-example below in Example 5.1.7. However, we can still build a graph epimorphism in the same way onto a digraph that is similar to $\widetilde{\mathbb{Z}}^k$ for some $k \leq n$. Further conditions on $\operatorname{Aut}(\Gamma)$ will then guarantee that this digraph is in fact $\widetilde{\mathbb{Z}}^n$.

Example 5.1.7. Let Γ_0 be the infinite directed tree with in-valency 1 and out-valency 2. Γ_0 is locally finite, connected and both vertex- and edge transitive. We will construct a new digraph Γ by adding edges to Γ_0. Let Δ be the subset of $V(\Gamma) \times V(\Gamma)$ such that

$(\alpha, \beta) \in \Delta$ if and only if there is a directed path of length two from α to β in Γ_0. Set $\Gamma := (V(\Gamma), E(\Gamma))$ with

$$V(\Gamma) := V(\Gamma_0) \quad \text{and} \quad E(\Gamma) := E(\Gamma_0) \cup \Delta.$$

Figure 1 shows parts of the digraphs Γ_0 and Γ, all edges are directed downwards. Of course both digraphs continue infinitely upwards and downwards.

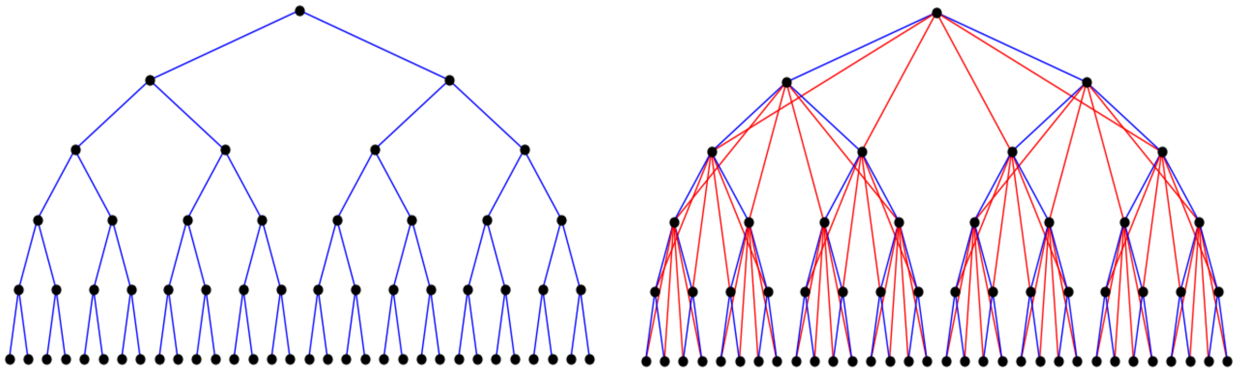

Figure 1: Vertex transitive graph with two orbits on edges

Here, Γ is not edge transitive because an edge from $E(\Gamma_0)$ can not be mapped to an edge from Δ with a graph automorphism. In fact, $\mathrm{Aut}(\Gamma)$ has exactly two orbits on edges, namely $E(\Gamma_0)$ and Δ (shown in blue and red, respectively). This digraph however can not be mapped onto $\widetilde{\mathbb{Z}}^2$ with a graph homomorphism because Γ contains triangles whereas $\widetilde{\mathbb{Z}}^2$ does not.

Definition 5.1.8. Let G be a group and $S \subseteq G$. The *Cayley digraph* of G with respect to S is defined as the directed graph,

$$\mathrm{Cay}(G, S) := (G, \Delta) \quad \text{where} \quad \Delta := \{(g, gs) : g \in G, s \in S\}.$$

We can identify many properties of the Cayley digraph of a group from the set S. Let $\Gamma := \mathrm{Cay}(G, S)$ for a group G and $S \subseteq G$. Then Γ is connected if and only if S generates G. If S contains the identity of G, then Γ has a loop at every vertex, otherwise it has no loops.

Example 5.1.9. The Cayley digraph of the additive group $\mathbb{Z}$ with respect to the set $S = \{1\}$ is $\widetilde{\mathbb{Z}}$. More generally, the Cayley digraph of the additive group $\mathbb{Z}^k$ with respect to the set $S = \{\mathbf{e}_1, \ldots, \mathbf{e}_k\}$ is $\widetilde{\mathbb{Z}}^k$ (here $\mathbf{e}_i \in \mathbb{Z}^k$ is the element with 1 as its i-th coordinate and 0 elsewhere).

Theorem 5.1.10. *Let Γ be an infinite, connected, locally finite, vertex transitive, digraph and let G be a subgroup of $\mathrm{Aut}(\Gamma)$ that has n orbits on $E(\Gamma)$, denoted by $\Delta_1, \ldots, \Delta_n$. For $i = 1, \ldots, n$, let $\Gamma_i := (V(\Gamma), \Delta_i)$ and u_i and v_i be the out-valency and in-valency of Γ_i, respectively. If ψ is the suborbit function defined by the action of G on Γ, then $\mathrm{Im}(\psi) \simeq \mathbb{Z}^k$ for some $k \leq n$ and there exists a graph epimorphism, $\varphi : \Gamma \to \mathrm{Cay}(\mathbb{Z}^k, \theta(S))$*

where $S = \{u_1/v_1, \ldots, u_n/v_n\}$ and θ is an isomorphism from $\mathrm{Im}(\psi)$ to $\mathbb{Z}^k$. Furthermore, every fiber of φ is infinite. (Here we suppose $\mathbb{Z}^0$ is the trivial additive group, $\{0\}$.)

Proof. Suppose first that $u_i = v_i$ for $i = 1, \ldots, n$. Then $\mathrm{Im}(\psi) \simeq \mathbb{Z}^0$ and $S = \mathrm{Im}(\psi)$ so $\mathrm{Cay}(\mathbb{Z}^0, \theta(S))$ is simply the graph with one vertex and a loop on that vertex. The map taking every vertex of Γ to this one vertex then clearly satisfies all our conditions.

Suppose $u_i \neq v_i$ for some i. If $(\alpha, \beta) \in \Delta_i$ is an edge and $\beta = \alpha^g$ we have

$$\psi(g) = \psi_{(\alpha,\beta)} = \frac{u_i}{v_i}.$$

Then, by the same argument as in Theorem 5.1.6, we have for arbitrary $\alpha, \beta \in V(\Gamma)$ such that $\beta = \alpha^g$:

$$\psi(g) = \left(\frac{u_1}{v_1}\right)^{m_1} \cdots \left(\frac{u_n}{v_n}\right)^{m_n}$$

with $m_i \in \mathbb{Z}$. Thus the group $\mathrm{Im}(\psi) = \langle u_1/v_1, \ldots, u_n/v_n \rangle$ is spanned by at most n elements in $\mathbb{Q}_+$ and so we have $\mathrm{Im}(\psi) \simeq \mathbb{Z}^k$ for some $k \in \{1, \ldots, n\}$. We can now construct a map in a similar way as in Theorem 5.1.6. Let θ be an isomorphism from $\mathrm{Im}(\psi)$ to $\mathbb{Z}^k$ and $S = \{u_1/v_1, \ldots, u_n/v_n\}$. We fix a vertex, $\alpha \in V(\Gamma)$ and define $\varphi : V(\Gamma) \to \mathbb{Z}^k$ by

$$\beta \mapsto \theta(\psi(g)) \quad \text{where} \quad \beta = \alpha^g.$$

It is clear that this map is surjective since $\mathrm{Im}(\psi) \simeq \mathbb{Z}^k$. Let $(\beta, \gamma) \in E(\Gamma)$, and $g, h \in G$ such that $\beta = \alpha^h$ and $\gamma = \beta^g$. Then $\varphi(\beta) = \theta(\psi(h))$ and

$$\varphi(\gamma) = \theta(\psi(hg)) = \theta(\psi(h)) + \theta(\psi(g)) = \varphi(\beta) + \theta(u_i/v_i)$$

for some $i = 1 \ldots, n$. Since $\theta(u_i/v_i) \in \theta(S)$ we have shown that $(\varphi(\beta), \varphi(\gamma))$ is an edge in $\mathrm{Cay}(\mathbb{Z}^k, \theta(S))$ and so φ is an epimorphism.

To show that every fiber of φ is infinite, we choose $i \in \{1, \ldots, n\}$ such that $u_i \neq v_i$. Then, $(z, z + \theta(u_i/v_i))$ is an edge for any $z \in \mathbb{Z}^k$. Furthermore, the number of edges in the orbital Δ_i with initial vertex in $\varphi^{-1}(z)$ is $u_i \cdot |\varphi^{-1}(z)|$, and the number of edges in Δ_i with terminal vertex in $\varphi^{-1}(z + \theta(u_i/v_i))$ is $v_i \cdot |\varphi^{-1}(z + \theta(u_i/v_i))|$ and these must be the same, so we have $u_i \cdot |\varphi^{-1}(z)| = v_i \cdot |\varphi^{-1}(z + \theta(u_i/v_i))|$, for all $z \in \mathbb{Z}^k$. Since $u_i \neq v_i$ this implies

$$|\varphi^{-1}(z)| < |\varphi^{-1}(z + \theta(u_i/v_i))| \quad \text{or} \quad |\varphi^{-1}(z)| > |\varphi^{-1}(z + \theta(u_i/v_i))|.$$

It is now clear that if $|\varphi^{-1}(z)|$ were finite for some z, we could find an element $z' \in \mathbb{Z}^k$ with $|\varphi^{-1}(z')| = 0$ which is impossible since φ is surjective. □

Example 5.1.11. Let Γ be the graph from Example 5.1.7 and set $\Gamma_1 := (V(\Gamma), E(\Gamma_0))$ and $\Gamma_2 := (V(\Gamma), \Delta)$. Then Γ_1 has out-valency $u_1 = 2$ and in-valency $v_1 = 1$, and Γ_2 has out-valency $u_2 = 4$ and in-valency $v_2 = 1$. The image of ψ is generated by $u_1/v_1 = 2$ and $u_2/v_2 = 4$, so $\mathrm{Im}(\psi) = \langle 2 \rangle \simeq \mathbb{Z}$. Let $S = \{2, 4\}$ and θ a group isomorphism from $\mathrm{Im}(\psi)$ to $\mathbb{Z}$. Then, by the theorem, we have a surjective graph homomorphism from Γ to $\mathrm{Cay}(\mathbb{Z}, \theta(S))$. Note that in this case the Cayley digraph is not the same as $\widetilde{\mathbb{Z}}$.

We now proceed to add conditions to our graph to guarantee that the Cayley digraph we construct is equal to $\widetilde{\mathbb{Z}}^k$, starting by proving the following well known fact.

Lemma 5.1.12. *Let G be a group acting transitively on a set Ω such that all suborbits are finite. Let $\alpha \in \Omega$ and assume that there exist elements $g_1, \ldots, g_n \in G$ such that $G = \langle G_\alpha, G_\alpha g_1, \ldots, G_\alpha g_n \rangle$. Then the digraph defined by $\Gamma := (\Omega, \Delta_1 \cup \cdots \cup \Delta_n)$, where $\Delta_i = (\alpha, \alpha^{g_i})^G$, is connected and has finite in- and out-valency.*

Proof. The finiteness of the in- and out valency is clear, since all suborbits are finite and Γ has finitely many orbits on edges.

We will show that there exists a path from any arbitrary vertex to α. Let $\beta \in \Omega$ and $g \in G$ such that $\beta = \alpha^g$. Since G is generated by the set $G_\alpha \cup G_\alpha g_1 \cup \cdots \cup G_\alpha g_n$ we can write

$$g = h_1 g_{k_1}^{\epsilon_1} \cdots h_m g_{k_m}^{\epsilon_m} h_{m+1}$$

where $k_i \in \{1, \ldots, n\}$, $\epsilon_i \in \{\pm 1\}$ and $h_i \in G_\alpha$ for all $i \in \{1, \ldots, m\}$.

Let $x_i := h_i g_{k_i}^{\epsilon_i} \ldots h_m g_{k_m}^{\epsilon_m} h_{m+1} \in G$ and $\beta_i := \alpha^{x_i}$ for $i \in \{1 \ldots m\}$. Then $\beta_1 = \beta$ and for every $i = 1, \ldots, m-1$ we have one of the following (depending on ϵ_i):

(i) $(\beta_{i+1}, \beta_i) = (\alpha, \alpha^{h_i g_{k_i}})^{x_{i+1}} = (\alpha, \alpha^{g_{k_i}})^{x_{i+1}} \in E(\Gamma)$

(ii) $(\beta_i, \beta_{i+1}) = (\alpha^{h_i g_{k_i}^{-1}}, \alpha)^{x_{i+1}} = (\alpha^{g_{k_i}^{-1}}, \alpha)^{x_{i+1}} \in E(\Gamma)$

In the same way either (α, β_m) or (β_m, α) is an edge and so $\alpha, \beta_m, \ldots, \beta_1 = \beta$ is a walk from α to β. $\square$

Theorem 5.1.13. *If we assume the hypotheses of the lemma and furthermore that*

$$\langle \psi(g_1), \ldots, \psi(g_n) \rangle = \mathrm{Im}(\psi) \simeq \mathbb{Z}^n,$$

then there exists a graph epimorphism $\varphi : \Gamma \to \widetilde{\mathbb{Z}}^n$ all of whose fibers are infinite.

Proof. By the lemma and the fact that $\psi(g_1), \ldots, \psi(g_n)$ generate $\mathrm{Im}(\psi)$ we have that Γ is connected, with finite but unequal in-valency and out-valency and has exactly n orbits on edges, namely $\Delta_1, \ldots, \Delta_n$. Furthermore, if u_i and v_i are in out-valency and in-valency of $\Gamma_i := (V(\Gamma), \Delta_i)$, respectively, then $\psi(g_i) = u_i/v_i$. Let $S := \{u_1/v_1, \ldots, u_n/v_n\}$. Then by Theorem 5.1.10 there exists a graph epimorphism, $\varphi : \Gamma \to \mathrm{Cay}(\mathbb{Z}^n, \theta(S))$, where θ is any isomorphism from $\mathrm{Im}(\psi)$ to $\mathbb{Z}^n$, with every fiber of φ infinite. We know that $\widetilde{\mathbb{Z}}^n$ is the Cayley digraph of $\mathbb{Z}^n$ with respect to the set $\{\mathbf{e}_1, \ldots, \mathbf{e}_n\}$, so it only remains to show that there is an isomorphism from $\mathrm{Im}(\psi)$ to $\mathbb{Z}^n$ taking u_i/v_i to $\mathbf{e}_i$. But this is clear, since the sets S and $\{\mathbf{e}_1, \ldots, \mathbf{e}_n\}$ are both bases for these groups as $\mathbb{Z}$-modules. $\square$

Now this generalization of Praeger's result (Theorem 5.1.6) is easily proved.

Corollary 5.1.14. *Let Γ be an infinite, connected, locally finite, vertex transitive, digraph* *and let G be a subgroup of* $\mathrm{Aut}(\Gamma)$ *that has n orbits on $E(\Gamma)$. If* $\mathrm{Im}(\psi) \simeq \mathbb{Z}^n$*, then there exists a graph epimorphism $\varphi : \Gamma \to \widetilde{\mathbb{Z}}^n$ all of whose fibers are infinite.*

Proof. Let $\alpha \in V(\Gamma)$ and $\Delta_1, \ldots, \Delta_n$ be the orbits on the edges. Then there exist $g_1, \ldots, g_n \in G$ such that $(\alpha, \alpha^{g_i}) \in \Delta_i$. It is not hard to see that $\langle G_\alpha, g_1, \ldots, g_n \rangle$ acts transitively on $V(\Gamma)$, because α can be mapped to any of its neighbors with $g_i h$ or $g_i^{-1} h$ for some $h \in G_\alpha$ and we generalize this by using the identity $G_{\alpha^g} = g^{-1} G_\alpha g$. We know that there is a one-to-one correspondence between $V(\Gamma)$ and G/G_α given by $\alpha^h \leftrightarrow G_\alpha h$, so we can look at the vertices of Γ as right cosets of G_α in G. Then, for any $h \in G$ there exists $h' \in \langle G_\alpha, g_1, \ldots, g_n \rangle$ such that $G_\alpha h = G_\alpha h'$, that is $h \in G_\alpha h' \subset \langle G_\alpha, g_1, \ldots, g_n \rangle$, and therefore $G = \langle G_\alpha, g_1, \ldots, g_n \rangle$. Now the result follows from Theorem 5.1.13. □

Corollary 5.1.15. *Under the assumptions of* Corollary 5.1.14*, Γ is bipartite.*

Proof. Clearly, $\widetilde{\mathbb{Z}}^n$ contains no cycles of odd lengths and so neither does Γ. This is equivalent to Γ being bipartite (see Proposition 1.6.1 in [6]) □

Example 5.1.16. Consider the infinite regular directed tree, Γ, with out-valency 5 and in-valency 2. Color its edges in two colors, such that every vertex has three blue edges going out and two red ones, and one of each color coming in.

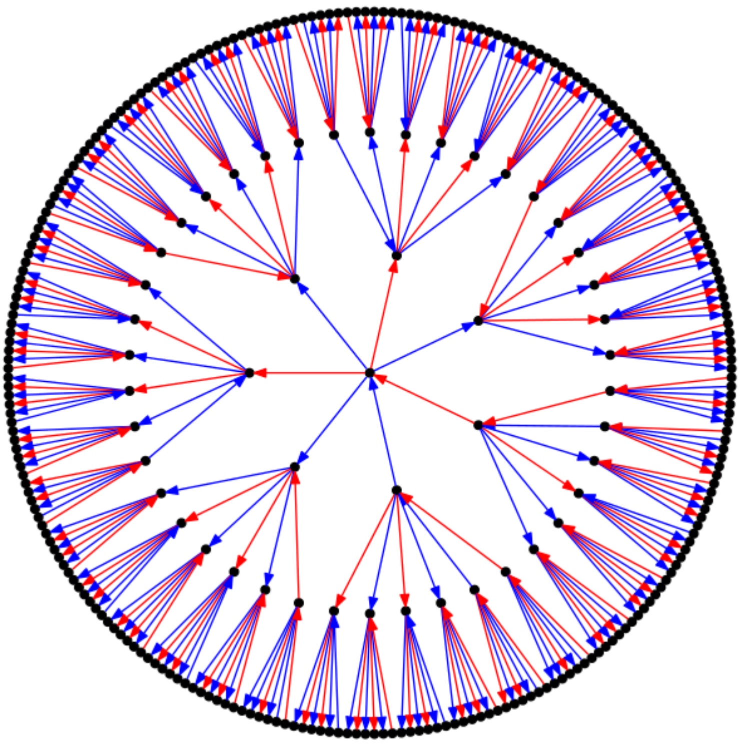

Figure 2: Infinite regular directed tree with two-colored edges

Define Δ_1 and Δ_2 as the sets of blue edges and red edges, respectively. Then $E(\Gamma) = \Delta_1 \cup \Delta_2$. Let $G \leq \mathrm{Aut}(\Gamma)$ be the subgroup that maps only red edges to red edges and blue edges to blue edges. G is clearly transitive on $V(\Gamma)$ since every vertex has both blue and red edges going in and out, and it has two orbits on $E(\Gamma)$, namely Δ_1 and Δ_2. As before we define $\Gamma_1 := (V(\Gamma), \Delta_1)$ and $\Gamma_2 := (V(\Gamma), \Delta_2)$. Then Γ_1 has out-valency $u_1 = 3$ and in-valency $v_1 = 1$, and Γ_2 has out-valency $u_2 = 2$ and in-valency $v_2 = 1$ so the image of ψ is the subgroup of $\mathbb{Q}_+$ generated by 2 and 3, that is $\mathrm{Im}(\psi) \simeq \mathbb{Z}^2$. Thus by Theorem 5.1.13, this digraph can be mapped onto $\widetilde{\mathbb{Z}}^2$ with an epimorphism all of whose fibers are infinite.

Example 5.1.17. We start with the same digraph as above. Adding edges and colors we can build similar epimorphisms onto $\widetilde{\mathbb{Z}}^n$ for any $n \in \mathbb{N}^*$. Suppose now Γ has countable in-valency and out-valency. Let p_i denote the i-th prime number and let $c_1, c_2, \ldots$ denote different colors. If we color the edges of Γ such that every vertex has one edge of each color coming in, and p_i edges of the color c_i going out, for every $i \in \mathbb{N}^*$, then the suborbit function is surjective onto $\mathbb{Q}_+$.

Of course, had we not colored the edges of the digraph in Example 5.1.16 in two different colors, we would simply have had a regular directed tree with unequal out-valency and in-valency, thus yielding a graph epimorphism onto $\widetilde{\mathbb{Z}}$. It even seems like a bit of cheat, taking a nice and edge transitive digraph and making it less nice so that it fits the conditions of our theorem. The fact of the matter is that it is not trivial to find a digraph such that taking G as the full automorphism group, it satisfies these conditions. However it is not impossible either, and we will see at the end of Chapter 6, when we have the proper tools, that we can in fact construct such digraphs for any $n \in \mathbb{N}^*$.

5.2 Highly arc transitive digraphs

Having gone from edge transitivity to finitely many orbits on edges, we now turn around and go in the other direction to consider a property of infinite digraphs that is even stronger than edge transitivity.

Definition 5.2.1. Let Γ be a digraph. An *s-arc* of Γ is a sequence of $s+1$ vertices, $\alpha_0, \alpha_1, \ldots, \alpha_s$ such that $(\alpha_{i-1}, \alpha_i) \in E(\Gamma)$ for every $i \in \{1, \ldots, s\}$. We say that Γ is *s-arc transitive* if the automorphism group, $\mathrm{Aut}(\Gamma)$, acts transitively on the set of s-arcs. A graph that is s-arc transitive for every $s \in \mathbb{N}$ is called *highly arc transitive.*

Remark. We note that 0-arcs and 1-arcs are simply vertices and edges respectively.

Example 5.2.2. (i) Every infinite, regular directed tree is highly arc transitive.

(ii) Let Γ_1 be the infinite, regular directed tree with in-valency 1 and out-valency 2. We will construct a new graph, Γ, by adding vertices and edges as follows: We duplicate each vertex of Γ_1 such that if α' and β' are the duplicates of α and β, respectively, and $(\alpha, \beta) \in E(\Gamma_1)$, then we add $(\alpha, \beta'), (\alpha', \beta)$ and (α', β') to the edge set of Γ. This digraph is highly arc transitive and part of it is shown in Figure 3, all edges directed downwards.

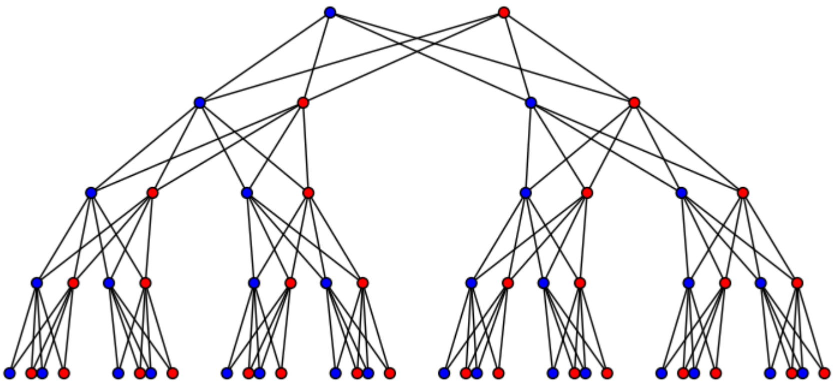

Figure 3: A highly arc transitive digraph

It is easily proved by induction that in an infinite, connected, vertex transitive digraph, s-arc transitivity implies $(s-1)$-arc transitivity. This naturally raises the question of whether there exist for any s, digraphs that are s-arc transitive but not $(s+1)$-arc transitive. We will consider two such examples, for $s = 1$ and $s = 2$.

Example 5.2.3. We construct a digraph Γ as follows: Let $V(\Gamma) := \mathbb{Z} \times \{0, 1\}$. Now we define the sets

$$\begin{aligned} \Delta_j &:= \{((2i-1, j), (2i, j)), ((2i+1, j), (2i, j)) : i \in \mathbb{Z}\} \quad \text{for} \quad j \in \{0,1\} \\ \Delta_0' &:= \{((2i, 0), (2i-1, 1)), ((2i, 0), (2i+1, 1)) : i \in \mathbb{Z}\} \\ \Delta_1' &:= \{((2i, 1), (2i-1, 0)), ((2i, 1), (2i+1, 0)) : i \in \mathbb{Z}\} \end{aligned}$$

and set $E(\Gamma) := \Delta_0 \cup \Delta_1 \cup \Delta_0' \cup \Delta_1'$. We can see part of the digraph Γ in figure 4.

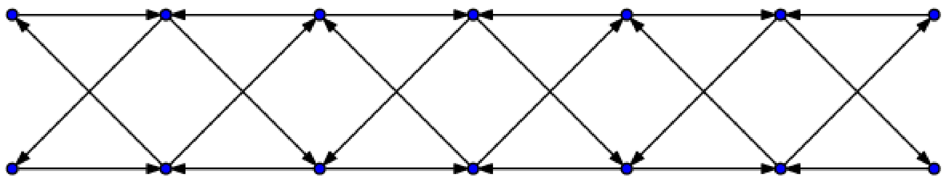

Figure 4: Edge transitive, non-2-arc transitive digraph

This digraph is edge transitive, but it is not 2-arc transitive, because the 2-arc $(1,0), (0,0), (1,1)$ can not be mapped to the 2-arc $(1,0), (0,0), (-1,1)$ with a graph homomorphism. We note that since the in-valency and the out-valency are equal, the suborbit function is trivial on this digraph.

Example 5.2.4. In 2007, Norbert Seifter conjectured that infinite, connected, locally finite, 2-arc transitive digraphs were always highly arc transitive [14]. This conjecture was disproved the same year by Sonia P. Mansilla [9], with an infinite family of 2-arc transitive digraphs that are not 3-arc transitive, namely digraphs, Γ_n, defined by:

$$V(\Gamma_n) := \mathbb{Z}_n \times \mathbb{Z}_n \times \mathbb{Z}$$

$$E(\Gamma_n) := \{((i,j,k),(j,i,k+1)),((i,j,k),(j,i+1,k+1)) : (i,j,k) \in V(\Gamma_n)\}$$
for $n \geq 3$.

We now start preparing for the key theorem of this section that gives conditions on a digraph that guarantee it to be highly arc transitive. We first need the following lemma.

Lemma 5.2.5. *If H and K are two subgroups of a group G such that $|G:H|$ and $|G:K|$ are finite and relatively prime then $|G:H\cap K| = |G:H||G:K|$.*

Proof. If we look at the action of G on the direct product of the coset spaces, $(G/H)\times(G/K)$, the stabilizer of the point (H,K) is $H\cap K$. Therefore

$$|G:H\cap K| = |(H,K)^G| \leq |(G/H)\times(G/K)| = |G:H||G:K|.$$

Since $H\cap K \leq H$ and $H\cap K\leq K$ we have

$$\begin{aligned}|G:H\cap K| &= |G:H||H:H\cap K|\\ &= |G:K||K:H\cap K|\end{aligned}$$

so $|G:K|$ divides $|H:H\cap K|$, and therefore $|G:H\cap K| = |G:H||G:K|k$ for some $k\in\mathbb{N}^*$. But since $|G:H\cap K|\leq |G:H||G:K|$ it is clear that $k=1$ and $|G:H\cap K| = |G:H||G:K|$. □

Theorem 5.2.6. *Let Γ be an infinite, connected, vertex transitive, edge transitive, digraph with finite out- and in-valency, u and v respectively. If u and v are relatively prime then Γ is highly arc transitive.*

Proof. We proceed by induction. Since the 1-arcs are simply edges, Γ is 1-arc transitive by the hypothesis.

Suppose Γ is s-arc transitive and let $\alpha_0,\alpha_1,\dots,\alpha_{s+1}$ be an $(s+1)$-arc. Since Γ is s-arc transitive, G_{α_s} acts transitively on the set of s-arcs having α_s as their terminal vertex. Then $|G_{\alpha_s}:G_{\alpha_0\dots\alpha_s}|$ is equal to the cardinality of this set, which is obviously v^s. In the same way we get $|G_{\alpha_s}:G_{\alpha_s\alpha_{s+1}}| = u$, because Γ is edge transitive. Since $G_{\alpha_0\cdots\alpha_s}\cap G_{\alpha_s\alpha_{s+1}} = G_{\alpha_0\cdots\alpha_{s+1}}$ we have by the lemma:

$$|G_{\alpha_s}:G_{\alpha_0\dots\alpha_{s+1}}| = |G_{\alpha_s}:G_{\alpha_0\dots\alpha_s}||G_{\alpha_s}:G_{\alpha_s\alpha_{s+1}}| = v^s u.$$

But then G_{α_s} acts transitively on the set of $(s+1)$-arcs having α_s as its second last vertex. Since G is transitive on vertices it can take α_s to any vertex, so this means that G is transitive on $(s+1)$-arcs. □

We get the following corollary that appeared in a paper by Norbert Seifter in 2008.

Corollary 5.2.7. [14, Proposition 3.2] *Let Γ be an infinite, connected vertex- and edge transitive digraph with prime out-valency, u and in-valency $v<u$. Then Γ is highly arc transitive.*

We can actually prove Theorem 5.2.6 using the suborbit function instead of Lemma 5.2.5. We will do this shortly, but in order to do that we need the following proposition.

Proposition 5.2.8. *Let Γ be an infinite, connected, vertex- and edge transitive digraph with finite out- and in-valencies u and v, respectively, and let $p, q \in \mathbb{N}^*$ be relatively prime numbers such that $p/q = u/v$. Then if $g \in G := \mathrm{Aut}(\Gamma)$ is such that $(\alpha, \alpha^g) \in E(\Gamma)$, we have for any $n \geq 1$*

$$p^n \leq |\alpha^{g^n G_\alpha}| \leq u^n$$

Proof. We have

$$\frac{|\alpha^{g^n G_\alpha}|}{|\alpha^{G_{\alpha g^n}}|} = \left(\frac{u}{v}\right)^n = \frac{p^n}{q^n}$$

where p^n/q^n is a reduced fraction. Therefore it is clear that $p^n \leq |\alpha^{g^n G_\alpha}|$.

We know that the number of n-arcs starting from α is u^n. It is also clear that every vertex in $\alpha^{g^n G_\alpha}$ is the last vertex of such an n-arc, and therefore $|\alpha^{g^n G_\alpha}| \leq u^n$. □

Second proof of Theorem 5.2.6. Let p and q be as in the proposition above. Then, since u and v are relatively prime, $p = u$, thus if (α, α^g) is an edge we have by the proposition $|\alpha^{g^n G_\alpha}| = u^n$ for all $n \in \mathbb{N}^*$. But since $|\alpha^{g^n G_\alpha}| \leq |G_{\alpha_0} : G_{\alpha_0 \alpha_1 \cdots \alpha_n}| \leq u^n$, we have that $|G_{\alpha_0} : G_{\alpha_0 \alpha_1 \cdots \alpha_n}| = u^n$. Thus G_α acts transitively on n-arcs with initial vertex α and since G acts transitively on vertices this means that G acts transitively on n-arcs. □

For the next two corollaries, recall Definition 4.2.9 of the scale function.

Corollary 5.2.9. *Let Γ be as in* Theorem 5.2.6 *and let* $\mathbf{s}$ *be the scale function defined by the permutation topology on $G := \mathrm{Aut}(\Gamma)$. If $g \in G$ is such that (α, α^g) is an edge for some $\alpha \in \Omega$, then*

$$\mathbf{s}(g) = \lim_{n \to \infty} |\alpha^{g^n G_\alpha}|^{1/n} = u.$$

Proof. This follows from Theorem 4.2.9 and the proof above. □

Corollary 5.2.10. *Let G be a totally disconnected, locally compact group and let $g \in G$. If there exists a compact open subgroup U of G such that the numbers $p := |U : U \cap (g^{-1}Ug)|$ and $q := |U : U \cap (gUg^{-1})|$ are relatively prime, then $\mathbf{s}(g) = p$ and $\mathbf{s}(g^{-1}) = q$. In this case U is tidy for g.*

Proof. We let Γ be the orbital digraph defined by the action of G on G/U, with respect to the orbital $\Delta := (\alpha, \alpha^g)^G$, where $\alpha \in V(\Gamma)$ is the vertex that corresponds to $U \in G/U$. Let $\beta := \alpha^g$. Then Γ is infinite, edge transitive and has out-valency

$$|\beta^{G_\alpha}| = |G_\alpha : G_{\alpha\beta}| = |U : U \cap (g^{-1}Ug)| = p$$

and in valency

$$|\alpha^{G_\beta}| = |G_\beta : G_{\alpha\beta}| = |U : U \cap (gUg^{-1})| = q$$

and these are relatively prime. We note that Γ is not necessarily connected, but this is not a problem because we can simply look at its connected components. Then we have by Corollary 5.2.9 that $\mathbf{s}(g) = p$, and by Corollary 4.2.7 we also have $\mathbf{s}(g^{-1}) = q$. By definition U is tidy for g. $\square$

Example 5.2.11. In an article from 2001, Reinhard Diestel and Imre Leader constructed a sequence of digraphs, using *line digraphs* [7]. We will describe these graphs quickly here. The *line digraph* of a digraph Γ, is the digraph whose vertex set is $E(\Gamma)$ and whose edges are $((\alpha, \beta), (\beta, \gamma))$ for $(\alpha, \beta), (\beta, \gamma) \in E(\Gamma)$. We now use this to inductively define a sequence of highly arc transitive digraphs. Let Γ_0 be the infinite regular directed tree with in-valency 2 and out-valency 3, and define Γ_{n+1} as the line digraph of Γ_n for $n \in \mathbb{N}$. Below we can see Γ_0, Γ_1 and Γ_2, with Γ_1 and Γ_2 drawn on top of the preceding digraphs in the sequence. All edges are directed downwards.

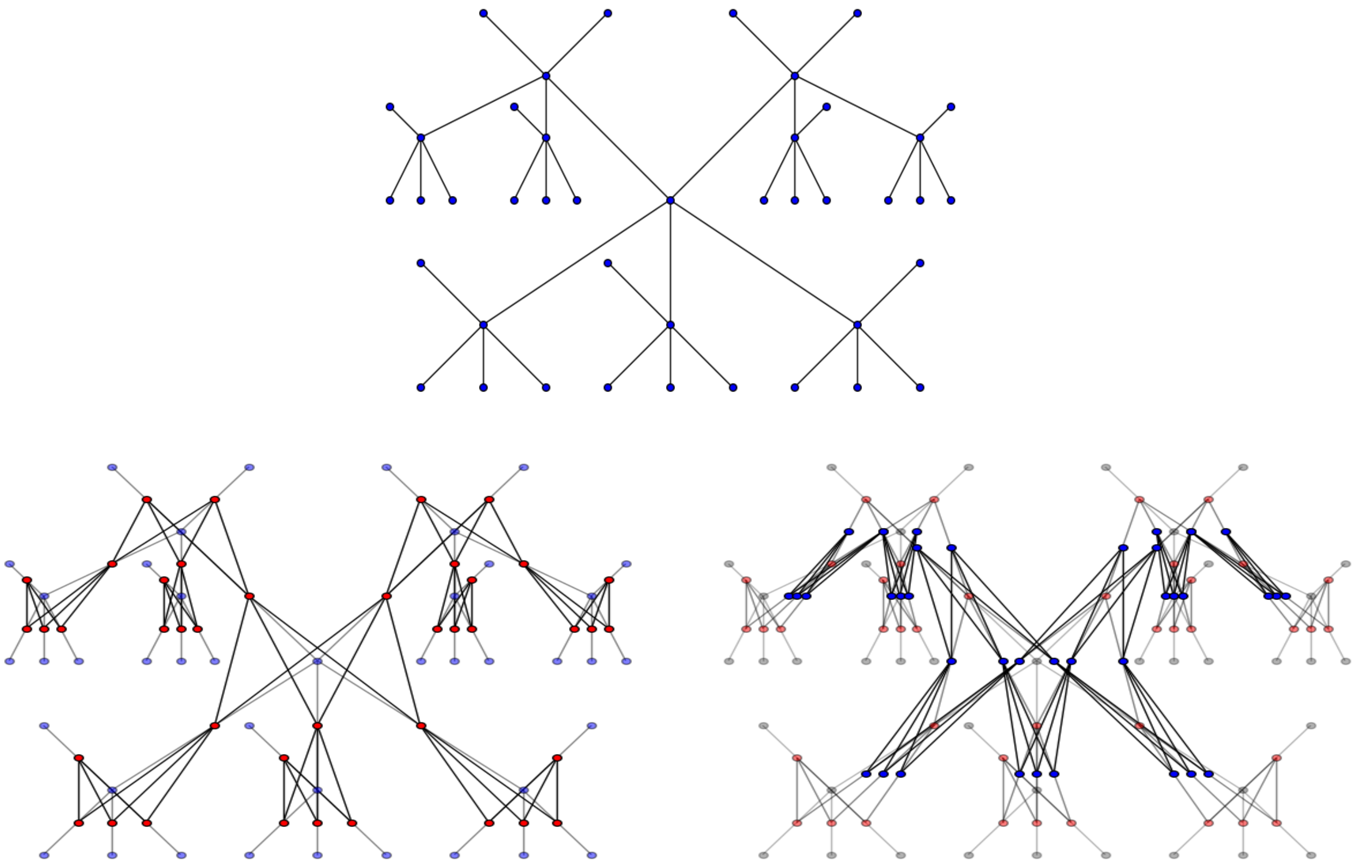

Figure 5: First three digraphs in the line digraph sequence

We know that Γ_0 is highly arc transitive. It can thus be shown by induction on n that Γ_n is connected, vertex- and edge transitive and has in-valency 2 and out-valency 3, for all $n \in \mathbb{N}$. Therefore, by Theorem 5.2.6, Γ_n is highly arc transitive for any $n \in \mathbb{N}$.

5.3 Cayley–Abels digraphs

We have already described Cayley digraphs of groups in Section 5.1. In this section we define similar digraphs for compactly generated, totally disconnected, locally compact groups, called Cayley–Abels digraphs.

Definition 5.3.1. A topological group G is *compactly generated* if it contains a compact subset that generates G.

Definition 5.3.2. Let G be a topological group. A connected digraph, Γ, is a *Cayley–Abels digraph* of G if there is a vertex transitive action of G on Γ such that every point stabilizer is a compact open subgroup of G.

Remark. We can define an undirected Cayley–Abels graph similarly, and in fact the main results of this section also apply to undirected Cayley–Abels graphs.

Let's look at how we can construct such a digraph. Let G be a compactly generated, totally disconnected, locally compact group and let S be a compact generating set of G. Since G is totally disconnected and locally compact, there exists a compact open subgroup U in G. The cosets of U are then an open covering of G, in particular an open covering of S, and since S is compact there exists a finite subcovering. Therefore, we can find finitely many elements, $g_1, \ldots, g_n \in G$, such that $\langle U, g_1, \ldots, g_n \rangle = G$. Define $\Gamma = (V(\Gamma), E(\Gamma))$ with

$$V(\Gamma) := G/U \quad \text{and} \quad E(\Gamma) := (\alpha, \alpha^{g_1})^G \cup \cdots \cup (\alpha, \alpha^{g_n})^G$$

where α is the vertex that corresponds to $U \in G/U$. Then G acts transitively on $V(\Gamma) = G/U$ and every point stabilizer is a conjugate of the compact open subgroup $G_\alpha = U$ and is therefore compact and open. Furthermore, Γ is connected by Lemma 5.1.12, so it is a Cayley–Abels digraph of G.

We note that given a compactly generated, totally disconnected, locally compact group, G, and an arbitrary locally finite Cayley–Abels digraph, Γ of G, we can describe Γ in the same way as above. We simply take a vertex α of Γ and define $U := G_\alpha$. Furthermore we let $\beta_1, \ldots, \beta_n$ be all the vertices of Γ such that $(\alpha, \beta_i) \in E(\Gamma)$ and let $g_i \in G$ such that $\alpha^{g_i} = \beta_i$. Then $G = \langle U, g_1, \ldots, g_n \rangle$ and $E(\Gamma) = (\alpha, \beta_1)^G \cup \cdots \cup (\alpha, \beta_n)^G$.

We now proceed to partly answer a question asked by George A. Willis on his visit to Reykjavík in 2014. Willis speculated whether the lowest possible valency of an undirected Cayley–Abels graph of a given compactly generated, totally disconnected, locally compact group, could tell us anything about the group. Conversely, whether properties of the group can tell us anything about a lowest valency of its Cayley–Abels graphs. We will give a partial answer to the latter question, giving a lower bound on the valency of Cayley–Abels digraphs.

Recall that for such a group G, the suborbit function defined by the action of G on G/U where U is a compact open subgroup, is independent of U (Remark 4.2.4.1).

Theorem 5.3.3. *Let G be a compactly generated, totally disconnected, locally compact group and let $\psi : G \to \mathbb{Q}_+$ be the suborbit function defined by the action of G on G/U for some compact open subgroup U. If $\mathrm{Im}(\psi)$ is cyclic and generated by the element p/q, with $p, q \in \mathbb{N}^*$ relatively prime, then every Cayley–Abels digraph of G has valency at least $p+q$.*

Proof. Let Γ be a locally finite Cayley–Abels digraph of G. Then G acts transitively on vertices of Γ and has finitely many orbits on its edges, $\Delta_1, \ldots, \Delta_n$. Define the subgraphs $\Gamma_i = (V(\Gamma), \Delta_i)$, for $i = 1, \ldots, n$ and let u_i and v_i be the out-valency and in-valency of Γ_i, respectively. Note that if d is the total valency of Γ, then $d = u_1 + \cdots + u_n + v_1 + \cdots + v_n$. For $\alpha \in V(\Gamma)$ we know that G_α is a compact open subgroup of G, and that we can look at the vertex set of Γ as the coset space G/G_α. The suborbit function defined by the action of G on $V(\Gamma)$ is therefore equal to ψ and so, as we have seen before, we have

$$\mathrm{Im}(\psi) = \left\langle \frac{p}{q} \right\rangle = \left\langle \frac{u_1}{v_1}, \ldots, \frac{u_n}{v_n} \right\rangle.$$

We can therefore write $u_i/v_i = (p/q)^{m_i}$ with $m_i \in \mathbb{Z}$ for every $i \in \{1, \ldots, n\}$ and at least one of the m_i is not 0, say $i = k$. Then we have

$$p + q \leq p^{m_k} + q^{m_k} \leq u_k + v_k \leq \sum_{i=1}^{n} (u_i + v_i) = d.$$

□

We can get a similar result when the image of ψ is not cyclic, but our problem here is finding a set of generators $\{p_1/q_1, \ldots, p_n/q_n\}$, with the smallest possible sum $p_1 + \cdots + p_n + q_1 + \cdots + q_n$. We can not necessarily find such a set, but we do however know that it exists.

Theorem 5.3.4. *Let G and ψ be as in Theorem 5.3.3. Define the set of sums:*

$$A := \left\{ \sum_{i=1}^{k} (p_i + q_i) : \left\langle \frac{p_1}{q_1}, \ldots, \frac{p_k}{q_k} \right\rangle = \mathrm{Im}(\psi) \right\} \subseteq \mathbb{N}^*.$$

Then every Cayley–Abels digraph of G has valency at least $\min(A)$.

Proof. Let Γ be a locally finite Cayley–Abels digraph of G and let $\Delta_1, \ldots, \Delta_n$ be the orbits of G on edges of Γ. Define the subgraphs $\Gamma_i = (V(\Gamma), \Delta_i)$, for $i = 1, \ldots, n$ and let u_i and v_i be the out-valency and in-valency of Γ_i, respectively. Then $d = u_1 + \cdots + u_n + v_1 + \cdots + v_n$ is the total valency of Γ. As before, we know that the suborbit function defined by the action of G on $V(\Gamma)$ is equal to ψ. Therefore

$$\mathrm{Im}(\psi) = \left\langle \frac{u_1}{v_1}, \ldots, \frac{u_n}{v_n} \right\rangle$$

so $d \in A$, that is $d \geq \min(A)$. □

5.4 Growth of graphs

Definition 5.4.1. We say that an undirected graph Γ, is *1-transitive* (sometimes called *arc transitive*) if for any two edges, $\{\alpha_1, \beta_1\}$ and $\{\alpha_2, \beta_2\}$ of Γ there exists an element $g \in \mathrm{Aut}(\Gamma)$ such that $\alpha_1^g = \alpha_2$ and $\beta_1^g = \beta_2$. Equivalently, Γ is 1-transitive if it is edge transitive and for every edge $\{\alpha, \beta\}$ of Γ there exists $g \in \mathrm{Aut}(\Gamma)$ such that $\alpha^g = \beta$ and $\beta^g = \alpha$.

Define a metric, d, on a connected graph Γ (directed or undirected) in the following way: for vertices, α and β of Γ we let $d(\alpha, \beta)$ be the number of edges in the shortest path between α and β. Clearly, any automorphism on Γ is an isometry with respect to this metric.

Definition 5.4.2. Let Γ be a connected graph (directed or undirected). We define the number $C_n(\alpha)$ as the number of vertices at distance n from α, that is

$$C_n(\alpha) := |\{\beta \in V(\Gamma) : d(\alpha, \beta) = n\}|, \quad n \in \mathbb{N}.$$

Furthermore we define $B_n(\alpha)$ as the number of vertices at distance less than or equal to n from α, that is

$$B_n(\alpha) := |\{\beta \in V(\Gamma) : d(\alpha, \beta) \leq n\}|, \quad n \in \mathbb{N}.$$

Definition 5.4.3. We say that an infinite connected graph Γ *grows exponentially* or *has exponential growth* if there exists a constant $a > 1$ and a number $N \in \mathbb{N}$ such that $a^n \leq B_n(\alpha)$ for all $n \geq N$ where α is some vertex of Γ. If Γ does not grow exponentially, we say that it *grows subexponentially.*

Lemma 5.4.4. *Let Γ be as in Theorem 5.1.6 and $\alpha \in V(\Gamma)$. Then $|\alpha^{g^n G_\alpha}| \leq C_n(\alpha)$ where $g \in G$ is such that $(\alpha, \alpha^g) \in E(\Gamma)$.*

Proof. Since (α, α^g) is an edge, then so is $(\alpha, \alpha^g)^{g^i} = (\alpha^{g^i}, \alpha^{g^{i+1}})$ for $i = 1, \ldots, n-1$. Then $\alpha, \alpha^g, \ldots, \alpha^{g^n}$ is a directed path of length n, and thus $d(\alpha, \alpha^{g^n}) \leq n$. But by Theorem 5.1.6 we have a graph homomorphism from Γ onto $\widetilde{\mathbb{Z}}$ where there is a unique path between every two vertices. Therefore every path from α to α^{g^n} must have the same length, that is $d(\alpha, \alpha^{g^n}) = n$.

Now let $\gamma \in \alpha^{g^n G_\alpha}$. Then $\gamma = \alpha^{g^n h}$ for some $h \in G_\alpha$ and since h is a graph automorphism, and therefore an isometry, we get:

$$n = d(\alpha, \alpha^{g^n}) = d(\alpha^h, \alpha^{g^n h}) = d(\alpha, \alpha^{g^n h}) = d(\alpha, \gamma).$$

Thus $\alpha^{g^n G_\alpha} \subset \{\beta \in V(\Gamma) : d(\alpha, \beta) \leq n\}$ and so $|\alpha^{g^n G_\alpha}| \leq C_n(\alpha)$. $\square$

Remark. In a vertex transitive graph, the numbers $C_n(\alpha)$ and $B_n(\alpha)$ do not depend on α. Since we are mainly concerned with vertex transitive graphs, we will from now on look at them as a function of n and denote them by $C(n)$ and $B(n)$.

We are now ready for the following theorem.

Theorem 5.4.5. *An infinite, connected, vertex transitive, edge transitive digraph of finite, unequal out-valency and in-valency has exponential growth.*

Proof. Let Γ be such a digraph and let $\alpha \in V(\Gamma)$. Since $u \neq v$ there exists $g \in G$ such that $(\alpha, \alpha^g) \in E(\Gamma)$ and $\psi(g) \neq 1$. Then we have for any $n \in \mathbb{N}^*$:

$$(\psi(g))^n = \psi(g^n) = \frac{\left|\alpha^{g^n G_\alpha}\right|}{\left|\alpha^{G_{\alpha g^n}}\right|} \leq \left|\alpha^{g^n G_\alpha}\right| \leq C(n) \leq B(n).$$

So we see that if $\psi(g) > 1$ then Γ has exponential growth and if $\psi(g) < 1$, we can replace g by g^{-1} to get the same result, because then $(\alpha^{g^{-1}}, \alpha) \in E(\Gamma)$. $\square$

We can use Theorem 5.4.5 to give another proof of a theorem of Thomassen and Watkins from 1989 [15].

Corollary 5.4.6. *Let Γ be an infinite, connected, vertex transitive, edge transitive (undirected) graph of odd valency. If the function $C(n)$ is subexponential, then Γ is 1-transitive.*

Proof. Suppose Γ is not 1-transitive. We can think of each edge, $\{\alpha, \beta\}$ as a pair of directed edges, $\{(\alpha, \beta), (\beta, \alpha)\}$. Then, since Γ is edge transitive but not 1-transitive, it has two orbits on the new directed edges, where two directed edges corresponding to the same undirected edge are in different orbits. Take one of these orbits, Δ, and define a new directed graph, $\Gamma' := (V(\Gamma), \Delta)$. It is clear that directing the edges in this way does not change the metric, d and thus the function $C(n)$ is still the same. Furthermore the out-valency, u, and in-valency, v, must differ since the total valency is odd. The result therefore follows from the theorem above. $\square$

6 Cartesian products

As with many other mathematical structures, we can define products of two or more given graphs. Graphs can be factored with respect to these products, and if they can not be factored non-trivially, they are said to be prime. These notions are all familiar (for example from number theory), but what do they mean for graphs? In fact there are many ways to define products of graphs, and in the first section of this chapter we will define three different graph products and look at some basic examples. However we are mainly concerned with one of these, called the Cartesian product, because it will prove useful for finding graphs with certain properties. In particular, we will construct graphs that satisfy the conditions of Corollary 5.1.14, for any number of orbits.

6.1 Definitions and first examples

We start with some basic definitions.

Definition 6.1.1. Let Γ_1, Γ_2 be digraphs and $\Omega := V(\Gamma_1) \times V(\Gamma_2)$. Define the sets:

$$\Delta_1 := \big\{\big((\alpha_1,\alpha_2),(\alpha_1,\beta_2)\big), \big((\gamma_1,\gamma_2),(\delta_1,\gamma_2)\big) : (\alpha_2,\beta_2) \in E(\Gamma_2), (\gamma_1,\delta_1) \in E(\Gamma_1)\big\}$$
$$\Delta_2 := \big\{\big((\alpha_1,\alpha_2),(\beta_1,\beta_2)\big) : (\alpha_1,\beta_1) \in E(\Gamma_1), (\alpha_2,\beta_2) \in E(\Gamma_2)\big\}.$$

We define the *Cartesian product* of Γ_1 and Γ_2 as $\Gamma_1 \square \Gamma_2 := (\Omega, \Delta_1)$, the *direct product* of Γ_1 and Γ_2 as $\Gamma_1 \times \Gamma_2 := (\Omega, \Delta_2)$, and the *strong product* of Γ_1 and Γ_2 as $\Gamma_1 \boxtimes \Gamma_2 := (\Omega, \Delta_1 \cup \Delta_2)$. If $\Gamma = \Gamma_1 * \Gamma_2$, where $*$ is any of the three products defined above, we say that Γ_1 and Γ_2 are *factors* of Γ, with respect to that particular product.

It is more common to define these products for undirected graphs, replacing every edge (α,β) in the definition, with an edge $\{\alpha,\beta\}$. For Cartesian products, this does not change the product, that is if Γ_1 and Γ_2 are digraphs and Γ_1' and Γ_2' are the corresponding undirected graphs, (where every edge (α,β) is replaced by $\{\alpha,\beta\}$) then $\Gamma_1' \square \Gamma_2'$ is the corresponding undirected graph for the digraph $\Gamma_1 \square \Gamma_2$. The same does not hold for direct and strong products because if $\{\alpha_1,\beta_1\} \in E(\Gamma_1)$ and $\{\alpha_2,\beta_2\} \in E(\Gamma_2)$ then we get two corresponding edges in Δ_2, namely $\big\{(\alpha_1,\alpha_2),(\beta_1,\beta_2)\big\}$ and $\big\{(\alpha_1,\beta_2),(\beta_1,\alpha_2)\big\}$. In these cases, the corresponding undirected graph for $\Gamma_1 \times \Gamma_2$ (resp. $\Gamma_1 \boxtimes \Gamma_2$) is only a subgraph of $\Gamma_1' \times \Gamma_2'$ (resp. $\Gamma_1' \boxtimes \Gamma_2'$).

Definition 6.1.2. Let $*$ be any of the three products, $\square$, $\times$ or $\boxtimes$. A digraph Γ is *prime* with respect to $*$ if it can not be factored non-trivially with this product. Two digraphs are said to be *relatively prime* with respect to a certain product, if they have no common factor with respect to that product.

Remark. A digraph Γ is prime with respect to the Cartesian product (resp. the strong product) if and only if $\Gamma = \Gamma_1 \square \Gamma_2$ (resp. $\Gamma = \Gamma_1 \boxtimes \Gamma_2$) implies that either Γ_1 or Γ_2 is trivial. For the direct product, this holds if we redefine the trivial graph to consist of one vertex and a loop on that vertex.

It is not hard to see that these three graph products are all associative. Therefore we can extend the definitions to products of n digraphs, $\Gamma_1 * \cdots * \Gamma_n$.

Example 6.1.3. Let Γ_1 be the infinite, regular directed tree with in-valency one and out-valency two and $\Gamma_2 := \widetilde{\mathbb{Z}}$. Parts of the digraphs $\Gamma_1 \square \Gamma_2$ and $\Gamma_1 \boxtimes \Gamma_2$ are shown below, all edges directed downwards and from left to right.

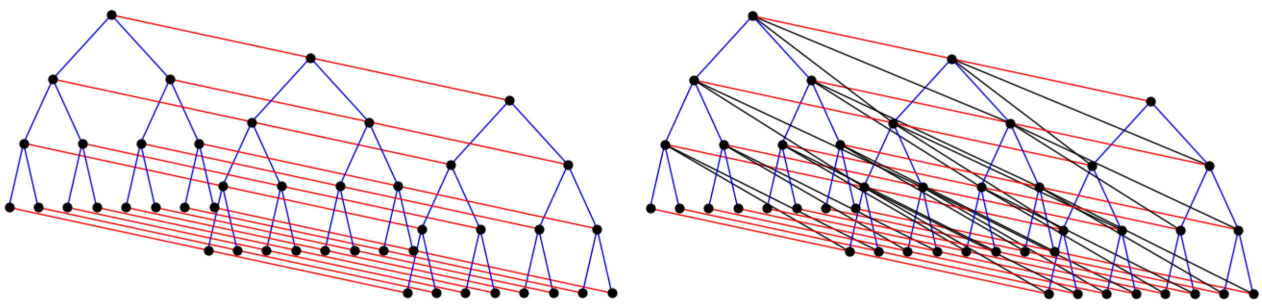

Figure 6: $\Gamma_1 \square \Gamma_2$ to the left and $\Gamma_1 \boxtimes \Gamma_2$ to the right

We exclude the direct product, because it is not very interesting. It is not connected; it is isomorphic to a disjoint union of countably many copies of Γ_1.

The digraph $\Gamma_1 \square \Gamma_2$ has two orbits on edges, namely

$$\Delta := \{((\alpha, i), (\beta, i)) : (\alpha, \beta) \in E(\Gamma_1), i \in \mathbb{Z}\} \quad \text{shown in blue, and}$$
$$\Delta' := \{((\alpha, i), (\alpha, i+1)) : \alpha \in V(\Gamma_1), i \in \mathbb{Z}\} \quad \text{shown in red.}$$

We note that $\Delta \cup \Delta' = \Delta_1$ from the definition. Here, the suborbit function gives us a graph homomorphism onto $\mathrm{Cay}(\mathbb{Z}, (0, 1))$, the naturally directed graph on $\mathbb{Z}$ that also has a loop on every vertex. However we can easily see that there exists a surjective graph homomorphism onto $\widetilde{\mathbb{Z}}^2$, because each subgraph, $V(\Gamma_1) \times \{i\}$, maps homomorphically onto $\widetilde{\mathbb{Z}}$. This is shown in Figure 7.

Of course, here (β_j, i) maps to (β, i) and (γ_j, i) maps to (γ, i) for $i \in \{0, 1\}$. It is also clear that every fiber of this homomorphism is infinite.

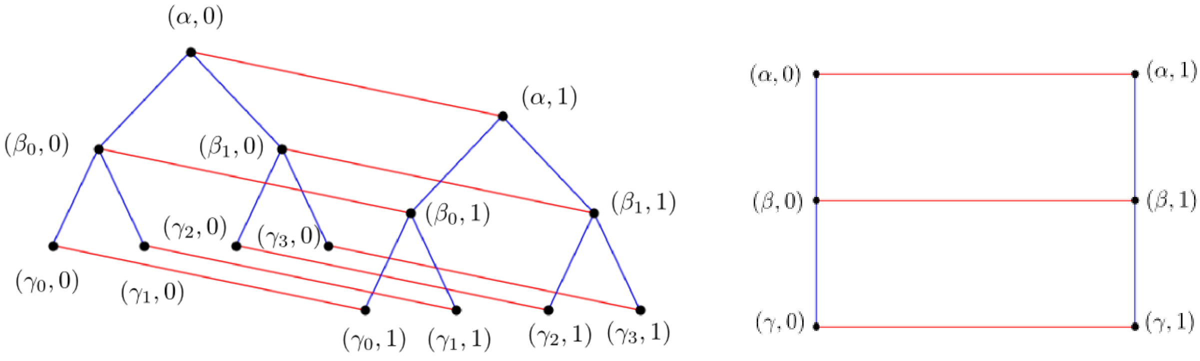

Figure 7: To the left we have $\Gamma_1 \square \Gamma_2$ and to the right its homomorphic image, $\widetilde{\mathbb{Z}}^2$.

The digraph $\Gamma_1 \boxtimes \Gamma_2$ has one more orbit on the edges (three in total), namely Δ_2 from the definition. These edges are shown in black in Figure 6.

6.2 Factors of Cartesian products

The rest of this chapter focuses on Cartesian products only. When we talk about factors, prime digraphs and relatively prime digraphs, we always mean with respect to Cartesian products.

We start by observing a few properties that can be passed on from the factors of a digraph Γ to the digraph itself. The following proposition is proved for finite undirected graphs in [8, Corollary 5.3]

Proposition 6.2.1. *A Cartesian product of n digraphs is connected if and only if all of its factors are.*

Proof. We prove the proposition for a Cartesian product of two digraphs, the rest is clear by induction. Let $\Gamma = \Gamma_1 \square \Gamma_2$ and suppose Γ is connected. Let $\alpha, \beta \in \Gamma_1$ and $\alpha', \beta' \in \Gamma_2$ be arbitrary vertices. Since Γ is connected there exists a walk,

$$(\alpha, \alpha') = (\alpha_0, \alpha'_0), \ldots, (\alpha_k, \alpha'_k) = (\beta, \beta')$$

from (α, α') to (β, β'). Then for $i = 1, \ldots, k-1$, either $(\alpha_i, \alpha_{i+1}) \in E(\Gamma_1)$ or $\alpha_i = \alpha_{i+1}$ so we get an induced walk in Γ_1 from α to β. In the same way we get an induced walk from α' to β' in Γ_2, and so both digraphs are connected.

Conversely, suppose Γ_1 and Γ_2 are connected and (α, α') and (β, β') are arbitrary vertices of Γ. Then we have walks, $\alpha = \alpha_0, \alpha_1, \ldots, \alpha_k = \beta$ and $\alpha' = \alpha'_0, \alpha'_1, \ldots, \alpha'_{k'} = \beta'$ in Γ_1 and Γ_2, respectively, and we use them to construct a walk from (α, α') to (β, β'), namely

$$(\alpha_0, \alpha'_0), (\alpha_1, \alpha'_0), \ldots, (\alpha_k, \alpha'_0), (\alpha_k, \alpha'_1), \ldots, (\alpha_k, \alpha'_{k'}),$$

so Γ is connected. $\square$

Proposition 6.2.2. *Let $\Gamma_1, \dots, \Gamma_n$ be infinite, regular digraphs with v_i and u_i the in-valency and out-valency of Γ_i, respectively. Then the Cartesian product, $\Gamma_1 \square \cdots \square \Gamma_n$ has in-valency $v_1 + \cdots + v_n$ and out-valency $u_1 + \cdots + u_n$.*

Proof. Let $\alpha = (\alpha_1, \dots, \alpha_n) \in V(\Gamma_1 \square \cdots \square \Gamma_n)$. The set of edges with α as an initial vertex is

$$\bigcup_{i=1}^{n} \big\{ \big(\alpha, (\alpha_1, \dots, \alpha_{i-1}, \beta, \alpha_{i+1}, \dots, \alpha_n)\big) : (\alpha_i, \beta) \in E(\Gamma_i) \big\},$$

and the cardinality of this set is clearly

$$\sum_{i=1}^{n} |\{\beta : (\alpha_i, \beta) \in E(\Gamma_i)\}| = \sum_{i=1}^{n} u_i.$$

Similarly we get that the total in-valency is $v_1 + \cdots + v_n$. $\square$

Definition 6.2.3. Let $\Gamma = \Gamma_1 \square \cdots \square \Gamma_n$ and fix a vertex $\alpha = (\alpha_1, \dots, \alpha_n)$ of Γ. We define the Γ_i*-layer trough* α as the induced subgraph of Γ

$$\Gamma_i^{\alpha} = \{\alpha_1\} \square \cdots \square \{\alpha_{i-1}\} \square \Gamma_i \square \{\alpha_{i+1}\} \square \cdots \square \{\alpha_n\}$$

where $\{\alpha_j\}$ is the trivial graph $(\{\alpha_j\}, \emptyset)$.

We note that for any vertex, $\alpha \in \Gamma$ we have $\Gamma_i^{\alpha} \simeq \Gamma_i$ because the map

$$\Gamma_i^{\alpha} \to \Gamma_i, \qquad (\alpha_1, \dots, \alpha_{i-1}, \beta, \alpha_{i+1}, \dots, \alpha_n) \mapsto \beta$$

is clearly a graph isomorphism. It is also easy to see that if two digraphs are isomorphic, their automorphism groups are isomorphic as well. Therefore, for every automorphism of Γ_i we have a corresponding automorphism on Γ_i^{α}, that translates to an automorphism on Γ fixing every vertex of every factor except for Γ_i. Now let $G_i := \mathrm{Aut}(\Gamma_i)$ for $i = 1, \dots, n$ and let $g = (g_1, \dots, g_n) \in G_1 \times \cdots \times G_n$. Then we get a corresponding automorphism on Γ, defined by $(\alpha_1, \dots, \alpha_n)^g := (\alpha_1^{g_1}, \dots, \alpha_n^{g_n})$, so we have $G_1 \times \cdots \times G_n \leq \mathrm{Aut}(\Gamma)$.

The following proposition is proved for finite undirected graphs in [8, Proposition 6.16]

Proposition 6.2.4. *A Cartesian product of connected, vertex transitive digraphs is vertex transitive.*

Proof. This is clear, simply because if $\Gamma = \Gamma_1 \square \cdots \square \Gamma_n$, then

$$\mathrm{Aut}(\Gamma_1) \times \cdots \times \mathrm{Aut}(\Gamma_n) \leq \mathrm{Aut}(\Gamma).$$

$\square$

Edge-transitivity, however, can not be passed to a digraph from its factors, except in the following special case.

We denote the n-th power of a digraph Γ, with respect to a Cartesian product, by $\Gamma^{\Box,n}$, for $n \in \mathbb{N}^*$.

Proposition 6.2.5. *Let Γ be an infinite, connected, digraph. If Γ is edge transitive then $\Gamma^{\Box,n}$ is edge transitive for any $n \in \mathbb{N}^*$.*

Proof. Let $H := \mathrm{Aut}(\Gamma)$ and $G := \mathrm{Aut}(\Gamma^{\Box,n})$. We have seen that

$$H^n = H \times H \times \cdots \times H \leq G.$$

Also, any map that permutes the coordinates of $V(\Gamma^{\Box,n})$ is clearly an automorphism on $\Gamma^{\Box,n}$.

Let $\alpha = (\alpha_1, \ldots, \alpha_n), \beta = (\beta_1, \ldots, \beta_n), \gamma = (\gamma_1, \ldots, \gamma_n), \delta = (\delta_1, \ldots, \delta_n) \in V(\Gamma^{\Box,n})$ such that (α, β) and (γ, δ) are edges of $\Gamma^{\Box,n}$, and let $k_1, k_2 \in \{1, \ldots, n\}$ be such that $\alpha_{k_1} \neq \beta_{k_1}$ and $\gamma_{k_2} \neq \delta_{k_2}$. Then $(\alpha_{k_1}, \beta_{k_1})$ and $(\gamma_{k_2}, \delta_{k_2})$ are both edges of Γ and since Γ is edge transitive, there exists an element, $h_{k_2} \in H$ taking $(\alpha_{k_1}, \beta_{k_1})$ to $(\gamma_{k_2}, \delta_{k_2})$. Let $h_{k_1} \in H$ such that $\alpha_{k_2}^{h_{k_1}} = \gamma_{k_1}$ and for $i \in \{1, \ldots, n\} \backslash \{k_1, k_2\}$ let $h_i \in H$ such that $\alpha_i^{h_i} = \gamma_i$. Then we have $(\alpha, \beta)^{gh} = (\gamma, \delta)$ where $h = (h_1, \ldots, h_n) \in H^n$ and g is the automorphism that interchanges the k_1-th and the k_2-th coordinates. $\Box$

Example 6.2.6. Proposition 6.2.5 yields a collection of examples of edge transitive digraphs. Let Γ be the infinite, regular, directed tree with finite in-valency v and out-valency u. Then $\Gamma^{\Box,n}$ is an infinite, connected, locally finite, vertex- and edge transitive digraph with in-valency nv and out-valency nu. If $v \neq u$, Theorem 5.1.6 gives a surjective graph homomorphism from $\Gamma^{\Box,n}$ to $\widetilde{\mathbb{Z}}$, all of whose fibers are infinite. Figure 8 shows part of $\Gamma \Box \Gamma$ with $v = 1$ and $u = 2$, edges directed downwards and from left to right.

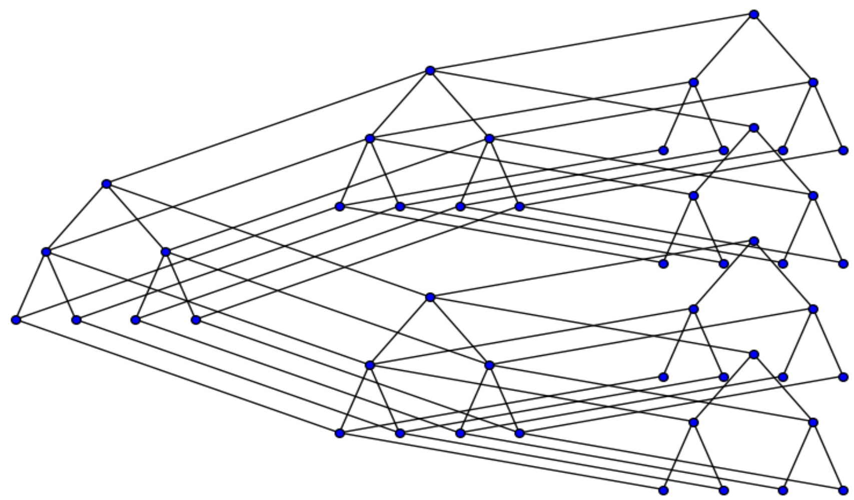

Figure 8: Edge transitive non-tree

6.3 Products of prime digraphs

We consider in particular Cartesian products of prime, pairwise non-isomorphic digraphs. In this case we can completely determine the automorphism group of the product by the automorphism groups of the factors, in fact we have

$$\mathrm{Aut}(\Gamma_1 \Box \cdots \Box \Gamma_n) = \mathrm{Aut}(\Gamma_1) \times \cdots \times \mathrm{Aut}(\Gamma_n).$$

To prove this though, we first need some more definitions.

Definition 6.3.1. Let $\Gamma = \Gamma_1 \Box \cdots \Box \Gamma_n$. A subgraph Δ of Γ is called a *box* if it is of the form $\Delta = \Delta_1 \Box \cdots \Box \Delta_n$ where Δ_i is a subgraph of Γ_i for all $i \in \{1, \dots, n\}$.

Definition 6.3.2. A subgraph Δ of a digraph Γ is *convex* in Γ if every shortest (undirected) path in Γ between two vertices of Δ lies within Δ.

Hammack, Imrich and Klavzar proved the next three lemmas for finite, undirected graphs in [8, Lemmas 6.3, 6.4, 6.5]. The proofs do not depend on the finiteness, thus we can use them for infinite graphs as well. Also, since our digraphs are without multiple edges, we can "forget" the direction of the edges to obtain a corresponding undirected graph. For what comes after, we only need these results for the underlying undirected graphs of our digraphs. We will therefore omit the proofs here, except for a part of the last one that is left as an exercise in [8].

Lemma 6.3.3. (Unique Square Lemma) *Let e and f be two incident edges of a Cartesian product $\Gamma_1 \Box \Gamma_2$ that are in different layers, that is, one in a Γ_1-layer and the other one in a Γ_2-layer. Then there exists exactly one square in $\Gamma_1 \Box \Gamma_2$ containing e and f. This square has no diagonals.*

Remark. Notice that this lemma holds for n factors.

Definition 6.3.4. We say a subgraph Δ of a Cartesian product Γ has the *square property* if for any two adjacent edges e and f of Δ that are in different layers, the unique square of Γ that contains them is also contained in Δ.

Lemma 6.3.5. *A connected subgraph of a Cartesian product is a box if and only if it has the square property.*

Lemma 6.3.6. *A subgraph Δ of a digraph $\Gamma = \Gamma_1 \Box \cdots \Box \Gamma_n$ is convex if and only if $\Delta = \Delta_1 \Box \cdots \Box \Delta_n$ where Δ_i is a convex subgraph of Γ_i for all i.*

Proof. We will prove that if $\Delta = \Delta_1 \Box \Delta_2$ with Δ_i convex in Γ_i for $i = 1, 2$, then Δ is convex in $\Gamma_1 \Box \Gamma_2$. The rest of the implication is then clear by induction and the converse is proved in [8, Lemma 6.5].

Suppose $\Delta = \Delta_1 \Box \Delta_2$ where Δ_i is a convex subgraph of Γ_i. Let (α, α') and (β, β') be vertices of Δ and let $(\alpha, \alpha') = (\alpha_0, \alpha'_0), \dots, (\alpha_n, \alpha'_n) = (\beta, \beta')$ be a shortest path between them in $\Gamma_1 \Box \Gamma_2$. Then for every $j \in \{0, \ldots, n-1\}$ we have either $\alpha_j = \alpha_{j+1}$ or $(\alpha_j, \alpha_{j+1}) \in E(\Gamma_1)$. By deleting repeated vertices we get a path, $\alpha = \gamma_0, \ldots \gamma_k = \beta$ from α to β in Γ_1. But this has to be the shortest path between α and β because a shorter one would yield a shorter path between (α, α') and (β, β'). Since Δ_1 is convex, the vertices $\{\gamma_0, \ldots, \gamma_k\} = \{\alpha_0, \ldots, \alpha_n\}$ are all contained in Δ_1. Similarly, α'_j are all contained in Δ_2, so the path $(\alpha_0, \alpha'_0), \ldots, (\alpha_n, \alpha'_n)$ is contained in $\Delta_1 \Box \Delta_2$ thus it is convex. $\square$

Lemma 6.3.6 implies that every convex subgraph is a box. Also, every Γ_i-layer in a Cartesian product $\Gamma_1 \Box \cdots \Box \Gamma_n$ is convex, because obviously every graph is a convex subgraph of itself and every trivial graph is convex.

We can now prove the main result of this section. This theorem is proved for finite, undirected graphs in [8, Theorem 6.13]

Theorem 6.3.7. *Let $\Gamma_1, \ldots, \Gamma_n$ be infinite, connected, locally finite digraphs with automorphism groups $G_1, \ldots, G_n$ respectively and let $\Gamma = \Gamma_1 \Box \cdots \Box \Gamma_n$. Suppose furthermore that the digraphs Γ_i are all prime and pairwise non-isomorphic. Then*

$$G := \mathrm{Aut}(\Gamma) = G_1 \times \cdots \times G_n.$$

Proof. Let $g \in G$ be an automorphism of Γ, fix a vertex $\alpha = (\alpha_1, \cdots, \alpha_n)$ of Γ and set $\beta = (\beta_1, \ldots, \beta_n) := \alpha^g$. For $k \in \{1, \ldots, n\}$ we know that the subgraph Γ_k^α is convex, and it is easy to see that this implies that $(\Gamma_k^\alpha)^g$ is also convex (and therefore a box). Let

$$\Delta := (\Gamma_k^\alpha)^g = \Delta_1 \Box \cdots \Box \Delta_n$$

where Δ_i is an induced subgraph of Γ_i. Since g is an automorphism on Γ we have $\Delta \simeq \Gamma_k^\alpha \simeq \Gamma_k$, so Δ is prime because Γ_k is. Then Δ_i is trivial for all but one $i \in \{1, \ldots, n\}$, in fact, since $\beta \in \Delta$, we have

$$\Delta = \{\beta_1\} \Box \cdots \Box \{\beta_{j-1}\} \Box \Delta_j \Box \{\beta_{j+1}\} \Box \cdots \Box \{\beta_n\}$$

for some $j \in \{1, \ldots, n\}$. This means that Δ is a subgraph of Γ_j^β, but similarly, because Γ_j^β is prime, we get that $(\Gamma_j^\beta)^{g^{-1}}$ is a subgraph of Γ_k^α. Therefore $(\Gamma_k^\alpha)^g = \Gamma_j^\beta$ but the Γ_i are pairwise non-isomorphic, so we must have $k = j$. Now, we know that $\Gamma_k^\alpha \simeq \Gamma_k^\beta \simeq \Gamma_k$, so by restricting g to the subgraph Γ_k^α we can assign to it an automorphism $g_k \in G_k$ of Γ_k. Then we have for an arbitrary vertex, a in Γ_k^α:

$$a^g = (\alpha_1, \ldots, \alpha_{k-1}, \omega, \alpha_{k+1}, \ldots, \alpha_n)^g = (\beta_1, \ldots, \beta_{k-1}, \omega^{g_k}, \beta_{k+1}, \ldots, \beta_n) =: b.$$

We want to show that the assigned automorphism g_k is independent of the vertex α. Let $\gamma = (\gamma_1, \ldots, \gamma_n)$ be another vertex of Γ and consider the Γ_k-layer, Γ_k^γ. Suppose first that (α, γ) is an edge (that is not contained in Γ_k^α). Then $\alpha_i = \gamma_i$ for all $i \in \{1, \ldots, n\}$ but one, say $i = j \neq k$, and we have $(\alpha_j, \gamma_j) \in E(\Gamma_j)$. Set $\delta = (\delta_1, \ldots, \delta_n) := \gamma^g$ and let $c \in \Gamma_k^\gamma$ be the vertex that corresponds to a in Γ_k^α. Then we have

$$c^g = (\gamma_1, \ldots, \gamma_{k-1}, \omega, \gamma_{k+1}, \ldots, \gamma_n)^g = (\delta_1, \ldots, \delta_{k-1}, \omega^{g'_k}, \delta_{k+1}, \ldots, \delta_n) =: d.$$

Since (α, γ) is an edge in a Γ_j-layer, then so are (a, c) and $(a, c)^g = (b, d)$, and so we have $\omega^{g_k} = \omega^{g'_k}$. Because Γ is connected, we get the same for arbitrary vertices α and γ and so $g_k \in G_k$ does not depend on α. Doing this for all $k \in \{1, \ldots, n\}$ we get $g = (g_1, \ldots, g_n) \in G_1 \times \cdots \times G_n$. $\square$

6.4 Arc-types of graphs

Now that we have identified the automorphism groups of products of prime, non-isomorphic digraphs, we want to use it to better determine the structure of these products. In order to do this, we define the arc-type of a digraph. This definition is analogous to a definition of arc-types for undirected graphs appearing in an article by Marston Conder, Tomaz Pisanski and Arjana Zitnik from 2015 [5].

Definition 6.4.1. Let Γ be an infinite, connected, locally finite, vertex transitive digraph with n orbits on edges, $\Delta_1, \ldots, \Delta_n$. Let v_i and u_i be the in-valency and out-valency of $\Gamma_i := (V(\Gamma), \Delta_i)$, respectively, for $i = 1, \ldots, n$. It is clear that $v = v_1 + \cdots + v_n$ and $u = u_1 + \cdots + u_n$ are the in-valency and out-valency of Γ, respectively. We define the *arc-type* of Γ as the partition Π of $u + v$ with

$$\Pi = (v_1 + u_1) + \cdots + (v_n + u_n).$$

We note that given the arc-type of a digraph, we can describe the image of the suborbit function, which then allows us to construct graph homomorphisms onto a Cayley digraph of $\mathbb{Z}^k$ for some $k \leq n$.

Theorem 6.4.2. *Let $\Gamma_1, \ldots, \Gamma_n$ be prime, infinite, connected, locally finite digraphs that are pairwise non-isomorphic. For $i = 1, \ldots, n$, let G_i be the automorphism group of Γ_i and Π_i its arc-type and let $\Gamma = \Gamma_1 \Box \cdots \Box \Gamma_n$. Then Γ has arc-type $\Pi = \Pi_1 + \cdots + \Pi_n$.*

Proof. We have already established that the arc-type of any digraph can be described as a sum of numbers $\left(|\alpha^{G_\beta}| + |\beta^{G_\alpha}|\right)$ where (α, β) are edges from different orbits on edges and that the number of orbits on edges is the same as the number of different suborbits of the form $|\beta^{G_\alpha}|$ where (α, β) is an edge.

Since $G = G_1 \times \cdots \times G_n$ we also have $G_\alpha = (G_1)_{\alpha_1} \times \cdots \times (G_n)_{\alpha_n}$ for $\alpha = (\alpha_1, \ldots, \alpha_n) \in V(\Gamma)$ where $(G_i)_{\alpha_i}$ is the stabilizer of α_i in G_i. Therefore, we have for two vertices, $\beta = (\beta_1, \ldots, \beta_n)$ and $\gamma = (\gamma_1, \ldots, \gamma_n)$ of Γ:

$$\gamma \in \beta^{G_\alpha} = \beta_1^{(G_1)_{\alpha_1}} \times \cdots \times \beta_n^{(G_n)_{\alpha_n}} \Leftrightarrow \gamma_i \in \beta_i^{(G_i)_{\alpha_i}}, \text{ for all } i = 1, \ldots, n. \tag{6.1}$$

So if there are m_i orbits on edges in Γ_i, the number of orbits on edges in Γ is exactly $m_1 + \cdots + m_n$.

Now suppose $(\alpha, \beta) \in E(\Gamma)$. Then $\alpha_i = \beta_i$ for all i but one, say $i = k$ and we have $(\alpha_k, \beta_k) \in E(\Gamma_k)$. Suppose the arc-type of Γ_k is

$$\Pi_k = \left(v_1^{(k)} + u_1^{(k)}\right) + \cdots + \left(v_{m_k}^{(k)} + u_{m_k}^{(k)}\right).$$

Then we know that $|\alpha_k^{G_{\beta_k}}| = v_j^{(k)}$ and $|\beta_k^{G_{\alpha_k}}| = u_j^{(k)}$ for some $j \in \{1, \dots, m_k\}$. Furthermore, we have

$$\begin{aligned} |\beta^{G_\alpha}| &= |\beta_1^{(G_1)_{\alpha_1}} \times \cdots \times \beta_n^{(G_n)_{\alpha_n}}| \\ &= |\beta_1^{(G_1)_{\alpha_1}}| \cdots |\beta_n^{(G_n)_{\alpha_n}}| \\ &= |\{\beta_1\}| \cdots |\{\beta_{k-1}\}| \cdot u_j^{(k)} \cdot |\{\beta_{k+1}\}| \cdots |\{\beta_n\}| = u_j^{(k)} \end{aligned}$$

and in the same way we get $|\alpha^{G_\beta}| = v_j^{(k)}$. Now it is clear from Equation (6.1) that $\Pi = \Pi_1 + \cdots + \Pi_n$. □

Remark. Let $\Pi = (v_1 + u_1) + \cdots + (v_n + u_n)$. Theorem 6.4.2 implies that if we can find connected, edge transitive digraphs, $\Gamma_1, \dots, \Gamma_n$ with in-valencies $v_1, \dots, v_n$ and out-valencies $u_1, \dots, u_n$, that are prime and pairwise non-isomorphic, then we can construct a digraph with arc-type Π.

Corollary 6.4.3. *Let $\Gamma_1, \dots, \Gamma_n$ be infinite, connected, locally finite vertex transitive digraphs and let $\Gamma = \Gamma_1 \Box \cdots \Box \Gamma_n$. If Γ has k orbits on edges, and Γ_i has k_i orbits on edges of for $i = 1, \dots, n$ then $k \leq k_1 + \cdots + k_n$.*

Proof. By the proof of Theorem 6.4.2, the group $\mathrm{Aut}(\Gamma_1) \times \cdots \times \mathrm{Aut}(\Gamma_n)$ has exactly $k_1 + \cdots + k_n$ orbits on edges, and we know that it is contained in $\mathrm{Aut}(\Gamma)$. □

Example 6.4.4. Let Γ_1 be the infinite regular directed tree with in-valency 1 and out-valency 2, and let Γ_2 be the digraph from Example 5.2.3. The Cartesian product of these two digraphs is shown in Figure 9. Of course, it continues infinitely to the sides, and each subtree goes infinitely up and down.

Both Γ_1 and Γ_2 are edge transitive, so Corollary 6.4.3 implies that $\Gamma_1 \Box \Gamma_2$ has at most two orbits on edges. In fact it is easy to see that it can not be edge transitive and therefore has exactly two orbits (shown in blue and black).

To be able to use Theorem 6.4.2 to construct digraphs with a given arc-type, we have to start with prime digraphs. But how do we know whether a given digraph is prime, and are we familiar with any prime digraphs?

Lemma 6.4.5. *If Γ is a connected digraph that is not prime, then Γ contains an undirected 4-cycle.*

Proof. Let $\Gamma = \Gamma_1 \Box \Gamma_2$ and suppose both factors are non-trivial. Since Γ_1 and Γ_2 are connected and non-trivial they both contain at least one edge. Let (α_1, β_1) and (α_2, β_2) be edges of Γ_1 and Γ_2, respectively. Then $(\alpha_1, \alpha_2), (\alpha_1, \beta_2), (\beta_1, \beta_2), (\beta_1, \alpha_2)$ is a 4-cycle in Γ. □

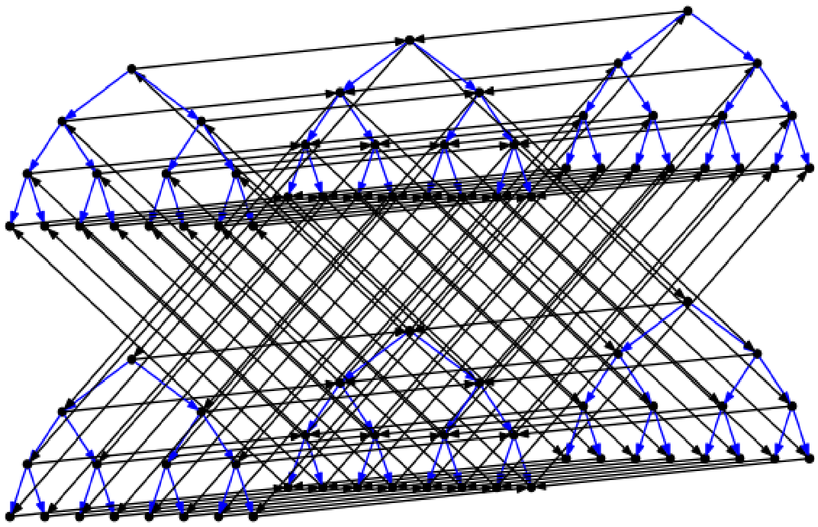

Figure 9: Cartesian product with two orbits on edges

Proposition 6.4.6. *Every directed tree is prime.*

Proof. This follows from Lemma 6.4.5 because directed trees have no cycles. □

Theorem 6.4.7. *Let* $\Pi = (v_1 + u_1) + \cdots + (v_n + u_n)$ *where for* $i, j \in \{1, \ldots, n\}$ *such that* $i \neq j$ *we have* $v_i \neq v_j$ *or* $u_i \neq u_j$*. Then there exists a digraph of arc-type* Π*.*

Proof. For $i = 1, \ldots, n$, let Γ_i be the infinite, regular directed tree with in-valency v_i and out-valency u_i. Then $\Gamma_1, \ldots, \Gamma_n$ are prime and pairwise non-isomorphic, so by Corollary 6.4.2, the Cartesian product $\Gamma_1 \square \cdots \square \Gamma_n$ has the given arc-type. □

Corollary 6.4.8. *Let* $n \in \mathbb{N}^*$*. Then there exists an infinite, connected, locally finite, vertex transitive digraph* Γ*, whose full automorphism group has* n *orbits on edges, such that there is an epimorphism from* Γ *to* $\widetilde{\mathbb{Z}}^n$ *with all fibers infinite.*

Proof. For $i = 1, \ldots, n$ let u_i be the i-th prime number and let Γ_i be the infinite, regular, directed tree with in-valency 1 and out-valency u_i. Then the Γ_i are prime and pairwise non-isomorphic, so the graph $\Gamma := \Gamma_1 \square \cdots \square \Gamma_n$ has arc-type

$$\Pi := (1 + u_1) + \cdots + (1 + u_n),$$

that is Γ is an infinite, connected, vertex transitive digraph with n orbits on edges. Because the u_i are all prime we also have $\mathrm{Im}(\psi) = \langle u_1, \ldots, u_n \rangle \simeq \mathbb{Z}^n$ so by Corollary 5.1.14 there exists a graph epimorphism $\varphi : \Gamma \to \widetilde{\mathbb{Z}}^n$ all of whose fibers are infinite. □

Example 6.4.9. We construct the digraph from Corollary 6.4.8 with $n = 2$. Part of this digraph is shown in Figure 10, the blue edges representing the orbit with out-valency 2 (edges directed from left to right) and the red edges representing the orbit with out-valency 3 (edges directed downwards).

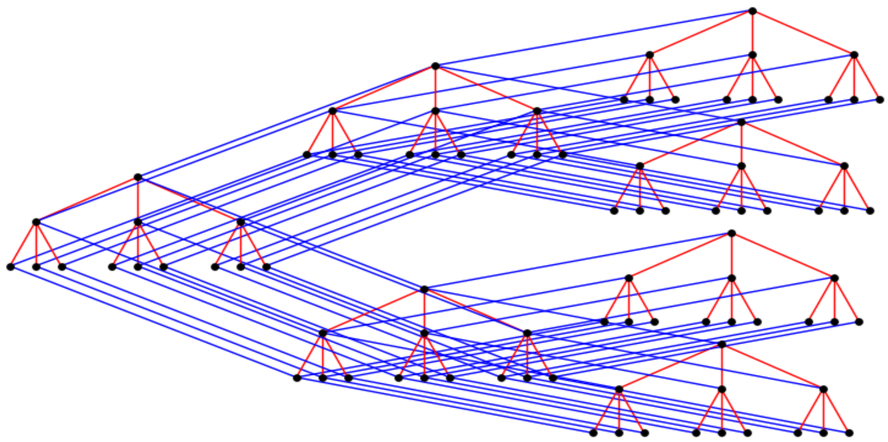

Figure 10: Cartesian product of two directed trees with arc-types $(1+2)$ *and* $(1+3)$

As we have shown, there exists a graph epimorphism from this graph onto $\widetilde{\mathbb{Z}}^2$, all of whose fibers are infinite.

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