University of Iceland **Faculty of Engineering**

An Optimized MIMO PID Controller

by

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UNIVERSITY OF ICELAND

A Thesis Presented in Partial Fulfillment of the
Requirements for the Degree
Master of Science in Electrical and Computer Engineering at the
University of Iceland

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Reykjavík
2010

Abstract

A method to optimize the zero locations for a PID controller for SISO systems to achive optimized tracking of a reference system has already been derived. In this thesis this method is expanded to work for MIMO systems. This is done by minimizing the difference between the impulse or the step response of the controlled system and the chosen reference system. The optimized zero locations can be found for the controller and the best tracking possible is a achived.

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Chapter 1

Introduction

PID (Proportional, Integral, Differential) controllers for single input single outputs (SISO) systems are the most common controllers in industry today. They are also a very interesting and popular research topic. Muliple input multiple output (MIMO) PID controllers are also very interesting and challenging but not as common in industry. When using MIMO PID controllers the number of coefficients that have to be determined grows fast with the number of inputs and outputs (I/O), and along with it the complexity and time needed to calculate them. Instead of calculating the coefficients from some mathematical model it is possible to tune the coefficients experimentally. Many tuning methods have been established for SISO PID controllers as well as a few methods for MIMO PID controllers, see e.g. [1]. Automatic tuning methods have also been developed to find MIMO PID controllers, see [2], [3] and [4]. MIMO PID controllers have furthermore been used to stabilize MIMO systems, see [5], [6] and [7]. Decoupling of MIMO systems has always been of great interest for industry and research, see [8], [9], [10] and [11]. Most decoupling methods provide decoupling, but no control of the system, in the sense of reference input tracking and disturbance rejection. Therefore, an outer loop is necessary to control the system with those methods, in fact similar to state feedback controllers.

Research on transfer function responses at the University of Iceland [12], [13], [14] and [15] for both continuous and discrete time, has lead to several research topics. The general problem on how to optimize zero locations, to get a system to track a reference system, is reported in [16], [17] and [18]. Optimized zeros locations are then applied in model reduction, see [19], [20] and [21]. That lead to an optimized PID controller to be derived, tracking a given open loop reference system resulting in a closed loop controlled system. The zeros locations of the PID controllers or generalized PID controllers with more than two zeros are optimized to get the best tracking of a reference system essentially containing the design requirements, see [22], [23], [24], [25], [26], [27] and [28]. The research has also led to a series of papers on Gramians, Lyapunov and Sylvester equations, see [29], [30] and [31].

In this thesis the optimized PID controller for a SISO system is expanded to an optimized MIMO PID controller. The relationship between inputs and outputs of a MIMO system is represented by a transfer function matrix (TFM) with many transfer functions contributing to one output. This makes the optimized MIMO PID controller more complex to find than the optimized PID controller for a SISO system. For a SISO system the optimization is done by choosing a reference system and minimizing the squared difference between the open loop impulse or step response of the system we want to control and the reference open loop system. For the MIMO system a reference system is chosen on a TFM form and the squared difference between the open loop impulse or step response of the element transfer functions in the system's TFM and the corresponding transfer functions in the reference system's TFM is again minimized.

It is shown in Chapter 2 how to find the optimized MIMO PID controller for a system that has the same number of control inputs and outputs, starting in Section 2.1 with a system with 3 control inputs and 3 outputs. The method from Section 2.1 is generalized in Section 2.2 for systems with p control inputs and p outputs. Then, in Chapter 3, it is shown how to find the optimized MIMO PID controller if the system does not have the same number of control inputs and outputs, beginning with a system having 4 control inputs and 3 outputs in Section 3.1. This method is then generalized for a systems with r control inputs and p outputs in Section 3.2. Finally, three examples in Chapter 4 show how effective the optimized MIMO PID controller is.

The MIMO PID controller in Chapters 2 and 3 can be optimized with respect to the impulse or the step response. A central task in finding an optimized PID controller for a SISO system is to find a controllability Gramian padded by system zeros on both sides. One approach to this is to derive a Lyapunov equation which includes this Gramian in its solution, which can then e.g be solved by using the Matlab's function `lyap`. The same turns out to be the case for a MIMO-system, and indeed one can make use of most of the basic results that have already been developed for SISO systems, even if the procedure becomes more complex.

Chapter 2

Square MIMO systems

We start by taking a look at the system in Figure 2.1, which we want to behave like the reference system in Figure 2.2, a multiple input multiple output (MIMO) system, assumed to have the same numbers of control inputs as outputs. Then the transfer function matrices (TFM) $C(s)$ and $G(s)$ are square matrices.

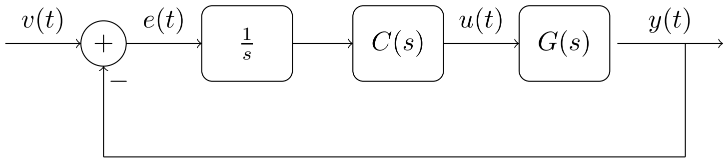

Figure 2.1: A closed loop MIMO system with a MIMO PID controller

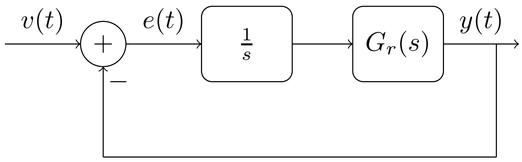

Figure 2.2: A MIMO reference system

2.1 MIMO systems with three inputs and three outputs

If the system $G(s)$ in Figure 2.1 has three inputs and three outputs (I/O), then it has the following transfer function matrix (TFM),

$$G(s) = \begin{bmatrix} G_{11}(s) & G_{12}(s) & G_{13}(s) \\ G_{21}(s) & G_{22}(s) & G_{23}(s) \\ G_{31}(s) & G_{32}(s) & G_{33}(s) \end{bmatrix} \tag{2.1}$$

The system $G(s)$ is assumed to be minimal, i.e. controllable and observable. The transfer functions $G_{ij}(s)$ in the TFM can be written as

$$G_{ij}(s) = \frac{b_{ij}(s)}{a(s)} = \frac{b_{m_{ij},ij}s^{m_{ij}} + b_{m_{ij}-1,ij}s^{m_{ij}-1} + \cdots + b_{1,ij}s + b_{0,ij}}{s^n + a_{n-1}s^{n-1} + \cdots + a_0}, \quad (2.2)$$

for $i = 1, 2, 3$ and $j = 1, 2, 3$. These transfer functions must satisfy $m_{ij} + 2 < n$, since the MIMO PID controller will add two zeros to each element transfer function and the TFM for the open loop controlled system must be causal.

The TFM for the reference system $G_r(s)$ in Figure 2.2 is taken to be of the diagonal form

$$G_r(s) = \begin{bmatrix} G_{r1}(s) & 0 & 0 \\ 0 & G_{r2}(s) & 0 \\ 0 & 0 & G_{r3}(s) \end{bmatrix}, \quad (2.3)$$

with the aim of making the controlled system as decoupled as possible. Choosing the open loop reference system on a nondiagonal form causes problems since the closed loop system is given by $G_r^{CL}(s) = (sI + G_r(s))^{-1}G_r(s)$ and it will then in general not be on a diagonal from. The transfer functions for the reference system are given by

$$G_{ri}(s) = \frac{b_{ri}(s)}{a_{ri}(s)} = \frac{b_{m_{ri},ri}s^{m_{ri}} + b_{m_{ri}-1,ri}s^{m_{ri}-1} + \cdots + b_{1,ri}s + b_{0,ri}}{s^{n_{ri}} + a_{n_{ri}-1,ri}s^{n_{ri}-1} + \cdots + a_{0,ri}}. \quad (2.4)$$

Since we assume the transfer functions for the reference systems to be causal they have to satisfy $m_{ri} = n_{ri} - 1$, if $m_{ri} < n_{ri} - 1$ the appropriate b_{ri} coefficients must be set to zero. For example selecting $G_{ri}(s)$ as the simple first order system of relative degree one

$$G_{ri}(s) = \frac{\omega_{ri}^2}{s + 2\zeta_{ri}\omega_{ri}} \quad (2.5)$$

the closed loop reference systems will all have transfer functions in the standard second order form

$$G_{ri}^{CL}(s) = \frac{\omega_{ri}^2}{s^2 + 2\zeta_{ri}\omega_{ri}s + \omega_{ri}^2}. \quad (2.6)$$

The PID (Proportional, Integral, Differential) MIMO controller $\frac{1}{s}C(s)$ in Figure 2.1 has the transfer function matrix

$$\frac{1}{s}C(s) = \frac{1}{s}\begin{bmatrix} c_{11}(s) & c_{12}(s) & c_{13}(s) \\ c_{21}(s) & c_{22}(s) & c_{23}(s) \\ c_{31}(s) & c_{32}(s) & c_{33}(s) \end{bmatrix} \quad (2.7)$$

where

$$c_{ij}(s) = K_{Dij}s^2 + K_{Pij}s + K_{Iij}. \quad (2.8)$$

We intend to derive the optimized MIMO PID controller first by minimizing the integral of the squared impulse response deviation between the system and the reference system. Then by minimizing the integral of the squared step response deviation between the system and the reference system. The impulse method does not take the DC gain into consideration unlike the step response method. The zeros in the impulse optimized PID controller are optimal but the PID controller does not have the correct DC gain. In the step response method, the optimized MIMO PID controller has both the zeros and the resulting open loop DC gain optimized. Since the step response method is based on the impulse response method, it is, however, useful to begin with a thorough coverage of impulse response optimization.

2.1.1 Impulse response

It is shown in [28] that the impulse response for $C(s)G(s) = \frac{c(s)b(s)}{a(s)}$ is given by

$$y_I(t) = \mathscr{L}^{-1}\left\{\frac{c(s)b(s)}{a(s)}\right\} = \mathcal{C}^T\mathcal{B}^TY_b(t) \tag{2.9}$$

where $\mathcal{C}$ is a vector with the PID coefficients, $\mathcal{C}^T = \begin{bmatrix} K_I & K_P & K_D \end{bmatrix}$and $\mathcal{B}$ is a convolution matrix with coefficients from $b(s)$, and $Y_b(t)$ is the vector

$$Y_b(t) = \begin{bmatrix} y_b(t) & y_b^{'}(t) & \cdots & y_b^{(n-1)}(t) \end{bmatrix}^T \tag{2.10}$$

where $y_b(t)$ is the basic impulse response for $\frac{1}{a(s)} = \frac{1}{s^n + a_{n-1}s^{n-1} + \cdots a_0}$, i.e. $y_b(t) = \mathscr{L}^{-1}\left\{\frac{1}{a(s)}\right\}$. For a minimal MIMO system, all transfer functions have the same denominator $a(s)$ in the TFM, so the basic impulse response $y_b(t)$ is the same for all transfer functions, and the vector $Y_b(t)$ is also the same for all elements in the TFM. For the three input/output (I/O) system we have nine $\mathcal{C}_{ij}$ vectors defined as

$$\mathcal{C}_{ij}^T = \begin{bmatrix} K_{Iij} & K_{Pij} & K_{Dij} \end{bmatrix}. \tag{2.11}$$

For a transfer function $G_{ij}(s)$ with $m_{ij} + 3 = n$, the convolution matrix $\mathcal{B}_{ij}$ becomes

$$\mathcal{B}_{ij} = \begin{bmatrix} b_{0,ij} & 0 & 0 \\ b_{1,ij} & b_{0,ij} & 0 \\ \vdots & b_{1,ij} & b_{0,ij} \\ b_{m_{ij},ij} & \vdots & b_{1,ij} \\ 0 & b_{m_{ij},ij} & \vdots \\ 0 & 0 & b_{m_{ij},ij} \end{bmatrix}_{n\times 3} \tag{2.12}$$

$\mathcal{B}_{ij}$ is always of size $n \times 3$ even if $m_{ij} + 3 < n$. If $m_{ij} + 3 < n$ then the corresponding b coefficients must be set to zero to fill in the matrix. All transfer functions $G_{ij}(s)$ have to be strictly stable, i.e. all eigenvalues strictly in the LHP. Thus in particular $a_0 \neq 0$.

We are interested in the impulse response for every transfer function in the

TFM $G(s)C(s)$, where $y_{I,ij}(t)$ denotes the impulse response for the element transfer function ij in the TFM $G(s)C(s)$. Using Equation (2.9) and the linear properties of the Laplace transform, all impulse responses can be found. Before writing the impulse response, we introduce the 9×1 vector

$$\mathcal{C}_{\cdot i} = \begin{bmatrix} \mathcal{C}_{1i} \\ \mathcal{C}_{2i} \\ \mathcal{C}_{3i} \end{bmatrix} \tag{2.13}$$

and the $n \times 9$ matrix

$$\mathcal{B}_{j\cdot} = \begin{bmatrix} \mathcal{B}_{j1} & \mathcal{B}_{j2} & \mathcal{B}_{j3} \end{bmatrix} \tag{2.14}$$

Using this notation it is possible to write the $1 \times n$ vector $(\mathcal{B}_{j\cdot}\mathcal{C}_{\cdot i})^T = \mathcal{C}_{\cdot i}^T\mathcal{B}_{j\cdot}^T$ as

$$\mathcal{C}_{\cdot i}^T\mathcal{B}_{j\cdot}^T = \mathcal{C}_{1i}^T\mathcal{B}_{j1}^T + \mathcal{C}_{2i}^T\mathcal{B}_{j2}^T + \mathcal{C}_{3i}^T\mathcal{B}_{j3}^T \tag{2.15}$$

Then, the impulse responses for all transfer functions of the open loop system are given by

$$\begin{aligned}
y_{I11}(t) &= \mathscr{L}^{-1}\left\{ \frac{1}{a(s)} \begin{bmatrix} b_{11}(s) & b_{12}(s) & b_{13}(s) \end{bmatrix} \begin{bmatrix} c_{11}(s) \\ c_{21}(s) \\ c_{31}(s) \end{bmatrix} \right\} \\
&= \mathscr{L}^{-1}\left\{ \frac{b_{11}(s)c_{11}(s) + b_{12}(s)c_{21}(s) + b_{13}(s)c_{31}(s)}{a(s)} \right\} \\
&= \left[\mathcal{C}_{11}^T\mathcal{B}_{11}^T + \mathcal{C}_{21}^T\mathcal{B}_{12}^T + \mathcal{C}_{31}^T\mathcal{B}_{13}^T\right] Y_b(t) = \mathcal{C}_{\cdot 1}^T\mathcal{B}_{1\cdot}^T Y_b(t) \\
y_{I12}(t) &= \mathcal{C}_{\cdot 2}^T\mathcal{B}_{1\cdot}^T Y_b(t) \\
y_{I13}(t) &= \mathcal{C}_{\cdot 3}^T\mathcal{B}_{1\cdot}^T Y_b(t) \\
y_{I21}(t) &= \mathcal{C}_{\cdot 1}^T\mathcal{B}_{2\cdot}^T Y_b(t) \\
y_{I22}(t) &= \mathcal{C}_{\cdot 2}^T\mathcal{B}_{2\cdot}^T Y_b(t) \\
y_{I23}(t) &= \mathcal{C}_{\cdot 3}^T\mathcal{B}_{2\cdot}^T Y_b(t) \\
y_{I31}(t) &= \mathcal{C}_{\cdot 1}^T\mathcal{B}_{3\cdot}^T Y_b(t) \\
y_{I32}(t) &= \mathcal{C}_{\cdot 2}^T\mathcal{B}_{3\cdot}^T Y_b(t) \\
y_{I33}(t) &= \mathcal{C}_{\cdot 3}^T\mathcal{B}_{3\cdot}^T Y_b(t).
\end{aligned} \tag{2.16}$$

For a SISO system the impulse response for the reference system is written as $y_{rI}(t) = \mathcal{B}_r^T Y_{rb}(t)$. Here $\mathcal{B}_r$ is a vector with the b coefficients, and $Y_{rb}(t)$ is a vector with the basic impulse response for the reference system and its derivatives. For our MIMO system the reference system's impulse response is written by

$$\begin{aligned}
y_{Ir1}(t) &= \mathcal{B}_{r1}^T Y_{b,r1}(t) \\
y_{Ir2}(t) &= \mathcal{B}_{r2}^T Y_{b,r2}(t) \\
y_{Ir3}(t) &= \mathcal{B}_{r3}^T Y_{b,r3}(t).
\end{aligned} \tag{2.17}$$

We assume there can be three different basic impulse responses $y_{b,ri}(t)$ for the reference system and thus three different $Y_{b,ri}(t)$ vectors. The basic impulse responses can all be different, as we might not want the outputs all to behave in the same way. We can then have three different transfer functions in the TFM $G_r(s)$. The vectors $Y_{b,ri}(t)$ are defined as

$$\begin{aligned} Y_{b,r1}(t) &= \begin{bmatrix} y_{b,r1}(t) & y'_{b,r1}(t) & \cdots & y_{b,r1}^{n_{r1}-1}(t) \end{bmatrix}^T \\ Y_{b,r2}(t) &= \begin{bmatrix} y_{b,r2}(t) & y'_{b,r2}(t) & \cdots & y_{b,r2}^{n_{r2}-1}(t) \end{bmatrix}^T \\ Y_{b,r3}(t) &= \begin{bmatrix} y_{b,r3}(t) & y'_{b,r3}(t) & \cdots & y_{b,r3}^{n_{r3}-1}(t) \end{bmatrix}^T . \end{aligned} \tag{2.18}$$

The $\mathcal{B}_{ri}^T$ vectors are then defined as

$$\begin{aligned} \mathcal{B}_{r1}^T &= \begin{bmatrix} b_{0,r1} & \cdots & b_{n_{r1}-1,r1} \end{bmatrix} \\ \mathcal{B}_{r2}^T &= \begin{bmatrix} b_{0,r2} & \cdots & b_{n_{r2}-1,r2} \end{bmatrix} \\ \mathcal{B}_{r3}^T &= \begin{bmatrix} b_{0,r3} & \cdots & b_{n_{r3}-1,r3} \end{bmatrix} . \end{aligned} \tag{2.19}$$

2.1.2 Impulse response cost function

The desired MIMO PID controller is the controller that gives us practically the same impulse response for the controlled system and the reference system. The MIMO PID controller can be found by minimizing the difference of the impulse response for the controlled system $G(s)C(s)$ and the reference system $G_r(s)$. We do not need to include the integral $\left(\frac{1}{s}\right)$ since we assume it is in both systems. This is done by setting up a cost function for the integral of the squared difference of the impulse response for all transfer functions in the transfer function matrix $G(s)C(s)$ and the corresponding transfer function $G_{ri}(s)$ in the reference system. Then the cost function is minimized. The cost function is given by

$$\begin{aligned} \mathcal{J}_I = \textstyle\int_0^\infty ((y_{I11} - y_{Ir1})^2 + \omega_{12}(y_{I12} - 0)^2 + \omega_{13}(y_{I13} - 0)^2 \\ \omega_{21}(y_{I21} - 0)^2 + (y_{I22} - y_{Ir2})^2 + \omega_{23}(y_{I23} - 0)^2 \\ \omega_{31}(y_{I31} - 0)^2 + \omega_{32}(y_{I32} - 0)^2 + (y_{I33} - y_{Ir3})^2)dt. \end{aligned} \tag{2.20}$$

The constants ω_{ij} are the decoupling weight coefficients. By increasing the value of ω_{ij}, the closed loop system is made more decoupled. No method existes to determine the value for the decoupling weight coefficients, other then practise and experience. We define a few matrices to simplify the notation:

$$\mathcal{A}_{n\times n} = \int_0^\infty Y_b(t) Y_b^T(t) dt, \tag{2.21}$$

$$\mathcal{G}_{i(9\times 9)} = \begin{bmatrix} \mathcal{B}_{i1}^T \\ \mathcal{B}_{i2}^T \\ \mathcal{B}_{i3}^T \end{bmatrix} \mathcal{A} \begin{bmatrix} \mathcal{B}_{i1} & \mathcal{B}_{i2} & \mathcal{B}_{i3} \end{bmatrix} = \mathcal{B}_{i\cdot}^T \mathcal{A} \mathcal{B}_{i\cdot}, \tag{2.22}$$

$$\mathcal{H}_{i(n\times n_{ri})} = \int_0^\infty Y_b(t) Y_{b,ri}^T(t) dt, \tag{2.23}$$

$$\mathcal{D}_{i(9\times 1)} = \mathcal{B}_{i\cdot}^T \mathcal{H}_i \mathcal{B}_{ri}, \tag{2.24}$$

$$\mathcal{A}_{ri(n_{ri}\times n_{ri})} = \int_0^\infty Y_{b,ri}(t) Y_{b,ri}^T(t) dt, \tag{2.25}$$

and

$$\mathcal{M}_{i(1\times 1)} = \mathcal{B}_{ri}^T \mathcal{A}_{ri} \mathcal{B}_{ri}. \tag{2.26}$$

Then $\int_0^\infty (y_{I11} - y_{I,r1})^2 dt$ can be written as

$$\begin{aligned} \int_0^\infty (y_{I11} - y_{Ir1})^2 dt &= \int_0^\infty \left(\left(\mathcal{C}_{\cdot 1}^T \mathcal{B}_{1\cdot}^T Y_b(t) \right)^2 - 2\mathcal{C}_{\cdot 1}^T \mathcal{B}_{1\cdot}^T Y_b(t) Y_{b,r1}^T(t) \mathcal{B}_{r1} + \left(\mathcal{B}_{r1}^T Y_{b,r1} \right)^2 \right) dt \\ &= \mathcal{C}_{\cdot 1}^T \mathcal{G}_1 \mathcal{C}_{\cdot 1} - 2\mathcal{C}_{\cdot 1}^T \mathcal{D}_1 + \mathcal{M}_1 \end{aligned} \tag{2.27}$$

and the rest of the cost function can similarly be written as

$$\begin{aligned} \int_0^\infty \omega_{12}(y_{I12} - 0)^2 dt &= \omega_{12} \mathcal{C}_{\cdot 2}^T \mathcal{G}_1 \mathcal{C}_{\cdot 2} \\ \int_0^\infty \omega_{13}(y_{I13} - 0)^2 dt &= \omega_{13} \mathcal{C}_{\cdot 3}^T \mathcal{G}_1 \mathcal{C}_{\cdot 3} \\ \int_0^\infty \omega_{21}(y_{I21} - 0)^2 dt &= \omega_{21} \mathcal{C}_{\cdot 1}^T \mathcal{G}_2 \mathcal{C}_{\cdot 1} \\ \int_0^\infty (y_{I22} - y_{Ir2})^2 dt &= \mathcal{C}_{\cdot 2}^T \mathcal{G}_2 \mathcal{C}_{\cdot 2} - 2\mathcal{C}_{\cdot 2}^T \mathcal{D}_2 + \mathcal{M}_2 \\ \int_0^\infty \omega_{23}(y_{I23} - 0)^2 dt &= \omega_{23} \mathcal{C}_{\cdot 3}^T \mathcal{G}_2 \mathcal{C}_{\cdot 3} \\ \int_0^\infty \omega_{31}(y_{I31} - 0)^2 dt &= \omega_{31} \mathcal{C}_{\cdot 1}^T \mathcal{G}_3 \mathcal{C}_{\cdot 1} \\ \int_0^\infty \omega_{32}(y_{I32} - 0)^2 dt &= \omega_{32} \mathcal{C}_{\cdot 2}^T \mathcal{G}_3 \mathcal{C}_{\cdot 2} \\ \int_0^\infty (y_{I33} - y_{Ir3})^2 dt &= \mathcal{C}_{\cdot 3}^T \mathcal{G}_3 \mathcal{C}_{\cdot 3} - 2\mathcal{C}_{\cdot 3}^T \mathcal{D}_3 + \mathcal{M}_3. \end{aligned} \tag{2.28}$$

Note that we are still using the vectors $\mathcal{C}_{\cdot i}^T$ and $\mathcal{B}_{i\cdot}^T$ to simplify the notation.

2.1.3 Impulse response minimization

In order to minimize the impulse cost function $\mathcal{J}_I$ all partial derivatives with respect to $\mathcal{C}_{\cdot i}$, $i = 1, 2, 3$ are set equal to zero, i.e

$$\frac{\partial \mathcal{J}_I}{\partial \mathcal{C}_{\cdot i}} = 0, \quad i = 1, 2, 3 \tag{2.29}$$

Writing them out gives

$$\begin{aligned}
\frac{\partial \mathcal{J}_I}{\partial \mathcal{C}_{\cdot 1}} &= 2\mathcal{G}_1\mathcal{C}_{\cdot 1} - 2\mathcal{D}_1 + 2\omega_{21}\mathcal{G}_2\mathcal{C}_{\cdot 1} + 2\omega_{31}\mathcal{G}_3\mathcal{C}_{\cdot 1} = 0 \\
\frac{\partial \mathcal{J}_I}{\partial \mathcal{C}_{\cdot 2}} &= 2\omega_{12}\mathcal{G}_1\mathcal{C}_{\cdot 2} + 2\mathcal{G}_2\mathcal{C}_{\cdot 2} - 2\mathcal{D}_2 + 2\omega_{32}\mathcal{G}_3\mathcal{C}_{\cdot 2} = 0 \\
\frac{\partial \mathcal{J}_I}{\partial \mathcal{C}_{\cdot 3}} &= 2\omega_{13}\mathcal{G}_1\mathcal{C}_{\cdot 3} + 2\omega_{23}\mathcal{G}_2\mathcal{C}_{\cdot 3} + 2\mathcal{G}_3\mathcal{C}_{\cdot 3} - 2\mathcal{D}_3 = 0
\end{aligned} \tag{2.30}$$

which we can rewrite as

$$\begin{aligned}
(\mathcal{G}_1 + \omega_{21}\mathcal{G}_2 + \omega_{31}\mathcal{G}_3)\,\mathcal{C}_{\cdot 1} &= \mathcal{D}_1 \\
(\omega_{12}\mathcal{G}_1 + \mathcal{G}_2 + \omega_{32}\mathcal{G}_3)\,\mathcal{C}_{\cdot 2} &= \mathcal{D}_2 \\
(\omega_{13}\mathcal{G}_1 + \omega_{23}\mathcal{G}_2 + \mathcal{G}_3)\,\mathcal{C}_{\cdot 3} &= \mathcal{D}_3.
\end{aligned} \tag{2.31}$$

These three 9×9 linear systems determine the 9-vectors $\mathcal{C}_{\cdot i}, \quad i = 1, 2, 3$, and note that the systems will be close to decoupled, which follows from our choice of the reference system.

2.1.4 Calculating the $\mathcal{A}$ matrix

The $\mathcal{A}$ matrix can be calculated in different ways. Here we present two methods. The first method makes use of the plaid structure of the matrix and the elements are calculated seperately. In the second method the matrix is obtained as a solution of a Lyapunov system. In the numerical examples in this thesis we shall always make use of method 2, as it makes the code very simple by making use of Matlab's `lyap` function.

Method 1

In [26], [32], [27], [29] and [30] it is shown that $\mathcal{A}$ has the following plaid structure

$$\mathcal{A} = \begin{bmatrix}
\mathcal{Y}_0 & 0 & -\mathcal{Y}_1 & 0 & \mathcal{Y}_2 & \cdots \\
0 & \mathcal{Y}_1 & 0 & -\mathcal{Y}_2 & 0 & \\
-\mathcal{Y}_1 & 0 & \mathcal{Y}_2 & 0 & \ddots & \\
0 & -\mathcal{Y}_2 & 0 & \mathcal{Y}_3 & \ddots & \\
\mathcal{Y}_2 & 0 & \ddots & \ddots & \ddots & 0 \\
\vdots & & & & 0 & \mathcal{Y}_{n-1}
\end{bmatrix}. \tag{2.32}$$

For stable systems $\mathcal{Y}_i$ is given by

$$\mathcal{Y}_i = \int_0^\infty \left(\left(y_b^{(i)}(t)\right)\right)^2 dt = (J^i\kappa)^T \int_0^\infty \mathcal{E}(t)\mathcal{E}^H(t)dt\overline{J^i\kappa}, \tag{2.33}$$

where J is a block diagonal Jordan matrix

$$J = \begin{bmatrix} J_1 & 0 & \cdots & 0 \\ 0 & J_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & J_\nu \end{bmatrix} \tag{2.34}$$

and the diagonal blocks are given by

$$J_i = \begin{bmatrix} \lambda_i & 1 & \cdots & \cdots & 0 \\ 0 & \lambda_i & 1 & & \vdots \\ 0 & 0 & \lambda_i & 1 & \vdots \\ \vdots & \ddots & \ddots & \ddots & 1 \\ 0 & \cdots & 0 & 0 & \lambda_i \end{bmatrix}_{d_i \times d_i} . \tag{2.35}$$

Here λ_i is a root of multiplicity d_i of the polynomial $a(s)$ and κ is the vector of the partial fraction expansion coefficients of $\frac{1}{a(s)}$ given by

$$\kappa = \begin{bmatrix} \kappa_{11} & \cdots & \kappa_{1d_1} & \cdots & \kappa_{1\nu_1} & \cdots & \kappa_{\nu d_v} \end{bmatrix}^T . \tag{2.36}$$

We know from [33] that these partial fraction expansion coefficients are given by the recursive formula

$$\kappa_{ij} = \begin{cases} \prod_{q=1, q\neq 1}^{\nu} \frac{1}{(\lambda_i - \lambda_q)^{d_q}} & , \quad j = d_i \\ \frac{1}{d_i - j} \sum_{q=1}^{d_i - j} (-1)^q \kappa_{i(j+q)} \times \sum_{p=1, p\neq i}^{\nu} \frac{d_p}{(\lambda_i - \lambda_p)^q} & , \quad j = d_i - 1, \ldots, 1. \end{cases} \tag{2.37}$$

The matrix $\mathcal{E}(t)$ is given by

$$\mathcal{E}(t) = \begin{bmatrix} \mathcal{E}_1(t) \\ \vdots \\ \mathcal{E}_\nu(t) \end{bmatrix} \tag{2.38}$$

where $\mathcal{E}_i(t)$ is

$$\mathcal{E}_i(t) = \begin{bmatrix} e^{\lambda_i t} \\ t e^{\lambda_i t} \\ \vdots \\ \frac{t^{d_i - 1}}{(d_i - 1)!} e^{\lambda_i t} \end{bmatrix} . \tag{2.39}$$

It is shown in [34] how the matrix $\int_0^\infty \mathcal{E}(t)\mathcal{E}^H(t)dt$ can be calculated, for stable systems. The (ρ, σ)-th element of the (k, j)-th subblock, $\rho = 1, 2, \ldots d_k, \quad \sigma = 1, 2, \ldots d_j, \quad k, j = 1, 2 \ldots \nu$, is given by

$$\left[\int_0^\infty \mathcal{E}_k(t)\mathcal{E}_j^H(t)dt \right]_{\rho,\sigma} = \frac{\binom{\rho+\sigma-2}{\rho-1}}{(-\lambda_k - \overline{\lambda_j})^{\rho+\sigma+1}} \quad , \quad -\lambda_k - \overline{\lambda_j} \neq 0. \tag{2.40}$$

where $-\lambda_k - \overline{\lambda_j} \neq 0$ is known as the Gantmacher condition, see [35]. Note that if $n > m = \overset{max}{ij}\,(m_{ij})$ zeros will be padded in the $\mathcal{B}_{ij}$ matrices from the bottom in Equation (2.12). Then we in fact only need to know the $(m+3)\times(m+3)$ principal submatrix of $\mathcal{A}$. We denote this submatrix with $\hat{\mathcal{A}}$. As an example $\mathcal{A}$ for $n = 6$ and the principal submatrix $\hat{\mathcal{A}}$ for $m = 1$ becomes this bracketed submatrix

$$\mathcal{A} = \left[\begin{array}{cccccc} \multicolumn{4}{c}{\left[\begin{array}{cccc} \mathcal{Y}_0 & 0 & -\mathcal{Y}_1 & 0 \\ 0 & \mathcal{Y}_1 & 0 & -\mathcal{Y}_2 \\ -\mathcal{Y}_1 & 0 & \mathcal{Y}_2 & 0 \\ 0 & -\mathcal{Y}_2 & 0 & \mathcal{Y}_3 \end{array}\right]} & \begin{array}{c} \mathcal{Y}_2 \\ 0 \\ -\mathcal{Y}_3 \\ 0 \end{array} & \begin{array}{c} 0 \\ \mathcal{Y}_3 \\ 0 \\ -\mathcal{Y}_4 \end{array} \\ \mathcal{Y}_2 & 0 & -\mathcal{Y}_3 & 0 & \mathcal{Y}_4 & 0 \\ 0 & \mathcal{Y}_3 & 0 & -\mathcal{Y}_4 & 0 & \mathcal{Y}_5 \end{array}\right]. \tag{2.41}$$

It is shown in [36], [37], [27] and [21] that for a stable system we can alternatively express

$$\mathcal{A} = \mathcal{K}^T \int_0^\infty \mathcal{E}(t)\mathcal{E}(t)^H dt\overline{\mathcal{K}}, \tag{2.42}$$

where $\mathcal{K}$ is

$$\mathcal{K} = \left[\begin{array}{cccc} \kappa & J\kappa & \cdots & J^{n-1}\kappa \end{array}\right]. \tag{2.43}$$

Note that $\mathcal{K}$ is computed most effectivly by first calculating κ and then the next column $J \cdot \kappa$, and so on recursively, $J^{i+1}\kappa = J(J^i\kappa), \quad i = 1, 2, \ldots, n-2$.

Also note, that if it is only necessary to compute $\hat{\mathcal{A}}$, it is easily done by trimming the columns of $\mathcal{K}$ down to $m + 3$, reducing the computation.

Finally note, that due to the structure of $\hat{\mathcal{A}}$, it is infact sufficent to calculate the first and the last column of $\hat{\mathcal{A}}$, i.e. $\mathcal{K}^T \int_0^\infty \mathcal{E}(t)\mathcal{E}(t)^H dt\overline{\kappa}$ and $\mathcal{K}^T \int_0^\infty \mathcal{E}(t)\mathcal{E}(t)^H dt\overline{J^{n-1}\kappa}$.

Method 2

It is shown in [30], [21] and [31] how it is possible to find $\mathcal{A}$ (for stable systems) by solving a Lyapunov equation. Consider $\mathcal{F}$ the $n \times n$ companion matrix for a state space system in the controller companion form

$$\mathcal{F} = \left[\begin{array}{ccccc} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -a_0 & -a_1 & -a_2 & -a_3 \cdots & -a_n \end{array}\right]. \tag{2.44}$$

With $\mathcal{A}$ defined in Equation (2.32), $Y_b(t)$ is given by

$$Y_b(t) = \left[\begin{array}{cccc} y_b(t) & y_b'(t) & \cdots & y_b^{(n-1)}(t) \end{array}\right]^T = e^{t\mathcal{F}}u_n \tag{2.45}$$

where u_n is an n-column unit vector with the n-th element as 1. Then the Gramian $\mathcal{A}$ is given by

$$\mathcal{A} = \int_0^\infty Y_b(t)Y_b^T(t)dt = \int_0^\infty e^{t\mathcal{F}} u_n u_n^T e^{t\mathcal{F}^T} dt. \tag{2.46}$$

It is the solution to the Lyapunov equation

$$\mathcal{F}\mathcal{A} + \mathcal{A}\mathcal{F}^T + u_n u_n^T = 0. \tag{2.47}$$

Note that Lyapunov's stability theorem states that $\mathcal{A}$ is positive definite if and only if the system $\mathcal{F}$ is strictly stable. It is possible to solve the Lyapunov equation by using the Matlab's function `lyap`. Matlab solves the equation by transforming $\mathcal{A}$ to a complex Schur form, solves the resulting triangular system and then transforms the solution back. Note that with this approach we have to solve for the full matrix $\mathcal{A}$ even if we only need to use the elements of a principal submatrix $\hat{\mathcal{A}}$ in the subsequent calculations.

It was shown in [37], [30], [31] and [21] that it is possible to solve for the $\mathcal{Y}_i$'s simultaneously from the linear system,

$$\begin{bmatrix} a_0 & a_2 & \cdots & \cdots & \cdots & 0 \\ 0 & a_1 & a_3 & \cdots & \cdots & 0 \\ 0 & a_0 & a_2 & \cdots & \cdots & 0 \\ 0 & 0 & a_1 & \cdots & \cdots & 0 \\ \vdots & \vdots & \ddots & & & \vdots \\ \vdots & \vdots & & \cdots & a_{n-2} & 1 \\ 0 & 0 & \cdots & \cdots & a_{n-3} & a_{n-1} \end{bmatrix} \times \begin{bmatrix} \mathcal{Y}_0 \\ -\mathcal{Y}_1 \\ \mathcal{Y}_2 \\ -\mathcal{Y}_3 \\ \vdots \\ \vdots \\ (-1)^{n-1}\mathcal{Y}_{n-1} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ (-1)^{n-1}1/2 \end{bmatrix}. \tag{2.48}$$

The transformation of this system of equations to upper triangular form can be shown to be analogous to the calculation of the Routh table for the polynonial $a(s)$.

Note that Equation (2.48) can be derived from the last line of the Lyapunov Equation (2.47), see [30], [31] and [21]. Further, the plaid structure in Equation (2.32) follows from the first $(n-1)$ lines of the Lyapunov equation [38].

Methods linked to Method 1 and 2, for solving Lyapunov and Sylvester equations may also be found in [25], [26] and [27].

2.1.5 Calculating The $\mathcal{H}_i$ matrix

It is shown in [21] and [28] for a SISO system that $\mathcal{H}$ has an alternating Hankel structure. In MIMO systems the matrix $\mathcal{H}_i$ is equivalent to the $\mathcal{H}$ matrix in SISO systems. $\mathcal{H}$ is defined as $\mathcal{H} = \int_0^\infty Y_b(t)Y_{b,r}(t)^T dt$, which is similar to how $\mathcal{H}_i$ is defined, see Equation (2.23). $\mathcal{H}_i$ will have the same alternating Hankel structure as $\mathcal{H}$. The $\mathcal{H}_i$ matrix is given as

$$\mathcal{H}_i = \begin{bmatrix} \mathcal{Z}_0 & -\mathcal{Z}_1 & \mathcal{Z}_2 & \cdots & (-1)^{n-1}\mathcal{Z}_{n_{r_i}-1} \\ \mathcal{Z}_1 & -\mathcal{Z}_2 & \mathcal{Z}_3 & \cdots & (-1)^{n-1}\mathcal{Z}_{n_{r_i}-1+1} \\ \mathcal{Z}_2 & -\mathcal{Z}_3 & \mathcal{Z}_4 & \cdots & (-1)^{n-1}\mathcal{Z}_{n_{r_i}-1+2} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \mathcal{Z}_{n-1} & -\mathcal{Z}_n & \mathcal{Z}_{n+1} & \cdots & (-1)^{n-1}\mathcal{Z}_{n_{r_i}-1+n-1} \end{bmatrix}_{n\times n_{ri}} \tag{2.49}$$

with

$$\mathcal{Z}_p = \begin{cases} \int_0^\infty y_b^{(p)}(t) y_{b,ri}(t)dt, & p = 0, 1 \ldots, n-1 \\ (-1)^k \int_0^\infty y_b^{(n-1)}(t) y_{b,ri}^{(k)}(t)dt, & p = n-1+k, \quad k = 1, 2, \ldots, n_{ri}-1 \end{cases} \tag{2.50}$$

We may only need to know a principal submatrix of $\mathcal{H}_i$, denoted by $\hat{\mathcal{H}}_i$ of dimension $(m+3) \times n_{ri}$, in the same way as we may only need the principal submatrix $\hat{\mathcal{A}}$ of $\mathcal{A}$, see for example for $n = 6$, $n_{ri} = 5$ and $m = 1$, we have

$$\mathcal{H}_i = \begin{bmatrix} \begin{bmatrix} \mathcal{Z}_0 & -\mathcal{Z}_1 & \mathcal{Z}_2 & -\mathcal{Z}_3 & \mathcal{Z}_4 \\ \mathcal{Z}_1 & -\mathcal{Z}_2 & \mathcal{Z}_3 & -\mathcal{Z}_4 & \mathcal{Z}_5 \\ \mathcal{Z}_2 & -\mathcal{Z}_3 & \mathcal{Z}_4 & -\mathcal{Z}_5 & \mathcal{Z}_6 \\ \mathcal{Z}_3 & -\mathcal{Z}_4 & \mathcal{Z}_5 & -\mathcal{Z}_6 & \mathcal{Z}_7 \end{bmatrix} \\ \begin{matrix} \mathcal{Z}_4 & -\mathcal{Z}_5 & \mathcal{Z}_6 & -\mathcal{Z}_7 & \mathcal{Z}_8 \\ \mathcal{Z}_5 & -\mathcal{Z}_6 & \mathcal{Z}_7 & -\mathcal{Z}_8 & \mathcal{Z}_9 \end{matrix} \end{bmatrix} \tag{2.51}$$

Closed form formulae for the elements of this matrix can be derived, similar to those derived for the elements of the $\mathcal{A}$ matrix above. Alternatively, $\mathcal{H}_i$ can be shown to be the solution to the Sylvester equation,

$$\mathcal{F}\mathcal{H}_i + \mathcal{H}_i\mathcal{F}_r^T + u_n u_{n_r}^T = 0 \tag{2.52}$$

where $\mathcal{F}_r$ is the controller or companion form matrix for the reference system. The Matlab function `lyap` can be used to solve the Sylvester equation.

It is shown in [37] how easily $\mathcal{D} = \mathcal{B}^T\mathcal{H}\mathcal{B}_r$ can be calculated for a reference system with only one pole and no zero. This method can be used to calculate $\mathcal{D}_i = \mathcal{B}_{i\cdot}^T\mathcal{H}_i\mathcal{B}_{r_i}$ for our optimized MIMO PID controller. First the reference system is chosen on a diagonal form with $G_{r_i}(s) = \frac{b_{0,r_i}}{s+a_{0,r_i}}$ on the diagonal line. $G_{r_i}(s)$ has the impulse response,

$$y_{I,r_i} = b_{0,r_i} e^{-a_{0,r_i}t}. \tag{2.53}$$

Using the impulse response it is possible to write

$$\mathcal{D}_i = \mathcal{B}_{i\cdot}^T \int_0^\infty Y_b(t) e^{-a_{0,r_i}t} dt b_{0,r_i} \tag{2.54}$$

using $Y_b(t)$ from Equation (2.10)

$$\mathcal{D}_i = \mathcal{B}_{i\cdot}^T \int_0^\infty \begin{bmatrix} y_b(t) \\ y_b^{'}(t) \\ \vdots \\ y_b^{(n-1)}(t) \end{bmatrix} e^{-a_{0,r_i}t} dt b_{0,r_i} \tag{2.55}$$

We denote the basic response with $F_b(s) = \frac{1}{a(s)}$ and noting the relation to the Laplace transform, then $\mathcal{D}_i$ becomes

$$\mathcal{D}_i = \mathcal{B}_{i\cdot}^T \begin{bmatrix} F_b(s) \\ sF_b(s) \\ \vdots \\ s^{n-1}F_b(s) \end{bmatrix} b_{0,r_i} \tag{2.56}$$

with $s = a_{0,r_i}$. Thus we get that

$$\mathcal{D}_i = \mathcal{B}_i^T \begin{bmatrix} 1 \\ a_{0,r_i} \\ a_{0,r_i}^2 \\ \vdots \\ a_{0,r_i}^{(n-1)} \end{bmatrix} F_b(a_{0,r_i}) b_{0,r_i} \tag{2.57}$$

It is further shown in [37] how this method can be used to solve for a more general reference system with more than one pole.

2.1.6 Step response

In many cases it is beneficial to minimize the step response rather than the impulse response. Then the DC gain of the controlled system and the reference system are taken into consideration. We define the step response for the transfer function ij in the open loop TFM $G(s)C(s)$ as

$$y_{S,ij}(t) = \int_0^t y_{I,ij}(u) du = \mathcal{C}_{\cdot j}^T \mathcal{B}_{i\cdot}^T \int_0^t Y_b(u) du. \tag{2.58}$$

Now we have to consider separately the transient part of this function denoted by $\hat{y}_{S,ij}(t)$ and the stationary part denoted by $\overline{y}_{S,ij}(t)$

$$y_{S,ij}(t) = \hat{y}_{S,ij}(t) + \overline{y}_{S,ij}(t). \tag{2.59}$$

Introduce the notation

$$y_b^{(-1)}(t) = \int_0^t y_b(u) du - \frac{1}{a_0} \tag{2.60}$$

and

$$Y_b^{(-1)}(t) = \begin{bmatrix} y_b^{(-1)}(t) & y_b(t) & y_b^{'}(t) & \cdots & y_b^{(n-2)}(t) \end{bmatrix}^T. \tag{2.61}$$

We then have that

$$\hat{y}_{S,ij}(t) = \mathcal{C}_{\cdot j}^T \mathcal{B}_{i\cdot}^T Y_b^{(-1)}(t) \tag{2.62}$$

and

$$\overline{y}_{S,ij}(t) = \mathcal{C}_{\cdot j}^T \mathcal{B}_{i\cdot}^T \left(\begin{bmatrix} \frac{1}{a_0} & 0 & \cdots & 0 \end{bmatrix}_{1\times n} \right)^T = \frac{b_{0,i\cdot}}{a_0} K_{I,\cdot j} \tag{2.63}$$

where $b_{0,i\cdot}$ and $K_{I,\cdot j}$ are defined according to

$$K_{I,\cdot i} = \begin{bmatrix} K_{I,1i} & K_{I,2i} & K_{I,3i} \end{bmatrix}^T \tag{2.64}$$

and

$$b_{0,i\cdot} = \begin{bmatrix} b_{0,i1} & b_{0,i2} & b_{0,i3} \end{bmatrix}. \tag{2.65}$$

Similarly for the reference system we have

$$y_{s,ri}(t) = \hat{y}_{S,ri}(t) + \overline{y}_{S,ri}, \tag{2.66}$$

$$\hat{y}_{S,ri}(t) = \mathcal{B}_{ri}^T Y_{b,ri}^{(-1)}(t), \tag{2.67}$$

$$\overline{y}_{S,ri} = \mathcal{B}_{ri}^T \left(\begin{bmatrix} \frac{1}{a_{0,ri}} & 0 & \cdots & 0 \end{bmatrix}_{1\times n_{ri}} \right)^T, \tag{2.68}$$

$$y_{b,ri}^{(-1)}(t) = \int_0^t y_{b,ri}(u)du - \frac{1}{a_{0,ri}}, \tag{2.69}$$

and

$$Y_{b,ri}^{(-1)}(t) = \begin{bmatrix} y_{b,ri}^{(-1)}(t) & y_{b,ri}(t) & y_{b,ri}'(t) & \cdots & y_{b,ri}^{(n_{ri}-2)}(t) \end{bmatrix}^T. \tag{2.70}$$

Further we can define $\mathcal{A}^{(-1)}$, $\mathcal{G}_i^{(-1)}$, $\mathcal{M}_i^{(-1)}$ and $\mathcal{H}_i^{(-1)}$ as

$$\mathcal{A}_{n\times n}^{(-1)} = \int_0^\infty Y_b^{(-1)}(t) Y_b^{(-1)}(t)^T dt \tag{2.71}$$

$$\mathcal{G}_{i(9\times 9)}^{(-1)} = \begin{bmatrix} \mathcal{B}_{i1}^T \\ \mathcal{B}_{i2}^T \\ \mathcal{B}_{i3}^T \end{bmatrix} \mathcal{A}^{(-1)} \begin{bmatrix} \mathcal{B}_{i1} & \mathcal{B}_{i2} & \mathcal{B}_{i3} \end{bmatrix} = \mathcal{B}_{i\cdot}^T \mathcal{A}^{(-1)} \mathcal{B}_{i\cdot} \tag{2.72}$$

$$\mathcal{H}_{i(n\times n_{ri})}^{(-1)} = \int_0^\infty Y_b^{(-1)}(t) Y_{b,ri}^{(-1)}(t)^T dt, \tag{2.73}$$

$$\mathcal{D}_{i(9\times 1)}^{(-1)} = \mathcal{B}_{i\cdot}^T \mathcal{H}_i^{(-1)} \mathcal{B}_{ri}, \tag{2.74}$$

$$\mathcal{A}^{(-1)}_{ri(n_{ri}\times n_{ri})} = \int_0^\infty Y^{(-1)}_{b,ri}(t) Y^{(-1)}_{b,ri}(t)^T dt, \tag{2.75}$$

and

$$\mathcal{M}^{(-1)}_{i(1\times 1)} = \mathcal{B}^T_{ri}\mathcal{A}^{(-1)}_{ri}\mathcal{B}_{ri}. \tag{2.76}$$

2.1.7 Step response cost function

When setting up the cost function using the step response instead of the impulse response, we want to minimize the difference in the step response for all transfer functions in the open loop system $G(s)C(s)$ and the corresponding transfer function in the reference system. We separate the cost function into a transient part and a stationary part. The transient part of the cost function is,

$$\begin{aligned}\hat{\mathcal{J}}_S = \textstyle\int_0^\infty ((\hat{y}_{S11}-\hat{y}_{Sr1})^2 + \omega_{12}(\hat{y}_{S12}-0)^2 + \omega_{13}(\hat{y}_{S13}-0)^2 \\ \omega_{21}(\hat{y}_{S21}-0)^2 + (\hat{y}_{S22}-\hat{y}_{Sr2})^2 + \omega_{23}(\hat{y}_{S23}-0)^2 \\ \omega_{31}(\hat{y}_{S31}-0)^2 + \omega_{32}(\hat{y}_{S32}-0)^2 + (\hat{y}_{S33}-\hat{y}_{Sr3})^2)dt.\end{aligned} \tag{2.77}$$

Now we have to treat the transient part $\hat{\mathcal{J}}_S$ and the stationary part separately. Similar to the cost function for the impulse we have for the transient part

$$\begin{aligned}
\int_0^\infty (\hat{y}_{S11}-\hat{y}_{Sr1})^2 dt &= \mathcal{C}^T_{\cdot 1}\mathcal{G}^{(-1)}_1\mathcal{C}_{\cdot 1} - 2\mathcal{C}^T_{\cdot 1}\mathcal{D}^{(-1)}_1 + \mathcal{M}^{(-1)}_1 \\
\int_0^\infty \omega_{12}(\hat{y}_{S12}-0)^2 dt &= \omega_{12}\mathcal{C}^T_{\cdot 2}\mathcal{G}^{(-1)}_1\mathcal{C}_{\cdot 2} \\
\int_0^\infty \omega_{13}(\hat{y}_{S13}-0)^2 dt &= \omega_{13}\mathcal{C}^T_{\cdot 3}\mathcal{G}^{(-1)}_1\mathcal{C}_{\cdot 3} \\
\int_0^\infty \omega_{21}(\hat{y}_{S21}-0)^2 dt &= \omega_{21}\mathcal{C}^T_{\cdot 1}\mathcal{G}^{(-1)}_2\mathcal{C}_{\cdot 1} \\
\int_0^\infty (\hat{y}_{S22}-\hat{y}_{Sr2})^2 dt &= \mathcal{C}^T_{\cdot 2}\mathcal{G}^{(-1)}_2\mathcal{C}_{\cdot 2} - 2\mathcal{C}^T_{\cdot 2}\mathcal{D}^{(-1)}_2 + \mathcal{M}^{(-1)}_2 \\
\int_0^\infty \omega_{23}(\hat{y}_{S23}-0)^2 dt &= \omega_{23}\mathcal{C}^T_{\cdot 3}\mathcal{G}^{(-1)}_2\mathcal{C}_{\cdot 3} \\
\int_0^\infty \omega_{31}(\hat{y}_{S31}-0)^2 dt &= \omega_{31}\mathcal{C}^T_{\cdot 1}\mathcal{G}^{(-1)}_3\mathcal{C}_{\cdot 1} \\
\int_0^\infty \omega_{32}(\hat{y}_{S32}-0)^2 dt &= \omega_{32}\mathcal{C}^T_{\cdot 2}\mathcal{G}^{(-1)}_3\mathcal{C}_{\cdot 2} \\
\int_0^\infty (\hat{y}_{S33}-\hat{y}_{Sr3})^2 dt &= \mathcal{C}^T_{\cdot 3}\mathcal{G}^{(-1)}_3\mathcal{C}_{\cdot 3} - 2\mathcal{C}^T_{\cdot 3}\mathcal{D}^{(-1)}_3 + \mathcal{M}^{(-1)}_3
\end{aligned} \tag{2.78}$$

In order for the cost function to remain finite, the stationary part has to be the same for the controlled system and the reference system. This implies

$$\begin{aligned}
\frac{b_{0,r1}}{a_{0,r1}} &= \frac{b_{0,11}}{a_0}K_{I11} + \frac{b_{0,12}}{a_0}K_{I21} + \frac{b_{0,13}}{a_0}K_{I31} = \frac{b_{0,1\cdot}}{a_0}K_{I,\cdot 1} \\
0 &= \frac{b_{0,1\cdot}}{a_0}K_{I\cdot 2} \\
0 &= \frac{b_{0,1\cdot}}{a_0}K_{I\cdot 3} \\
0 &= \frac{b_{0,2\cdot}}{a_0}K_{I\cdot 1} \\
\frac{b_{0,r2}}{a_{0,r2}} &= \frac{b_{0,2\cdot}}{a_0}K_{I\cdot 2} \\
0 &= \frac{b_{0,2\cdot}}{a_0}K_{I\cdot 3} \\
0 &= \frac{b_{0,3\cdot}}{a_0}K_{I\cdot 1} \\
0 &= \frac{b_{0,3\cdot}}{a_0}K_{I\cdot 2} \\
\frac{b_{0,r3}}{a_{0,r3}} &= \frac{b_{0,3\cdot}}{a_0}K_{I\cdot 3}.
\end{aligned} \tag{2.79}$$

It is possible to include these constraints by augmenting $\hat{\mathcal{J}}_S$ with a Lagrangian multiplier which can be expressed in a simple way as

$$\begin{aligned}
\mathcal{J}_{S,\lambda} = \hat{\mathcal{J}}_S \quad &+ \quad \lambda_{11}\left(\frac{b_{0,1\cdot}}{a_0}K_{I\cdot 1} - \frac{b_{0,r1}}{a_{0,r1}}\right) + \lambda_{12}\left(\frac{b_{0,1\cdot}}{a_0}K_{I\cdot 2}\right) + \lambda_{13}\left(\frac{b_{0,1\cdot}}{a_0}K_{I\cdot 3}\right) \\
&+ \quad \lambda_{21}\left(\frac{b_{0,2\cdot}}{a_0}K_{I\cdot 1}\right) + \lambda_{22}\left(\frac{b_{0,2\cdot}}{a_0}K_{I\cdot 2} - \frac{b_{0,r2}}{a_{0,r2}}\right) + \lambda_{23}\left(\frac{b_{0,2\cdot}}{a_0}K_{I\cdot 3}\right) \\
&+ \quad \lambda_{31}\left(\frac{b_{0,3\cdot}}{a_0}K_{I\cdot 1}\right) + \lambda_{32}\left(\frac{b_{0,3\cdot}}{a_0}K_{I\cdot 2}\right) + \lambda_{33}\left(\frac{b_{0,3\cdot}}{a_0}K_{I\cdot 3} - \frac{b_{0,r3}}{a_{0,r3}}\right).
\end{aligned} \tag{2.80}$$

Here λ_{ij} are the Lagrange multipliers.

2.1.8 Step response minimization

In order to minimize the cost function $\mathcal{J}_{S,\lambda}$ we find the partial derivatives with respect to C_{ij} and λ_{ij} and set them equal to zero

$$\frac{\partial \mathcal{J}_{S,\lambda}}{\partial \mathcal{C}_{\cdot i}} = 0 \tag{2.81}$$

and

$$\frac{\partial \mathcal{J}_{S,\lambda}}{\partial \lambda_{\cdot i}} = 0. \tag{2.82}$$

Writing these out gives

$$
\begin{aligned}
\frac{\partial \mathcal{J}_{S,\lambda}}{\partial \mathcal{C}_{\cdot 1}} &= 2\mathcal{G}_1^{(-1)}\mathcal{C}_{\cdot 1} - 2\mathcal{D}_1^{(-1)} + 2\omega_{21}\mathcal{G}_2^{(-1)}\mathcal{C}_{\cdot 1} + 2\omega_{31}\mathcal{G}_3^{(-1)}\mathcal{C}_{\cdot 1} \\
&\quad + \frac{1}{a_0}\left(b_{0,\cdot 1}u_1^T + b_{0,\cdot 2}u_4^T + b_{0,\cdot 3}u_7^T\right)^T \lambda_{\cdot 1} = 0 \\
\frac{\partial \mathcal{J}_{S,\lambda}}{\partial \mathcal{C}_{\cdot 2}} &= 2\omega_{12}\mathcal{G}_1^{(-1)}\mathcal{C}_{\cdot 2} + 2\mathcal{G}_2^{(-1)}\mathcal{C}_{\cdot 2} - 2\mathcal{D}_2^{(-1)} + 2\omega_{32}\mathcal{G}_3^{(-1)}\mathcal{C}_{\cdot 2} \\
&\quad + \frac{1}{a_0}\left(b_{0,\cdot 1}u_1^T + b_{0,\cdot 2}u_4^T + b_{0,\cdot 3}u_7^T\right)^T \lambda_{\cdot 2} = 0 \\
\frac{\partial \mathcal{J}_{S,\lambda}}{\partial \mathcal{C}_{\cdot 3}} &= 2\omega_{13}\mathcal{G}_1^{(-1)}\mathcal{C}_{\cdot 3} + 2\omega_{23}\mathcal{G}_2^{(-1)}\mathcal{C}_{\cdot 3} + 2\mathcal{G}_2^{(-1)}\mathcal{C}_{\cdot 3} - 2\mathcal{D}_3^{(-1)} \\
&\quad + \frac{1}{a_0}\left(b_{0,\cdot 1}u_1^T + b_{0,\cdot 2}u_4^T + b_{0,\cdot 3}u_7^T\right)^T \lambda_{\cdot 3} = 0 \\
\frac{\partial \mathcal{J}_{S,\lambda}}{\partial \lambda_{\cdot 1}} &= \frac{1}{a_0}\left(b_{0,\cdot 1}u_1^T + b_{0,\cdot 2}u_4^T + b_{0,\cdot 3}u_7^T\right)\mathcal{C}_{\cdot 1} - \begin{bmatrix} \frac{b_{0,r1}}{a_{0,r1}} \\ 0 \\ 0 \end{bmatrix} = 0 \\
\frac{\partial \mathcal{J}_{S,\lambda}}{\partial \lambda_{\cdot 2}} &= \frac{1}{a_0}\left(b_{0,\cdot 1}u_1^T + b_{0,\cdot 2}u_4^T + b_{0,\cdot 3}u_7^T\right)\mathcal{C}_{\cdot 2} - \begin{bmatrix} 0 \\ \frac{b_{0,r2}}{a_{0,r2}} \\ 0 \end{bmatrix} = 0 \\
\frac{\partial \mathcal{J}_{S,\lambda}}{\partial \lambda_{\cdot 3}} &= \frac{1}{a_0}\left(b_{0,\cdot 1}u_1^T + b_{0,\cdot 2}u_4^T + b_{0,\cdot 3}u_7^T\right)\mathcal{C}_{\cdot 3} - \begin{bmatrix} 0 \\ 0 \\ \frac{b_{0,r3}}{a_{0,r3}} \end{bmatrix} = 0.
\end{aligned}
\tag{2.83}
$$

Here u_k is a column vector of size 9 with element k as 1. Let us take a closer look at how $\frac{\partial \mathcal{J}_{S,\lambda}}{\partial \mathcal{C}_{\cdot 1}}$ is found:

$$
\begin{aligned}
\frac{\partial \mathcal{J}_{S,\lambda}}{\partial \mathcal{C}_{\cdot 1}} &= 2\mathcal{G}_1^{(-1)}\mathcal{C}_{\cdot 1} - 2\mathcal{D}_1^{(-1)} + 2\omega_{21}\mathcal{G}_2^{(-1)}\mathcal{C}_{\cdot 1} + 2\omega_{31}\mathcal{G}_3^{(-1)}\mathcal{C}_{\cdot 1} \\
&\quad + \frac{\partial}{\partial \mathcal{C}_{\cdot 1}}\left(\frac{1}{a_0}\left(\lambda_{11}b_{0,1\cdot} + \lambda_{21}b_{0,2\cdot} + \lambda_{31}b_{0,3\cdot}\right)K_{I,\cdot 1}\right) \\
&= 2\mathcal{G}_1^{(-1)}\mathcal{C}_{\cdot 1} - 2\mathcal{D}_1^{(-1)} + 2\omega_{21}\mathcal{G}_2^{(-1)}\mathcal{C}_{\cdot 1} + 2\omega_{31}\mathcal{G}_3^{(-1)}\mathcal{C}_{\cdot 1} \\
&\quad + \frac{1}{a_0}\begin{bmatrix} \lambda_{11}b_{0,11} + \lambda_{21}b_{0,21} + \lambda_{31}b_{0,31} \\ 0 \\ 0 \\ \lambda_{11}b_{0,12} + \lambda_{21}b_{0,22} + \lambda_{31}b_{0,32} \\ 0 \\ 0 \\ \lambda_{11}b_{0,13} + \lambda_{21}b_{0,23} + \lambda_{31}b_{0,33} \\ 0 \\ 0 \end{bmatrix} \\
&= 2\mathcal{G}_1^{(-1)}\mathcal{C}_{\cdot 1} - 2\mathcal{D}_1^{(-1)} + 2\omega_{21}\mathcal{G}_2^{(-1)}\mathcal{C}_{\cdot 1} + 2\omega_{31}\mathcal{G}_3^{(-1)}\mathcal{C}_{\cdot 1} \\
&\quad + \frac{1}{a_0}\left(u_1 b_{0,\cdot 1}^T\lambda_{\cdot 1} + u_4 b_{0,\cdot 2}^T\lambda_{\cdot 1} + u_7 b_{0,\cdot 3}^T\lambda_{\cdot 1}\right) \\
&= 2\mathcal{G}_1^{(-1)}\mathcal{C}_{\cdot 1} - 2\mathcal{D}_1^{(-1)} + 2\omega_{21}\mathcal{G}_2^{(-1)}\mathcal{C}_{\cdot 1} + 2\omega_{31}\mathcal{G}_3^{(-1)}\mathcal{C}_{\cdot 1} \\
&\quad + \frac{1}{a_0}\left(b_{0,\cdot 1}u_1^T + b_{0,\cdot 2}u_4^T + b_{0,\cdot 3}u_7^T\right)^T \lambda_{\cdot 1}
\end{aligned}
\tag{2.84}
$$

Similarly, $\frac{\partial \mathcal{J}_{S,\lambda}}{\partial \lambda_{\cdot 1}}$ becomes

$$\begin{aligned}
\frac{\partial \mathcal{J}_{S,\lambda}}{\partial \lambda_{\cdot 1}} &= \frac{\partial}{\partial \lambda_{\cdot 1}} \left(\lambda_{11} \left(\frac{b_{0,1\cdot}}{a_0} K_{I\cdot 1} - \frac{b_{0,r1}}{a_{0,r1}} \right) + \lambda_{21} \frac{b_{0,2\cdot}}{a_0} K_{I\cdot 1} + \lambda_{31} \frac{b_{0,3\cdot}}{a_0} K_{I\cdot 1} \right) \\
&= \begin{bmatrix} \frac{b_{0,1\cdot}}{a_0} K_{I\cdot 1} - \frac{b_{0,r2}}{a_{r0}} & \frac{b_{0,2\cdot}}{a_0} K_{I\cdot 1} & \frac{b_{0,3\cdot}}{a_0} K_{I\cdot 1} \end{bmatrix}^T \\
&= \frac{1}{a_0} \begin{bmatrix} b_{0,11} & b_{0,12} & b_{0,13} \\ b_{0,21} & b_{0,22} & b_{0,23} \\ b_{0,31} & b_{0,32} & b_{0,33} \end{bmatrix} \begin{bmatrix} K_{I11} \\ K_{I21} \\ K_{I31} \end{bmatrix} - \begin{bmatrix} \frac{b_{0,r1}}{a_{0,r1}} \\ 0 \\ 0 \end{bmatrix} \\
&= \frac{1}{a_0} \begin{bmatrix} b_{0,11} & 0 & 0 & b_{0,12} & 0 & 0 & b_{0,13} & 0 & 0 \\ b_{0,21} & 0 & 0 & b_{0,22} & 0 & 0 & b_{0,23} & 0 & 0 \\ b_{0,31} & 0 & 0 & b_{0,32} & 0 & 0 & b_{0,33} & 0 & 0 \end{bmatrix} \begin{bmatrix} K_{I11} \\ K_{P11} \\ K_{D11} \\ K_{I21} \\ K_{P21} \\ K_{D21} \\ K_{I31} \\ K_{P31} \\ K_{D31} \end{bmatrix} - \begin{bmatrix} \frac{b_{0,r1}}{a_{0,r1}} \\ 0 \\ 0 \end{bmatrix} \\
&= \frac{1}{a_0} \left[b_{0,\cdot 1} u_1^T + b_{0,\cdot 2} u_4^T + b_{0,\cdot 3} u_7^T \right] \begin{bmatrix} \mathcal{C}_{11} \\ \mathcal{C}_{21} \\ \mathcal{C}_{31} \end{bmatrix} - \begin{bmatrix} \frac{b_{0,r1}}{a_{0,r1}} \\ 0 \\ 0 \end{bmatrix} \qquad (2.85)
\end{aligned}$$

The rest of Equation (2.83) is then found in similar manner. The derivatives can then be set up in a very compact matrix form that can be easily solved using computer programs like Matlab, i.e.

$$\begin{aligned}
&\begin{bmatrix} \left[\mathcal{G}_1^{(-1)} + \omega_{21} \mathcal{G}_2^{(-1)} + \omega_{31} \mathcal{G}_3^{(-1)} \right]_{9\times 9} & \frac{1}{a_0} \left[b_{0,1\cdot} u_1^T + b_{0,2\cdot} u_4^T + b_{0,3\cdot} u_7^T \right]_{9\times 3}^T \\ \frac{1}{a_0} \left[b_{0,1\cdot} u_1^T + b_{0,2\cdot} u_4^T + b_{0,3\cdot} u_7^T \right]_{3\times 9} & \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}_{3\times 3} \end{bmatrix} \begin{bmatrix} [C_{\cdot 1}]_{9\times 1} \\ [\lambda_{\cdot 1}]_{3\times 1} \end{bmatrix} \\
&\qquad = \begin{bmatrix} \left[\mathcal{D}_1^{(-1)} \right]_{9\times 1} \\ \begin{bmatrix} \frac{b_{0,r1}}{a_{0,r1}} \\ 0 \\ 0 \end{bmatrix}_{3\times 1} \end{bmatrix} \\
&\begin{bmatrix} \omega_{12} \mathcal{G}_1^{(-1)} + \mathcal{G}_2^{(-1)} + \omega_{32} \mathcal{G}_3^{(-1)} & \frac{1}{a_0} \left[b_{0,1\cdot} u_1^T + b_{0,2\cdot} u_4^T + b_{0,3\cdot} u_7^T \right]^T \\ \frac{1}{a_0} \left[b_{0,1\cdot} u_1^T + b_{0,2\cdot} u_4^T + b_{0,3\cdot} u_7^T \right] & 0 \end{bmatrix} \begin{bmatrix} C_{\cdot 2} \\ \lambda_{\cdot 2} \end{bmatrix} \\
&\qquad = \begin{bmatrix} \mathcal{D}_2^{(-1)} \\ 0 \\ \frac{b_{0,r2}}{a_{0,r2}} \\ 0 \end{bmatrix} \qquad (2.86) \\
&\begin{bmatrix} \omega_{13} \mathcal{G}_1^{(-1)} + \omega_{23} \mathcal{G}_2^{(-1)} + \mathcal{G}_3^{(-1)} & \frac{1}{a_0} \left[b_{0,1\cdot} u_1^T + b_{0,2\cdot} u_4^T + b_{0,3\cdot} u_7^T \right]^T \\ \frac{1}{a_0} \left[b_{0,1\cdot} u_1^T + b_{0,2\cdot} u_4^T + b_{0,3\cdot} u_7^T \right] & 0 \end{bmatrix} \begin{bmatrix} C_{\cdot 3} \\ \lambda_{\cdot 3} \end{bmatrix} \\
&\qquad = \begin{bmatrix} \mathcal{D}_3^{(-1)} \\ 0 \\ 0 \\ \frac{b_{0,r3}}{a_{0,r3}} \end{bmatrix}
\end{aligned}$$

The second and the third matrices in Equation (2.86) have the same dimensions as the first matrix. Equation (2.86) can be written in more detail as

$$
\left[\begin{array}{cc}
\dfrac{\begin{bmatrix}\mathcal{B}_{11}^T\\ \mathcal{B}_{12}^T\\ \mathcal{B}_{13}^T\end{bmatrix}\mathcal{A}^{(-1)}[\mathcal{B}_{11}\mathcal{B}_{12}\mathcal{B}_{13}]+\omega_{21}\begin{bmatrix}\mathcal{B}_{21}^T\\ \mathcal{B}_{22}^T\\ \mathcal{B}_{23}^T\end{bmatrix}\mathcal{A}^{(-1)}[\mathcal{B}_{21}\mathcal{B}_{22}\mathcal{B}_{23}]+\omega_{31}\begin{bmatrix}\mathcal{B}_{31}^T\\ \mathcal{B}_{32}^T\\ \mathcal{B}_{33}^T\end{bmatrix}\mathcal{A}^{(-1)}[\mathcal{B}_{31}\mathcal{B}_{32}\mathcal{B}_{33}]}{\frac{1}{a_0}\begin{bmatrix}b_{0,11}u_1^T+b_{0,12}u_4^T+b_{0,13}u_7^T\\ b_{0,21}u_1^T+b_{0,22}u_4^T+b_{0,23}u_7^T\\ b_{0,31}u_1^T+b_{0,32}u_4^T+b_{0,33}u_7^T\end{bmatrix}^T} & \cdots \\
\dfrac{\frac{1}{a_0}\begin{bmatrix}b_{0,11}u_1^T+b_{0,12}u_4^T+b_{0,13}u_7^T\\ b_{0,21}u_1^T+b_{0,22}u_4^T+b_{0,23}u_7^T\\ b_{0,31}u_1^T+b_{0,32}u_4^T+b_{0,33}u_7^T\end{bmatrix}}{\begin{bmatrix}0&0&0\\0&0&0\\0&0&0\end{bmatrix}} &
\end{array}\right]
\begin{bmatrix}\begin{bmatrix}\mathcal{C}_{11}\\ \mathcal{C}_{21}\\ \mathcal{C}_{31}\end{bmatrix}\\ \hline \lambda_{11}\\ \lambda_{21}\\ \lambda_{31}\end{bmatrix}
=\begin{bmatrix}\begin{bmatrix}\mathcal{B}_{11}^T\\ \mathcal{B}_{12}^T\\ \mathcal{B}_{13}^T\end{bmatrix}\mathcal{H}_1^{(-1)}\mathcal{B}_{r1}\\ \frac{b_{0,r1}}{a_{0,r1}}\\ 0\\ 0\end{bmatrix}
$$

$$
\left[\begin{array}{cc}
\dfrac{\omega_{12}\begin{bmatrix}\mathcal{B}_{11}^T\\ \mathcal{B}_{12}^T\\ \mathcal{B}_{13}^T\end{bmatrix}\mathcal{A}^{(-1)}[\mathcal{B}_{11}\mathcal{B}_{12}\mathcal{B}_{13}]+\begin{bmatrix}\mathcal{B}_{21}^T\\ \mathcal{B}_{22}^T\\ \mathcal{B}_{23}^T\end{bmatrix}\mathcal{A}^{(-1)}[\mathcal{B}_{21}\mathcal{B}_{22}\mathcal{B}_{23}]+\omega_{32}\begin{bmatrix}\mathcal{B}_{31}^T\\ \mathcal{B}_{32}^T\\ \mathcal{B}_{33}^T\end{bmatrix}\mathcal{A}^{(-1)}[\mathcal{B}_{31}\mathcal{B}_{32}\mathcal{B}_{33}]}{\frac{1}{a_0}\begin{bmatrix}b_{0,11}u_1^T+b_{0,12}u_4^T+b_{0,13}u_7^T\\ b_{0,21}u_1^T+b_{0,22}u_4^T+b_{0,23}u_7^T\\ b_{0,31}u_1^T+b_{0,32}u_4^T+b_{0,33}u_7^T\end{bmatrix}^T} & \cdots \\
\dfrac{\frac{1}{a_0}\begin{bmatrix}b_{0,11}u_1^T+b_{0,12}u_4^T+b_{0,13}u_7^T\\ b_{0,21}u_1^T+b_{0,22}u_4^T+b_{0,23}u_7^T\\ b_{0,31}u_1^T+b_{0,32}u_4^T+b_{0,33}u_7^T\end{bmatrix}}{\begin{bmatrix}0&0&0\\0&0&0\\0&0&0\end{bmatrix}} &
\end{array}\right]
\begin{bmatrix}\begin{bmatrix}\mathcal{C}_{12}\\ \mathcal{C}_{22}\\ \mathcal{C}_{32}\end{bmatrix}\\ \hline \lambda_{12}\\ \lambda_{22}\\ \lambda_{32}\end{bmatrix}
=\begin{bmatrix}\begin{bmatrix}\mathcal{B}_{21}^T\\ \mathcal{B}_{22}^T\\ \mathcal{B}_{23}^T\end{bmatrix}\mathcal{H}_2^{(-1)}\mathcal{B}_{r2}\\ \frac{b_{0,r2}}{a_{0,r2}}\\ 0\\ 0\end{bmatrix}
\qquad (2.87)
$$

$$
\left[\begin{array}{cc}
\dfrac{\omega_{13}\begin{bmatrix}\mathcal{B}_{11}^T\\ \mathcal{B}_{12}^T\\ \mathcal{B}_{13}^T\end{bmatrix}\mathcal{A}^{(-1)}[\mathcal{B}_{11}\mathcal{B}_{12}\mathcal{B}_{13}]+\omega_{23}\begin{bmatrix}\mathcal{B}_{21}^T\\ \mathcal{B}_{22}^T\\ \mathcal{B}_{23}^T\end{bmatrix}\mathcal{A}^{(-1)}[\mathcal{B}_{21}\mathcal{B}_{22}\mathcal{B}_{23}]+\begin{bmatrix}\mathcal{B}_{31}^T\\ \mathcal{B}_{32}^T\\ \mathcal{B}_{33}^T\end{bmatrix}\mathcal{A}^{(-1)}[\mathcal{B}_{31}\mathcal{B}_{32}\mathcal{B}_{33}]}{\frac{1}{a_0}\begin{bmatrix}b_{0,11}u_1^T+b_{0,12}u_4^T+b_{0,13}u_7^T\\ b_{0,21}u_1^T+b_{0,22}u_4^T+b_{0,23}u_7^T\\ b_{0,31}u_1^T+b_{0,32}u_4^T+b_{0,33}u_7^T\end{bmatrix}^T} & \cdots \\
\dfrac{\frac{1}{a_0}\begin{bmatrix}b_{0,11}u_1^T+b_{0,12}u_4^T+b_{0,13}u_7^T\\ b_{0,21}u_1^T+b_{0,22}u_4^T+b_{0,23}u_7^T\\ b_{0,31}u_1^T+b_{0,32}u_4^T+b_{0,33}u_7^T\end{bmatrix}}{\begin{bmatrix}0&0&0\\0&0&0\\0&0&0\end{bmatrix}} &
\end{array}\right]
\begin{bmatrix}\begin{bmatrix}\mathcal{C}_{13}\\ \mathcal{C}_{23}\\ \mathcal{C}_{33}\end{bmatrix}\\ \hline \lambda_{13}\\ \lambda_{23}\\ \lambda_{33}\end{bmatrix}
=\begin{bmatrix}\begin{bmatrix}\mathcal{B}_{31}^T\\ \mathcal{B}_{32}^T\\ \mathcal{B}_{33}^T\end{bmatrix}\mathcal{H}_3^{(-1)}\mathcal{B}_{r3}\\ \frac{b_{0,r3}}{a_{0,r3}}\\ 0\\ 0\end{bmatrix}
$$

2.1.9 Calculating the $\mathcal{A}^{(-1)}$ matrix

We have two methods to find $\mathcal{A}^{(-1)}$, which are similar to the methods used to find $\mathcal{A}$.

Method 1

The almost plaid structure for the matrix $\mathcal{A}^{(-1)}$ and how it can be derived is shown in [21] and [28]. The almost plaid structure for $\mathcal{A}^{(-1)}$ is given by

$$\mathcal{A}^{(-1)} = \begin{bmatrix} \mathcal{Y}_{-1} & -\frac{1}{2a_0^2} & -\mathcal{Y}_0 & 0 & \mathcal{Y}_1 & \cdots \\ -\frac{1}{2a_0^2} & \mathcal{Y}_0 & 0 & -\mathcal{Y}_1 & 0 & \\ -\mathcal{Y}_0 & 0 & \mathcal{Y}_1 & 0 & \ddots & \\ 0 & -\mathcal{Y}_1 & 0 & \mathcal{Y}_2 & \ddots & \\ \mathcal{Y}_1 & 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & & & & 0 & \mathcal{Y}_{n-2} \end{bmatrix}. \tag{2.88}$$

For a stable system the $\mathcal{Y}_i$ are given by

$$\mathcal{Y}_i = \int_0^\infty \left(\left(y_b^{(i)}(t)\right)\right)^2 dt = (J^i\kappa)^T \int_0^\infty \mathcal{E}(t)\mathcal{E}^H(t)dt\overline{J^i\kappa}, \tag{2.89}$$

The matrix J_i and the vectors κ and $\mathcal{E}(t)$ are defind the same way as in Section 2.1.4, see Equations (2.34) - (2.40).

The same applies here as in Section 2.1.4, that if $n > m = {}^{max}_{ij}(m_{ij})$, resulting in some zeros to be padded in the $\mathcal{B}_{ij}$, then we only need to know the $(m+3) \times (m+3)$ principal submatrix of $\mathcal{A}^{(-1)}$, which we denote, similary as before by $\hat{\mathcal{A}}^{(-1)}$.

Alternatively we can express

$$\mathcal{A}^{(-1)} = \left(J^{-1}\mathcal{K}\right)^T \int_0^\infty \mathcal{E}(t)\mathcal{E}(t)^H dt \left(J^{-1}\mathcal{K}\right) \tag{2.90}$$

where $J^{-1}\mathcal{K}$ is given by

$$J^{-1}\mathcal{K} = \begin{bmatrix} J^{-1}\kappa & \kappa & J\kappa & \cdots & J^{n-2}\kappa \end{bmatrix} \tag{2.91}$$

Note again here that if it is only necessary to compute $\hat{A}^{(-1)}$, then it is easily done by trimming the columns of $J^{-1}\mathcal{K}$ down to $m+3$, reducing the computation. Further, columns $2, 3, \ldots, n$ can be computed recursively as before. The first column is calculated most effectively by solving the system $J\tilde{x} = \kappa$ by back substitution. Finally, all the elements in $\mathcal{A}^{(-1)}$ can be found in its first and last column, $(J^{-1}\mathcal{K})^T \int_0^\infty \mathcal{E}(t)\mathcal{E}(t)^H dt J^{-1}\kappa$ and $(J^{-1}\mathcal{K})^T \int_0^\infty \mathcal{E}(t)\mathcal{E}(t)^H dt\overline{J^{n-2}\kappa}$, respectively.

Method 2

Similarly as before it can be shown that $\mathcal{A}^{(-1)}$ is the solution of the following Lyapunov equation

$$\mathcal{F}\mathcal{A}^{(-1)} + \mathcal{A}^{(-1)}\mathcal{F}^T + \frac{1}{a_0^2}u_1u_1^T = 0, \tag{2.92}$$

where $\mathcal{F}$ is defined as before and $Y_b^{(-1)}(t)$ is given by

$$\begin{aligned} Y_b^{(-1)}(t) &= \int Y_b(t)dt = \int e^{t\mathcal{F}} u_n dt \\ &= e^{t\mathcal{F}}\mathcal{F}^{-1}u_n = e^{t\mathcal{F}}\left(-\frac{1}{a_0}\right)u_1 \end{aligned} \tag{2.93}$$

or

$$Y_b^{(-1)}(t) = \left[\begin{array}{cccc} y_b^{(-1)}(t) & y_b(t) & \cdots & y_b^{(n-2)}(t) \end{array}\right] = e^{t\mathcal{F}}\left(\frac{-1}{a_0}\right)u_1. \tag{2.94}$$

The Lyapunov equation can e.g. be solved by using the Matlab's function `lyap`.

It is easily shown from the last row in Equation (2.92) that the elements of $\mathcal{A}^{(-1)}$ can be found by solving the linear system of equations

$$\left[\begin{array}{ccccccc} a_0 & a_2 & \cdots & \cdots & \cdots & \cdots & 0 \\ 0 & a_1 & a_3 & \cdots & \cdots & \cdots & 0 \\ 0 & a_0 & a_2 & \cdots & \cdots & \cdots & 0 \\ 0 & 0 & a_1 & \cdots & \cdots & \cdots & 0 \\ \vdots & \vdots & \ddots & & & & \vdots \\ \vdots & \vdots & & 0 & & & a_n \\ 0 & 0 & \vdots & 0 & \ddots & \cdots & a_{n-1} \end{array}\right] \left[\begin{array}{c} \mathcal{Y}_{-1} \\ -\mathcal{Y}_0 \\ \mathcal{Y}_1 \\ -\mathcal{Y}_2 \\ \vdots \\ \vdots \\ (-1)^{n-1}\mathcal{Y}_{(n-2)} \end{array}\right] = \left[\begin{array}{c} -\frac{a_1}{2a_0^2} \\ -\frac{a_1}{2a_0} \\ \vdots \\ \vdots \\ 0 \end{array}\right], \tag{2.95}$$

using the plaid structure of $\mathcal{A}^{(-1)}$, see [38].

2.1.10 Calculating the $\mathcal{H}_i^{(-1)}$ matrix

To find the $\mathcal{H}_i^{(-1)} = \int_0^\infty Y_b^{(-1)}(t)(Y_{b,ri}^{(-1)}(t))^T dt$ matrices the same methods that are used in [21] and [28] for SISO systems can be used. Only a principal submartix of this matrix may be needed $\hat{\mathcal{H}}_i^{(-1)}$, similar to $\hat{\mathcal{A}}^{(-1)}$ and $\mathcal{A}^{(-1)}$. $\mathcal{H}_i^{(-1)}$ has the alternating sign (almost) Hermitian structure

$$\mathcal{H}_i^{(-1)} = \left[\begin{array}{ccccc} -\mathcal{Z}_{-2} & \mathcal{Z}_{-1} & -\mathcal{Z}_0 & \cdots & (-1)^{n-1}\mathcal{Z}_{n_{ri}-3} \\ -\mathcal{Z}_{-1} - \frac{1}{a_0 a_{r,0}} & \mathcal{Z}_0 & -\mathcal{Z}_1 & \cdots & (-1)^{n-1}\mathcal{Z}_{n_{ri}-3+1} \\ -\mathcal{Z}_0 & \mathcal{Z}_1 & -\mathcal{Z}_2 & \cdots & (-1)^{n-1}\mathcal{Z}_{n_{ri}-3+2} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ -\mathcal{Z}_{n-3} & \mathcal{Z}_{n-2} & -\mathcal{Z}_{n-1} & \cdots & (-1)^{n-1}\mathcal{Z}_{n_{ri}-3+n-1} \end{array}\right]_{n\times n_{ri}} \tag{2.96}$$

with

$$\begin{aligned} \mathcal{Z}_{-2} &= \int_0^\infty y_b^{(-1)}(t)y_{b,ri}^{(-1)}(t)dt \\ \mathcal{Z}_{-1} &= \int_0^\infty y_b^{(-1)}(t)y_{b,ri}(t)dt \end{aligned} \tag{2.97}$$

$\mathcal{Z}_i$ for $i = 1, 2, \ldots$ are found in the same way as for the impulse case. $\mathcal{H}_i^{(-1)}$ can be found by solving the Sylvester equation

$$\mathcal{F}\mathcal{H}_i^{(-1)} + \mathcal{H}_i^{(-1)}\mathcal{F}_r^T + \frac{1}{a_0 a_{r,0}} [u_1]_{n\times 1} \left[u_1^T\right]_{1\times n_{ri}} = 0 \tag{2.98}$$

using e.g. the `lyap` function in Matlab.

2.2 General case: p control inputs and p outputs

We now generalize our results and assume that the system in Figure (2.1) has p numbers of inputs and p numbers of outputs, i.e., a general square MIMO system. The TFM for the open loop system $G(s)$, the TFM for the reference system $G_r(s)$ and the TFM for the PID controller $C(s)$ are generalized to

$$G(s) = \begin{bmatrix} G_{11}(s) & G_{12}(s) & \cdots & G_{1p}(s) \\ G_{21}(s) & G_{22}(s) & \cdots & G_{2p}(s) \\ \vdots & \vdots & \ddots & \vdots \\ G_{p1}(s) & G_{p2}(s) & \cdots & G_{pp}(s) \end{bmatrix}, \tag{2.99}$$

$$G_r(s) = \begin{bmatrix} G_{r1}(s) & 0 & \cdots & 0 \\ 0 & G_{r2}(s) & & \vdots \\ \vdots & & \ddots & \vdots \\ 0 & 0 & 0 & G_{rp}(s) \end{bmatrix} \tag{2.100}$$

and

$$\frac{1}{s}C(s) = \frac{1}{s}\begin{bmatrix} c_{11}(s) & c_{12}(s) & \cdots & c_{1p}(s) \\ c_{21}(s) & c_{22}(s) & \cdots & c_{2p}(s) \\ \vdots & \vdots & \ddots & \vdots \\ c_{p1}(s) & c_{p2}(s) & \cdots & c_{pp}(s) \end{bmatrix}. \tag{2.101}$$

Here, $G_{ij}(s)$, $G_{ri}(s)$ and $C_{ij}(s)$ are all defined in the same way as before, i.e.

$$G_{ij}(s) = \frac{b_{ij}(s)}{a(s)} = \frac{b_{m_{ij},ij}s^{m_{ij}} + b_{m_{ij}-1,ij}s^{m_{ij}-1} + \cdots + b_{1,ij}s + b_{0,ij}}{s^n + a_{n-1}s^{n-1} + \cdots + a_0} \tag{2.102}$$

and

$$G_{ri}(s) = \frac{b_{m_{ri},ri}s^{m_{ri}} + b_{m_{ri}-1,ri}s^{m_{ri}-1} + \cdots + b_{1,ri}s + b_{0,ri}}{s^{n_{ri}} + a_{n_{ri}-1,ri}s^{n_{ri}-1} + \cdots + a_{0,ri}} \tag{2.103}$$

Again the most convenient transfer function for the reference system is

$$G_{ri} = \frac{\omega_{ri}^2}{s + 2\zeta_{ri}\omega_{ri}}. \tag{2.104}$$

and the controller elements are

$$c_{ij}(s) = K_{D,ij}s^2 + K_{P,ij}s + K_{I,ij}. \tag{2.105}$$

2.2.1 Impulse response

For the three I/O systems the impulse response for all transfer functions in the TFM $G(s)C(s)$ where found using Equation (2.9) and the linear properties of the Laplace transform. The same method is used for a p I/O system. The only difference is the size of the vectors $\mathcal{C}_{\cdot i}$ and $\mathcal{B}_{j\cdot}$, they are now given by

$$\mathcal{C}_{\cdot i} = \begin{bmatrix} \mathcal{C}_{1i} \\ \mathcal{C}_{2i} \\ \vdots \\ \mathcal{C}_{pi} \end{bmatrix}_{3p \times 1} \tag{2.106}$$

and

$$\mathcal{B}_{j\cdot} = \begin{bmatrix} \mathcal{B}_{j1} & \mathcal{B}_{j2} & \cdots & \mathcal{B}_{jp} \end{bmatrix}_{n \times 3p}. \tag{2.107}$$

It is then possible to write

$$\mathcal{C}_{\cdot i}^T \mathcal{B}_{j\cdot}^T = \mathcal{C}_{1i}^T \mathcal{B}_{j1}^T + \mathcal{C}_{2i}^T \mathcal{B}_{j2}^T + \cdots + \mathcal{C}_{pi}^T \mathcal{B}_{jp}^T. \tag{2.108}$$

Using Equation (2.108), the impulse response can be written in the same way as before

$$\begin{aligned}
y_{I11} &= \mathscr{L}^{-1}\left\{ \frac{1}{a(s)} \begin{bmatrix} b_{11}(s) & b_{12}(s) & \cdots & b_{1p}(s) \end{bmatrix} \begin{bmatrix} c_{11}(s) \\ c_{21}(s) \\ \vdots \\ c_{p1}(s) \end{bmatrix} \right\} \\
&= \mathscr{L}^{-1}\left\{ \frac{b_{11}(s)c_{11}(s) + b_{12}(s)c_{21}(s) + \cdots + b_{1p}c_{p1}(s)}{a(s)} \right\} \\
&= \left[\mathcal{C}_{11}^T \mathcal{B}_{11}^T + \mathcal{C}_{21}^T \mathcal{B}_{12}^T + \cdots + \mathcal{C}_{p1}^T \mathcal{B}_{1p}^T\right] Y_b(t) = \mathcal{C}_{\cdot 1}^T \mathcal{B}_{1\cdot}^T Y_b(t) \\
y_{I12} &= \mathcal{C}_{\cdot 2}^T \mathcal{B}_{1\cdot}^T Y_b(t) \\
&\vdots \\
y_{I1p} &= \mathcal{C}_{\cdot p}^T \mathcal{B}_{1\cdot}^T Y_b(t) \\
y_{I21} &= \mathcal{C}_{\cdot 1}^T \mathcal{B}_{2\cdot}^T Y_b(t) \\
&\vdots \\
y_{I2p} &= \mathcal{C}_{\cdot p}^T \mathcal{B}_{2\cdot}^T Y_b(t) \\
&\vdots \\
y_{Ip1} &= \mathcal{C}_{\cdot 1}^T \mathcal{B}_{p\cdot}^T Y_b(t) \\
&\vdots \\
y_{Ipp} &= \mathcal{C}_{\cdot p}^T \mathcal{B}_{p\cdot}^T Y_b(t).
\end{aligned} \tag{2.109}$$

$Y_b(t)$ is defined in the same way as before as a vector including the basic impulse response and its derivatives. The impulse responses for the reference systems are again defined in the same way as before as

$$\begin{aligned} y_{Ir1} &= \mathcal{B}_{r1}^T Y_{b,r1}(t) \\ y_{Ir2} &= \mathcal{B}_{r2}^T Y_{b,r2}(t) \\ &\vdots \\ y_{Irp} &= \mathcal{B}_{rp}^T Y_{b,rp}(t) \end{aligned} \tag{2.110}$$

where $\mathcal{B}_{ri}$ is given by

$$\begin{aligned} \mathcal{B}_{r1}^T &= \begin{bmatrix} b_{0,r1} & \cdots & b_{m_{r1},r1} \end{bmatrix} \\ \mathcal{B}_{r2}^T &= \begin{bmatrix} b_{0,r2} & \cdots & b_{m_{r2},r2} \end{bmatrix} \\ &\vdots \\ \mathcal{B}_{rp}^T &= \begin{bmatrix} b_{0,rp} & \cdots & b_{m_{rp},rp} \end{bmatrix} \end{aligned} \tag{2.111}$$

2.2.2 Impulse response cost function

For the p I/O system as for the three I/O system we want to minimize the difference between the impulse response for all transfer functions in the TFM $G(s)C(s)$ and the reference system's TFM. The matrices $\mathcal{C}_{ij(3\times1)}$, $\mathcal{B}_{ri(n_{ri}\times1)}$, $\mathcal{A}_{n\times n}$, $\mathcal{H}_{i(n\times n_{ri})}$, $\mathcal{A}_{ri(n_{ri}\times n_{ri})}$ and $\mathcal{M}_{i(1\times1)}$ are all defined precisely in the same way as before, see Equations (2.11), (2.19), (2.21), (2.23), (2.25) and (2.26). The matrices $\mathcal{B}_{i\cdot(n\times3p)}$, $\mathcal{G}_{i(3p\times3p)}$ and $\mathcal{D}_{i(3p\times1)}$ are all expanded for the general case but basically defined in the same way as for the three I/O system,

$$\mathcal{B}_{i\cdot(n\times3p)} = \begin{bmatrix} \mathcal{B}_{i1} & \mathcal{B}_{i2} & \cdots & \mathcal{B}_{ip} \end{bmatrix}, \tag{2.112}$$

$$\mathcal{G}_{i(3p\times3p)} = \mathcal{B}_{i\cdot}^T \mathcal{A} \mathcal{B}_{i\cdot}, \tag{2.113}$$

$$\mathcal{D}_i = \mathcal{B}_{i\cdot}^T \mathcal{H}_i \mathcal{B}_{ri}. \tag{2.114}$$

The cost function for the p I/O system now becomes

$$\begin{aligned} \mathcal{J}_I &= \int_0^\infty [(y_{I11}(t) - y_{Ir1}(t))^2 + \omega_{12}(y_{I12}(t) - 0)^2 + \cdots + \omega_{1p}(y_{I1p}(t) - 0)^2 \\ &+ \omega_{21}(y_{I21}(t) - 0)^2 + (y_{I22}(t) - y_{Ir2}(t))^2 + \cdots + \omega_{2p}(y_{I2p}(t) - 0)^2 \\ &\vdots \\ &+ \omega_{p1}(y_{Ip1}(t) - 0)^2 + \omega_{p2}(y_{Ip2}(t) - 0)^2 + \cdots + (y_{Ipp}(t) - y_{Irp}(t))^2]dt. \end{aligned} \tag{2.115}$$

with $\int_0^\infty (y_{I,ii}(t) - y_{I,ri}(t))^2$ given by

$$\int_0^\infty (y_{Iii}(t) - y_{Iri}(t))^2 dt = \mathcal{C}_{\cdot i}^T \mathcal{G}_i \mathcal{C}_{\cdot i} - 2\mathcal{C}_{\cdot i}^T \mathcal{D}_i + \mathcal{M}_i \tag{2.116}$$

and $\int_0^\infty \omega_{ij} y_{I,ij}(t)^2$ when $i \neq j$ given by

$$\int_0^\infty \omega_{ij}(y_{Iij}(t) - 0)^2 dt = \omega_{ij}\mathcal{C}_{\cdot j}^T \mathcal{G}_i \mathcal{C}_{\cdot j} \tag{2.117}$$

2.2.3 Impulse response minimization

Minimizing the impulse response cost function $\mathcal{J}_I$ for the p I/O system, we find the partial derivatives for all $\mathcal{C}_{\cdot i}$ and set them equal to zero,

$$\frac{\partial \mathcal{J}_I}{\partial \mathcal{C}_{\cdot i}} = 0. \tag{2.118}$$

The derivatives are then written out in the same way as before

$$\begin{aligned}
\frac{\partial \mathcal{J}_I}{\partial \mathcal{C}_{\cdot 1}} &= 2\mathcal{G}_1\mathcal{C}_{\cdot 1} + 2\omega_{21}\mathcal{G}_2\mathcal{C}_{\cdot 1} + \cdots + 2\omega_{p1}\mathcal{G}_p\mathcal{C}_{\cdot 1} - 2\mathcal{D}_1 = 0 \\
\frac{\partial \mathcal{J}_I}{\partial \mathcal{C}_{\cdot 2}} &= 2\omega_{12}\mathcal{G}_1\mathcal{C}_{\cdot 2} + 2\mathcal{G}_2\mathcal{C}_{\cdot 2} + \cdots + 2\omega_{p2}\mathcal{G}_p\mathcal{C}_{\cdot 2} - 2\mathcal{D}_2 = 0 \\
&\vdots \\
\frac{\partial \mathcal{J}_I}{\partial \mathcal{C}_{\cdot p}} &= 2\omega_{1p}\mathcal{G}_1\mathcal{C}_{\cdot p} + 2\omega_{2p}\mathcal{G}_2\mathcal{C}_{\cdot p} + \cdots + 2\mathcal{G}_p\mathcal{C}_{\cdot p} - 2\mathcal{D}_p = 0
\end{aligned} \tag{2.119}$$

which we rewrite as

$$\begin{aligned}
(\mathcal{G}_1 + \omega_{21}\mathcal{G}_2 + \cdots + \omega_{p1}\mathcal{G}_p)\,\mathcal{C}_{\cdot 1} &= \mathcal{D}_1 \\
(2\omega_{12}\mathcal{G}_1 + \mathcal{G}_2 + \cdots + \omega_{p2}\mathcal{G}_p)\,\mathcal{C}_{\cdot 2} &= \mathcal{D}_2 \\
&\vdots \\
(\omega_{1p}\mathcal{G}_1 + \omega_{2p}\mathcal{G}_2 + \cdots + \mathcal{G}_p)\,\mathcal{C}_{\cdot p} &= \mathcal{D}_p.
\end{aligned} \tag{2.120}$$

2.2.4 Step response

For the general case the step response for the element transfer function ij in the open loop TFM $G(s)C(s)$ is defined in the same way as for the 2 I/O system previously,

$$\begin{aligned}
y_{S,ij}(t) &= \int_0^t y_{I,ij}(u)du \\
&= \mathcal{C}_{\cdot j}^T \mathcal{B}_{i\cdot}^T \int_0^t Y_b(u)dt \\
&= \hat{y}_{S,ij}(t) + \overline{y}_{S,ij}(t).
\end{aligned} \tag{2.121}$$

The same goes for the transient and stationary parts of the step response

$$\hat{y}_{S,ij}(t) = \mathcal{C}_{\cdot j}^T \mathcal{B}_{i\cdot}^T Y_b^{(-1)}(t), \tag{2.122}$$

$$\overline{y}_{S,ij}(t) = \mathcal{C}_{\cdot j}^T \mathcal{B}_{i\cdot}^T \left(\left[\begin{array}{cccc} \frac{1}{a_0} & 0 & \cdots & 0 \end{array} \right]_{1\times n} \right)^T = \frac{b_{0,i\cdot}}{a_0} K_{I,\cdot j}. \tag{2.123}$$

Now $b_{0,i\cdot}$ and $K_{I,\cdot j}$ are defined as

$$K_{I\cdot i} = \left[\begin{array}{cccc} K_{I1i} & K_{I2i} & \cdots & K_{Ipi} \end{array} \right]^T \tag{2.124}$$

and

$$b_{0,i\cdot} = \left[\begin{array}{cccc} b_{0,i1} & b_{0,i2} & \cdots & b_{0,ip} \end{array} \right]. \tag{2.125}$$

The matrices $\mathcal{A}_{n\times n}^{(-1)}$, $\mathcal{A}_{ri(n_{ri}\times n_{ri})}^{(-1)}$, $\mathcal{H}_{i(n\times n_{ri})}^{(-1)}$ and $\mathcal{M}_{i(1\times 1)}^{(-1)}$ are all defined in the same way as before, see Equations (2.71), (2.75), (2.73) and (2.76). The matrices $\mathcal{D}_{i(3p\times 1)}^{(-1)}$ and $\mathcal{G}_{i(3p\times 3p)}^{(-1)}$ are expanded for the general case.

$$\mathcal{D}_{i(3p\times 1)}^{(-1)} = \mathcal{B}_{i\cdot}^T \mathcal{H}^{(-1)} \mathcal{B}_{ri} \tag{2.126}$$

$$\mathcal{G}_{i(3p\times 3p)}^{(-1)} = \mathcal{B}_{i\cdot}^T \mathcal{A}^{(-1)} \mathcal{B}_{i\cdot} \tag{2.127}$$

2.2.5 Step response cost function

When finding the cost function for the step response, the same applies for the p I/O system as for the 3 I/O system. We want to minimize the difference between every transfer function in the transfer function matrix $G(s)C(s)$ and the reference system transfer function matrix. We separate the transient and stationary parts of the cost function, the transient part becomes

$$\begin{aligned} \hat{\mathcal{J}}_S \;=\; & \int_0^\infty [(\hat{y}_{S11}(t) - \hat{y}_{Sr1}(t))^2 + \omega_{12}(\hat{y}_{S12}(t) - 0)^2 + \cdots + \omega_{1p}(\hat{y}_{S1p}(t) - 0)^2 \\ + \; & \omega_{21}(\hat{y}_{S21}(t) - 0)^2 + (\hat{y}_{S22}(t) - \hat{y}_{Sr2}(t))^2 + \cdots + \omega_{2p}(\hat{y}_{S2p}(t) - 0)^2 \\ & \vdots \\ + \; & \omega_{p1}(\hat{y}_{Sp1}(t) - 0)^2 + \omega_{p2}(\hat{y}_{Sp2}(t) - 0)^2 + \cdots + (\hat{y}_{Spp}(t) - \hat{y}_{Srp}(t))^2]dt, \end{aligned} \tag{2.128}$$

with $\int_0^\infty (\hat{y}_{Sii} - \hat{y}_{S,ri})^2 dt$ given by

$$\int_0^\infty (\hat{y}_{Sii} - \hat{y}_{S,ri})^2 dt = \mathcal{C}_{\cdot i}^T \mathcal{G}_i^{(-1)} \mathcal{C}_{\cdot i} - 2\mathcal{C}_{\cdot i}^T \mathcal{D}_i^{(-1)} + \mathcal{M}_i^{(-1)} \tag{2.129}$$

and $\int_0^\infty \omega_{ij}(\hat{y}_{Sij}(t) - 0)^2 dt$ given by

$$\int_0^\infty \omega_{ij}(\hat{y}_{Sij}(t) - 0)^2 dt = \omega_{ij} \mathcal{C}_{\cdot j}^T \mathcal{G}_i^{(i)} \mathcal{C}_{\cdot j} \tag{2.130}$$

The stationary parts of the controlled system and the reference system have to be the same for the cost function to remain finite. These constraints are given by

$$\begin{aligned} \frac{b_{0,i\cdot}}{a_0} K_{I,\cdot i} &= \frac{b_{0,ri}}{a_{0,ri}} \\ \frac{b_{0,i\cdot}}{a_0} K_{I,\cdot j} &= 0, \quad i \neq j. \end{aligned} \tag{2.131}$$

These constraints are then included by augmenting $\hat{\mathcal{J}}_S$ with a Lagrangian function,

$$\begin{aligned} \mathcal{J}_{S,\lambda} = \hat{\mathcal{J}}_S \quad & + \quad \lambda_{11}\left(\frac{b_{0,1\cdot}}{a_0}K_{I\cdot 1} - \frac{b_{0,r1}}{a_{0,r1}}\right) + \lambda_{12}\left(\frac{b_{0,1\cdot}}{a_0}K_{I\cdot 2}\right) + \cdots + \lambda_{1p}\left(\frac{b_{0,1\cdot}}{a_0}K_{I\cdot p}\right) \\ & + \quad \lambda_{21}\left(\frac{b_{0,2\cdot}}{a_0}K_{I\cdot 1}\right) + \lambda_{22}\left(\frac{b_{0,2\cdot}}{a_0}K_{I\cdot 2} - \frac{b_{0,r2}}{a_{0,r2}}\right) + \cdots + \lambda_{2p}\left(\frac{b_{0,2\cdot}}{a_0}K_{I\cdot p}\right) \\ & \quad \vdots \\ & + \quad \lambda_{p1}\left(\frac{b_{0,p\cdot}}{a_0}K_{I\cdot 1}\right) + \lambda_{p2}\left(\frac{b_{0,p\cdot}}{a_0}K_{I\cdot 2}\right) + \cdots + \lambda_{pp}\left(\frac{b_{0,p\cdot}}{a_0}K_{I\cdot p} - \frac{b_{0,rp}}{a_{0,rp}}\right). \end{aligned} \tag{2.132}$$

2.2.6 Step response minimization

The next step in finding the PID coefficients is minimizing the Lagrangian step response cost function $\mathcal{J}_{S,\lambda}$. That is done by finding all partial derivatives of $\mathcal{J}_{S,\lambda}$ with respect to all $\mathcal{C}_{\cdot i}$ and all $\lambda_{\cdot i}$. Then we set them all to zero

$$\frac{\partial \mathcal{J}_{S,\lambda}}{\partial \mathcal{C}_{\cdot i}} = 0 \tag{2.133}$$

and

$$\frac{\partial \mathcal{J}_{S,\lambda}}{\partial \lambda_{\cdot i}} = 0. \tag{2.134}$$

Writing them out gives,

$$
\begin{aligned}
\frac{\partial \mathcal{J}_{S,\lambda}}{\partial \mathcal{C}_{\cdot 1}} &= 2\mathcal{G}_1^{(-1)}\mathcal{C}_{\cdot 1} - 2\mathcal{D}_1^{(-1)} + 2\omega_{21}\mathcal{G}_2^{(-1)}\mathcal{C}_{\cdot 1} + \cdots + 2\omega_{p1}\mathcal{G}_p^{(-1)}\mathcal{C}_{\cdot 1} \\
&\quad + \frac{1}{a_0}\left(b_{0,\cdot 1}u_1^T + b_{0,\cdot 2}u_4^T + \cdots + b_{0,\cdot p}u_{3p-2}^T\right)^T \lambda_{\cdot 1} = 0 \\
\frac{\partial \mathcal{J}_{S,\lambda}}{\partial \mathcal{C}_{\cdot 2}} &= 2\omega_{12}\mathcal{G}_1^{(-1)}\mathcal{C}_{\cdot 2} + 2\mathcal{G}_2^{(-1)}\mathcal{C}_{\cdot 2} - 2\mathcal{D}_2^{(-1)} + \cdots + 2\omega_{p2}\mathcal{G}_p^{(-1)}\mathcal{C}_{\cdot 2} \\
&\quad + \frac{1}{a_0}\left(b_{0,\cdot 1}u_1^T + b_{0,\cdot 2}u_4^T + \cdots + b_{0,\cdot p}u_{3p-2}^T\right)^T \lambda_{\cdot 2} = 0 \\
&\;\;\vdots \\
\frac{\partial \mathcal{J}_{S,\lambda}}{\partial \mathcal{C}_{\cdot p}} &= 2\omega_{1p}\mathcal{G}_1^{(-1)}\mathcal{C}_{\cdot p} + 2\omega_{2p}\mathcal{G}_2^{(-1)}\mathcal{C}_{\cdot p} + \cdots + 2\mathcal{G}_p^{(-1)}\mathcal{C}_{\cdot p} - 2\mathcal{D}_p^{(-1)} \qquad (2.135) \\
&\quad + \frac{1}{a_0}\left(b_{0,\cdot 1}u_1^T + b_{0,\cdot 2}u_4^T + \cdots + b_{0,\cdot p}u_{3p-2}^T\right)^T \lambda_{\cdot p} = 0 \\
\frac{\partial \mathcal{J}_{S,\lambda}}{\partial \lambda_{\cdot 1}} &= \frac{1}{a_0}\left(b_{0,\cdot 1}u_1^T + b_{0,\cdot 2}u_4^T + \cdots + b_{0,\cdot p}u_{3p-2}^T\right)\mathcal{C}_{\cdot 1} - \begin{bmatrix} \frac{b_{0,r1}}{a_{0,r1}} \\ 0 \\ 0 \end{bmatrix} = 0 \\
\frac{\partial \mathcal{J}_{S,\lambda}}{\partial \lambda_{\cdot 2}} &= \frac{1}{a_0}\left(b_{0,\cdot 1}u_1^T + b_{0,\cdot 2}u_4^T + \cdots + b_{0,\cdot h}u_{3p-2}^T\right)\mathcal{C}_{\cdot 2} - \begin{bmatrix} \frac{b_{0,r2}}{a_{0,r2}} \\ 0 \\ 0 \end{bmatrix} = 0 \\
&\;\;\vdots \\
\frac{\partial \mathcal{J}_{S,\lambda}}{\partial \lambda_{\cdot p}} &= \frac{1}{a_0}\left(b_{0,\cdot 1}u_1^T + b_{0,\cdot 2}u_4^T + \cdots + b_{0,\cdot p}u_{3p-2}^T\right)\mathcal{C}_{\cdot p} - \begin{bmatrix} \frac{b_{0,rp}}{a_{0,rp}} \\ 0 \\ 0 \end{bmatrix} = 0.
\end{aligned}
$$

As in the three I/O system, the partial derivatives in Equation (2.135) can be written in the very compact matrix form as

$$\begin{bmatrix} \left[\mathcal{G}_1^{(-1)}+\omega_{21}\mathcal{G}_2^{(-1)}+\cdots+\omega_{p1}\mathcal{G}_p^{(-1)}\right]_{3p\times 3p} & \frac{1}{a_0}\left[b_{\cdot 1,0}u_1^T+b_{\cdot 2,0}u_4^T+\cdots+b_{\cdot p,0}u_{3p-2}^T\right]_{3p\times p}^T \\ \frac{1}{a_0}\left[b_{\cdot 1,0}u_1^T+b_{\cdot 2,0}u_4^T+\cdots+b_{\cdot p,0}u_{3p-2}\right]_{p\times 3p} & [0]_{p\times p} \end{bmatrix} \begin{bmatrix} [C_{\cdot 1}]_{3p\times 1} \\ [\lambda_{\cdot 1}]_{p\times 1} \end{bmatrix}$$

$$= \begin{bmatrix} \left[\mathcal{D}_1^{(-1)}\right]_{3p\times 1} \\ \begin{bmatrix} \frac{b_{0,r1}}{a_{0,r1}} \\ 0 \\ \vdots \\ 0 \end{bmatrix}_{p\times 1} \end{bmatrix}$$

$$\begin{bmatrix} \omega_{12}\mathcal{G}_1^{(-1)}+\mathcal{G}_2^{(-1)}+\cdots+\omega_{p2}\mathcal{G}_p^{(-1)} & \frac{1}{a_0}\left[b_{\cdot 1,0}u_1^T+b_{\cdot 2,0}u_4^T+\cdot+b_{\cdot p,0}u_{3p-2}\right]^T \\ \frac{1}{a_0}\left[b_{\cdot 1,0}u_1^T+b_{\cdot 2,0}u_4^T+\cdot+b_{\cdot p,0}u_{3p-2}\right] & 0 \end{bmatrix} \begin{bmatrix} C_{\cdot 2} \\ \lambda_{\cdot 2} \end{bmatrix} = \begin{bmatrix} \mathcal{D}_2^{(-1)} \\ 0 \\ \frac{b_{0,r2}}{a_{0,r}} \\ 0 \\ \vdots \\ 0 \end{bmatrix} \qquad (2.136)$$

$$\vdots$$

$$\begin{bmatrix} \omega_{1p}\mathcal{G}_1^{(-1)}+\omega_{2p}\mathcal{G}_2^{(-1)}+\cdots+\mathcal{G}_p^{(-1)} & \frac{1}{a_0}\left[b_{\cdot 1,0}u_1^T+b_{\cdot 2,0}u_4^T+\cdot+b_{\cdot p,0}u_{3p-2}\right]^T \\ \frac{1}{a_0}\left[b_{\cdot 1,0}u_1^T+b_{\cdot 2,0}u_4^T+\cdot+b_{\cdot p,0}u_{3p-2}\right] & 0 \end{bmatrix} \begin{bmatrix} C_{\cdot p} \\ \lambda_{\cdot p} \end{bmatrix} = \begin{bmatrix} \mathcal{D}_p^{(-1)} \\ 0 \\ \vdots \\ \frac{b_{0,rp}}{a_{0,r3}} \end{bmatrix}$$

These matrices can be used to find a MIMO PID controller for a linear MIMO system with a square TFM and results in a close to a decoupled system. See the Appendix for the Matlab code use to set up these matrices and to solve the resulting linear system of equations.

Chapter 3

Non square systems

In Chapter 2 a MIMO controller was designed for a system with p I/O. We now assume that the system does not have equal numbers of control inputs and outputs, rather that it has r control inputs and p outputs. We want to design an optimized MIMO PID controller that makes the controlled closed loop system in Figure (3.1) behave like the closed loop reference system in Figure (3.2).

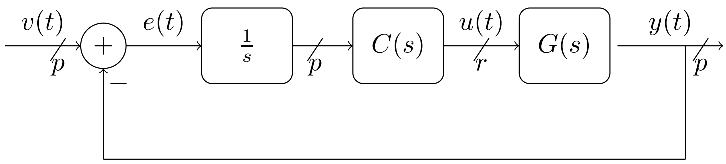

Figure 3.1: A controlled closed loop MIMO system with r control inputs and p outputs.

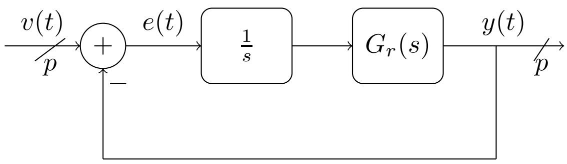

Figure 3.2: A MIMO reference system.

3.1 Systems with three control inputs and two outputs

We begin by looking at a system with three control inputs ($r = 3$) and two outputs ($p = 2$), which has the following TFM

$$G(s) = \begin{bmatrix} G_{11}(s) & G_{12}(s) & G_{13}(s) \\ G_{21}(s) & G_{22}(s) & G_{23}(s) \end{bmatrix} \tag{3.1}$$

The controller $C(s)$ is then chosen as

$$C(s) = \begin{bmatrix} c_{11}(s) & c_{12}(s) \\ c_{21}(s) & c_{22}(s) \\ c_{31}(s) & c_{32}(s) \end{bmatrix}. \tag{3.2}$$

With this controller the open loop system $G(s)C(s)$ is a square system with a 2×2 TFM given by

$$G(s)C(s) =$$

$$\begin{bmatrix} G_{11}(s)c_{11}(s) + G_{12}(s)c_{21}(s) + G_{13}(s)c_{31}(s) & G_{11}(s)c_{12}(s) + G_{12}(s)c_{22}(s) + G_{13}(s)c_{32}(s) \\ G_{21}(s)c_{11}(s) + G_{22}(s)c_{21}(s) + G_{23}(s)c_{31}(s) & G_{21}(s)c_{12}(s) + G_{22}(s)c_{22}(s) + G_{23}(s)c_{32}(s) \end{bmatrix}. \tag{3.3}$$

Since $G(s)C(s)$ has a square TFM the closed loop system will have the same numbers of reference inputs and outputs. In this case the closed loop controlled system will be a 2 I/O system. The open loop reference system has to have a TFM of the same size as the open loop system $G(s)C(s)$. Since we have a 2×2 TFM for $G(s)C(s)$ the reference system will also be a 2×2 TFM

$$G_r(s) = \begin{bmatrix} G_{r1}(s) & 0 \\ 0 & G_{r2}(s) \end{bmatrix}. \tag{3.4}$$

The TFM for the reference system is chosen to be on a diagonal form since we want reference input 1 only to effect output 1 and reference input 2 only to effect output 2. The transfer functions $G_{ij}(s)$, $G_{ri}(s)$ and $c_{ij}(s)$ are all defined in the same way as before as

$$G_{ij}(s) = \frac{b_{ij}(s)}{a(s)} = \frac{b_{m_{ij},ij}s^{m_{ij}} + b_{m_{ij}-1,ij}s^{m_{ij}-1} + \cdots + b_{1,ij}s + b_{0,ij}}{s^n + a_{n-1}s^{n-1} + \cdots + a_0}, \tag{3.5}$$

$$G_{ri}(s) = \frac{b_{m_{ri},ri}s^{m_{ri}} + b_{m_{ri}-1,ri}s^{m_{ri}-1} + \cdots + b_{1,ri}s + b_{0,ri}}{s^{n_{ri}} + a_{n_{ri}-1,ri}s^{n_{ri}-1} + \cdots + a_{0,ri}}, \tag{3.6}$$

$$G_{ri} = \frac{b_{ri}(s)}{a_{ri}(s)} = \frac{\omega_{ri}^2}{s + 2\zeta_{ri}\omega_{ri}}, \tag{3.7}$$

and

$$c_{ij}(s) = K_{Dij}s^2 + K_{Pij}s + K_{Iij}. \tag{3.8}$$

The same applies to the nonsquare systems and to the square systems, the transfer functions $G_{ij}(s)$ have to satisfy $m_{ij} + 2 < n$ and G_{r_i} have to satisfy $m_{r_i} = n_{ri} - 1$.

3.1.1 Impulse response

The impulse responses for the TFM $G(s)C(s)$ are found in the same way as before. The vectors $\mathcal{C}_{\cdot i}$ and $\mathcal{B}_{j\cdot}$ can be written as

$$\mathcal{C}_{\cdot i} = \begin{bmatrix} \mathcal{C}_{1i} \\ \mathcal{C}_{2i} \\ \mathcal{C}_{3i} \end{bmatrix}_{3r\times 1} \tag{3.9}$$

$$\mathcal{B}_{j\cdot} = \begin{bmatrix} \mathcal{B}_{j1} & \mathcal{B}_{j2} & \mathcal{B}_{j3} \end{bmatrix}_{n\times 3r}, \tag{3.10}$$

where C_{ij} is a vector with the PID coefficients and B_{ij} is the convolution matrix both defined as before. Then we can write

$$\mathcal{C}_{\cdot i}^T \mathcal{B}_{j\cdot}^T = \mathcal{C}_{1i}^T \mathcal{B}_{j1}^T + \mathcal{C}_{2i}^T \mathcal{B}_{j2}^T + \mathcal{C}_{3i}^T \mathcal{B}_{j3}^T. \tag{3.11}$$

The impulse response is

$$\begin{aligned}
y_{I11} &= \mathscr{L}^{-1}\left\{ \frac{1}{a(s)} \begin{bmatrix} b_{11}(s) & b_{12}(s) & b_{13}(s) \end{bmatrix} \begin{bmatrix} c_{11}(s) \\ c_{21}(s) \\ c_{31}(s) \end{bmatrix} \right\} \\
&= \mathscr{L}^{-1}\left\{ \frac{b_{11}(s)c_{11}(s) + b_{12}(s)c_{21}(s) + b_{13}c_{31}(s)}{a(s)} \right\} \\
&= \left[\mathcal{C}_{11}^T \mathcal{B}_{11}^T + \mathcal{C}_{21}^T \mathcal{B}_{12}^T + \mathcal{C}_{31}^T \mathcal{B}_{13}^T\right] Y_b(t) = \mathcal{C}_{\cdot 1}^T \mathcal{B}_{1\cdot}^T Y_b(t) \\
y_{I12} &= \mathcal{C}_{\cdot 2}^T \mathcal{B}_{1\cdot}^T Y_b(t) \\
y_{I21} &= \mathcal{C}_{\cdot 1}^T \mathcal{B}_{2\cdot}^T Y_b(t) \\
y_{I22} &= \mathcal{C}_{\cdot 2}^T \mathcal{B}_{2\cdot}^T Y_b(t)
\end{aligned} \tag{3.12}$$

The impulse response for the reference system is given by

$$\begin{aligned}
y_{Ir1}(t) &= \mathcal{B}_{r1}^T Y_{b,r1}(t) \\
y_{Ir2}(t) &= \mathcal{B}_{r2}^T Y_{b,r2}(t),
\end{aligned} \tag{3.13}$$

where $Y_{br1}(t)$ and $Y_{br2}(t)$ are defined as

$$\begin{aligned}
Y_{br1}(t) &= \begin{bmatrix} y_{br1}(t) & y_{br1}^{'}(t) & \cdots & y_{br1}^{n_{r1}-1}(t) \end{bmatrix}^T \\
Y_{br2}(t) &= \begin{bmatrix} y_{br2}(t) & y_{br2}^{'}(t) & \cdots & y_{br2}^{n_{r2}-1}(t) \end{bmatrix}^T
\end{aligned} \tag{3.14}$$

3.1.2 Impulse response cost function

With the impulse response known for both the closed loop system and the reference system it is possible to set up the cost function as

$$\begin{aligned}\mathcal{J}_I = \textstyle\int_0^\infty [(y_{I11} - y_{Ir1})^2 &+ \omega_{12}(y_{I12} - 0)^2 \\ +\omega_{21}(y_{I21} - 0)^2 &+ (y_{I22} - y_{Ir2})^2]dt\end{aligned} \tag{3.15}$$

where the ω_{ij} constants are weight coefficients controlling the decoupling as before. Further, the following matrices do not change

$$\mathcal{A}_{n\times n} = \int_0^\infty Y_b(t)Y_b^T(t)dt, \tag{3.16}$$

$$\mathcal{G}_{i(9\times 9)} = \begin{bmatrix} \mathcal{B}_{i1}^T \\ \mathcal{B}_{i2}^T \\ \mathcal{B}_{i3}^T \end{bmatrix} \mathcal{A} \begin{bmatrix} \mathcal{B}_{i1} & \mathcal{B}_{i2} & \mathcal{B}_{i3} \end{bmatrix} = \mathcal{B}_{i\cdot}^T \mathcal{A}\mathcal{B}_{i\cdot}, \tag{3.17}$$

$$\mathcal{H}_{i(n\times n_{ri})} = \int_0^\infty Y_b(t)Y_{b,ri}^T(t)dt, \tag{3.18}$$

$$\mathcal{D}_{i(9\times 1)} = \mathcal{B}_{i\cdot}^T \mathcal{H}_i \mathcal{B}_{ri}, \tag{3.19}$$

$$\mathcal{A}_{ri(n_{ri}\times n_{ri})} = \int_0^\infty Y_{b,ri}(t)Y_{b,ri}^T(t)dt, \tag{3.20}$$

and

$$\mathcal{M}_{i(1\times 1)} = \mathcal{B}_{ri}^T \mathcal{A}_{ri} \mathcal{B}_{ri}, \tag{3.21}$$

Parts of the cost function can be written as

$$\begin{aligned}\int_0^\infty (y_{I11}(t) - y_{Ir1}(t))^2\, dt &= \mathcal{C}_{\cdot 1}^T \mathcal{G}_1 \mathcal{C}_{\cdot 1} - 2\mathcal{C}_{\cdot 1}^T \mathcal{D}_1 + \mathcal{M}_1 \\ \omega_{12}\int_0^\infty (y_{I12}(t) - 0)^2\, dt &= \omega_{12}\mathcal{C}_{\cdot 2}^T \mathcal{G}_1 \mathcal{C}_{\cdot 2} \\ \omega_{21}\int_0^\infty (y_{I21}(t) - 0)^2\, dt &= \omega_{21}\mathcal{C}_{\cdot 1}^T \mathcal{G}_2 \mathcal{C}_{\cdot 1} \\ \int_0^\infty (y_{I22}(t) - y_{Ir2}(t))^2\, dt &= \mathcal{C}_{\cdot 2}^T \mathcal{G}_2 \mathcal{C}_{\cdot 2} - 2\mathcal{C}_{\cdot 2}^T \mathcal{D}_2 + \mathcal{M}_2\end{aligned} \tag{3.22}$$

3.1.3 Impulse response minimization

The minimization is done by finding the partial derivatives of the cost function with respect to the PID controller coefficients and setting them equal to zero,

$$\frac{\partial \mathcal{J}_I}{\partial \mathcal{C}_{\cdot i}} = 0. \tag{3.23}$$

Writing them out gives

$$\begin{aligned}\frac{\partial \mathcal{J}_I}{\partial \mathcal{C}_{\cdot 1}} &= 2\mathcal{G}_1\mathcal{C}_{\cdot 1} - 2\mathcal{D}_1 + 2\omega_{21}\mathcal{G}_2\mathcal{C}_{\cdot 1} = 0, \\ \frac{\partial \mathcal{J}_I}{\partial \mathcal{C}_{\cdot 2}} &= 2\mathcal{G}_2\mathcal{C}_{\cdot 2} - 2\mathcal{D}_2 + 2\omega_{12}\mathcal{G}_1\mathcal{C}_{\cdot 2} = 0,\end{aligned} \tag{3.24}$$

resulting in the following linear equations

$$\begin{aligned}(\mathcal{G}_1+\omega_{21}\mathcal{G}_2)\,\mathcal{C}_{\cdot 1} &= \mathcal{D}_1\\ (\omega_{12}\mathcal{G}_1+\mathcal{G}_2)\,\mathcal{C}_{\cdot 2} &= \mathcal{D}_2\end{aligned} \tag{3.25}$$

This determines all six PID controller coefficients.

3.1.4 Step response

We define the step response for the transfer function ij in the open loop TFM $G(s)C(s)$ as

$$\begin{aligned}y_{S,ij} &= \int_0^t y_{I,ij}(u)du\\ &= \mathcal{C}_{\cdot j}^T\mathcal{B}_{i\cdot}^T\int_0^t Y_b(u)du\\ &= \hat{y}_{S,ij}(t)+\overline{y}_{S,ij}(t),\end{aligned} \tag{3.26}$$

with $\hat{y}_{S,ij}(t)$ and $\overline{y}_{S,ij}(t)$ denoting the transient and stationary parts, respectively. They are defined in the same way as before as

$$\hat{y}_{S,ij}(t)=\mathcal{C}_{\cdot j}^T\mathcal{B}_{i\cdot}^T Y_b^{(-1)}(t) \tag{3.27}$$

and

$$\overline{y}_{S,ij}(t)=\mathcal{C}_{\cdot j}^T\mathcal{B}_{i\cdot}^T\left(\left[\begin{array}{cccc}\frac{1}{a_0} & 0 & \cdots & 0\end{array}\right]_{1\times n}\right)^T=\frac{b_{0,i\cdot}}{a_0}K_{I,\cdot j} \tag{3.28}$$

where $b_{0,i\cdot}$ and $K_{I,\cdot j}$ are defined as

$$K_{I,\cdot i}=\left[\begin{array}{ccc}K_{I,1i} & K_{I,2i} & K_{I,3i}\end{array}\right]_{3\times 1}^T \tag{3.29}$$

and

$$b_{0,i\cdot}=\left[\begin{array}{ccc}b_{0,i1} & b_{0,i2} & b_{0,i3}\end{array}\right]_{1\times 3}. \tag{3.30}$$

$y_b^{(-1)}(t)$, $Y_b^{(-1)}(t)$, $\hat{y}_{S,ri}(t)$, $y_{b,ri}^{(-1)}(t)$ and $Y_{b,ri}^{(-1)}(t)$ are all defined as before in Equations (2.60), (2.61), (2.67), (2.69) and (2.70). Finally we define $\mathcal{A}^{(-1)}$, $\mathcal{G}_i^{(-1)}$, $\mathcal{M}_i^{(-1)}$ and $\mathcal{H}_i^{(-1)}$ as

$$\mathcal{A}_{n\times n}^{(-1)}=\int_0^\infty Y_b^{(-1)}(t)Y_b^{(-1)}(t)^T dt \tag{3.31}$$

$$\mathcal{G}_{i(9\times 9)}^{(-1)}=\left[\begin{array}{c}B_{i1}^T\\ B_{i2}^T\\ B_{i3}^T\end{array}\right]\mathcal{A}^{(-1)}\left[\begin{array}{ccc}B_{i1} & B_{i2} & B_{i3}\end{array}\right]=\mathcal{B}_{i\cdot}^T\mathcal{A}^{(-1)}\mathcal{B}_{i\cdot} \tag{3.32}$$

$$\mathcal{H}^{(-1)}_{i(n\times n_{ri})} = \int_0^\infty Y_b^{(-1)}(t) Y_{b,ri}^{(-1)}(t)^T dt, \tag{3.33}$$

$$\mathcal{D}^{(-1)}_{i(9\times 1)} = \mathcal{B}_{i\cdot}^T \mathcal{H}_i^{(-1)} B_{ri}, \tag{3.34}$$

$$\mathcal{A}^{(-1)}_{ri((n_{ri}-1)\times(n_{ri}-1))} = \int_0^\infty Y_{b,ri}^{(-1)}(t) Y_{b,ri}^{(-1)}(t)^T dt, \tag{3.35}$$

and

$$\mathcal{M}^{(-1)}_{i(1\times 1)} = \mathcal{B}_{ri}^T \mathcal{A}_{ri}^{(-1)} \mathcal{B}_{ri}. \tag{3.36}$$

3.1.5 Step response cost function

The cost function is split up into the transient and stationary part in the same way as for square systems. The transient part is

$$\begin{array}{rcl} \hat{\mathcal{J}}_S = \int_0^\infty [(\hat{y}_{S11} - \hat{y}_{Sr1})^2 & + & \omega_{12}(\hat{y}_{S12} - 0)^2 \\ +\omega_{21}(\hat{y}_{S21} - 0)^2 & + & (\hat{y}_{S22} - \hat{y}_{Sr2})^2] dt \end{array}, \tag{3.37}$$

where the parts of the cost function are given by

$$\begin{aligned} \int_0^\infty (\hat{y}_{S11}(t) - \hat{y}_{Sr1}(t))^2\, dt &= \mathcal{C}_{\cdot 1}^T \mathcal{G}_1^{(-1)} \mathcal{C}_{\cdot 1} - 2\mathcal{C}_{\cdot 1}^T \mathcal{D}_1^{(-1)} + \mathcal{M}_1^{(-1)} \\ \omega_{12} \int_0^\infty (\hat{y}_{S12}(t) - 0)^2\, dt &= \omega_{12} \mathcal{C}_{\cdot 2}^T \mathcal{G}_1^{(-1)} \mathcal{C}_{\cdot 2} \\ \omega_{21} \int_0^\infty (\hat{y}_{S21}(t) - 0)^2\, dt &= \omega_{21} \mathcal{C}_{\cdot 1}^T \mathcal{G}_2^{(-1)} \mathcal{C}_{\cdot 1} \\ \int_0^\infty (\hat{y}_{S22}(t) - \hat{y}_{Sr2}(t))^2\, dt &= \mathcal{C}_{\cdot 2}^T \mathcal{G}_2^{(-1)} \mathcal{C}_{\cdot 2} - 2\mathcal{C}_{\cdot 2}^T \mathcal{D}_2^{(-1)} + \mathcal{M}_2^{(-1)}. \end{aligned} \tag{3.38}$$

In order for this cost function to be finite the stationary part has to be the same for the controlled system and the reference system. This means we have the following constraints

$$\begin{aligned} \frac{b_{0,r1}}{a_{0,r1}} &= \frac{b_{0,11}}{a_0} K_{I11} + \frac{b_{0,12}}{a_0} K_{I21} + \frac{b_{0,13}}{a_0} K_{I31} = \frac{b_{0,1\cdot}}{a_0} K_{I,\cdot 1} \\ 0 &= \frac{b_{0,1\cdot}}{a_0} K_{I\cdot 2} \\ 0 &= \frac{b_{0,2\cdot}}{a_0} K_{I\cdot 1} \\ \frac{b_{0,r2}}{a_{0,r2}} &= \frac{b_{0,2\cdot}}{a_0} K_{I\cdot 2} \end{aligned} \tag{3.39}$$

These constaints are included in the same way as before by augmenting $\hat{\mathcal{J}}_S$ with a Lagrangian function,

$$\begin{aligned} \mathcal{J}_{S,\lambda} = \hat{\mathcal{J}}_S &+ \lambda_{11} \left(\frac{b_{0,1\cdot}}{a_0} K_{I\cdot 1} - \frac{b_{0,r1}}{a_{0,r1}} \right) + \lambda_{12} \left(\frac{b_{0,1\cdot}}{a_0} K_{I\cdot 2} \right) \\ &+ \lambda_{21} \left(\frac{b_{0,2\cdot}}{a_0} K_{I\cdot 1} \right) + \lambda_{22} \left(\frac{b_{0,2\cdot}}{a_0} K_{I\cdot 2} - \frac{b_{0,r2}}{a_{0,r2}} \right). \end{aligned} \tag{3.40}$$

3.1.6 Step response minimization

Minimizing the cost function $\mathcal{J}_{S,\lambda}$ is done in the same way as before by finding all partial derivatives and setting them equal to zero

$$\begin{aligned} \frac{\partial \mathcal{J}_{S,\lambda}}{\partial \mathcal{C}_{\cdot i}} &= 0 \\ \frac{\partial \mathcal{J}_{S,\lambda}}{\partial \lambda_{\cdot i}} &= 0. \end{aligned} \tag{3.41}$$

These derivatives can be written as

$$\begin{aligned} \frac{\partial \mathcal{J}_{S,\lambda}}{\partial \mathcal{C}_{\cdot 1}} &= 2\mathcal{G}_1^{(-1)}\mathcal{C}_{\cdot 1} - 2\mathcal{D}_1^{(-1)} + 2\omega_{21}\mathcal{G}_2^{(-1)}\mathcal{C}_{\cdot 1} \\ &\quad + \frac{1}{a_0}\left(b_{0,\cdot 1}u_1^T + b_{0,\cdot 2}u_4^T + b_{0,\cdot 3}u_7^T\right)\lambda_{\cdot 1} = 0 \\ \frac{\partial \mathcal{J}_{S,\lambda}}{\partial \mathcal{C}_{\cdot 2}} &= 2\omega_{12}\mathcal{G}_1^{(-1)}\mathcal{C}_{\cdot 2} + 2\mathcal{G}_2^{(-1)}\mathcal{C}_{\cdot 2} - 2\mathcal{D}_2^{(-1)} \\ &\quad + \frac{1}{a_0}\left(b_{0,\cdot 1}u_1^T + b_{0,\cdot 2}u_4^T + b_{0,\cdot 3}u_7^T\right)\lambda_{\cdot 2} = 0 \\ \frac{\partial \mathcal{J}_{S,\lambda}}{\partial \lambda_{\cdot 1}} &= \frac{1}{a_0}\left(b_{0,\cdot 1}u_1^T + b_{0,\cdot 2}u_4^T + b_{0,\cdot 3}u_7^T\right)\mathcal{C}_{\cdot 1} - \begin{bmatrix} \frac{b_{0,r1}}{a_{0,r1}} \\ 0 \end{bmatrix} = 0 \\ \frac{\partial \mathcal{J}_{S,\lambda}}{\partial \lambda_{\cdot 2}} &= \frac{1}{a_0}\left(b_{0,\cdot 1}u_1^T + b_{0,\cdot 2}u_4^T + b_{0,\cdot 3}u_7^T\right)\mathcal{C}_{\cdot 2} - \begin{bmatrix} 0 \\ \frac{b_{0,r2}}{a_{0,r2}} \end{bmatrix} = 0. \end{aligned} \tag{3.42}$$

Here u_k is a unit column vector of size 9 with the k-th element as 1. Equation (3.42) can then be rewritten in a compact matrix form as

$$\begin{aligned} &\begin{bmatrix} \left[\mathcal{G}_1^{(-1)} + \omega_{21}\mathcal{G}_2^{(-1)}\right]_{9\times 9} & \frac{1}{a_0}\left[b_{0,1\cdot}u_1^T + b_{0,2\cdot}u_4^T + b_{0,3\cdot}u_7^T\right]^T_{9\times 2} \\ \frac{1}{a_0}\left[b_{0,1\cdot}u_1^T + b_{0,2\cdot}u_4^T + b_{0,3\cdot}u_7^T\right]_{2\times 9} & \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}_{2\times 2} \end{bmatrix} \begin{bmatrix} [C_{\cdot 1}]_{9\times 1} \\ [\lambda_{\cdot 1}]_{2\times 1} \end{bmatrix} \\ &\qquad = \begin{bmatrix} \left[\mathcal{D}_1^{(-1)}\right]_{9\times 1} \\ \begin{bmatrix} \frac{b_{0,r1}}{a_{0,r1}} \\ 0 \end{bmatrix}_{2\times 1} \end{bmatrix} \\ &\begin{bmatrix} \omega_{12}\mathcal{G}_1^{(-1)} + \mathcal{G}_2^{(-1)} & \frac{1}{a_0}\left[b_{0,1\cdot}u_1^T + b_{0,2\cdot}u_4^T + b_{0,3\cdot}u_7^T\right]^T \\ \frac{1}{a_0}\left[b_{0,1\cdot}u_1^T + b_{0,2\cdot}u_4^T + b_{0,3\cdot}u_7^T\right] & 0 \end{bmatrix} \begin{bmatrix} C_{\cdot 2} \\ \lambda_{\cdot 2} \end{bmatrix} \\ &\qquad = \begin{bmatrix} \mathcal{D}_2^{(-1)} \\ 0 \\ \frac{b_{0,r2}}{a_{0,r2}} \end{bmatrix} \end{aligned} \tag{3.43}$$

3.2 General case: r control inputs and p outputs

We now look at the general case with r control inputs and p outputs where the TFM $G(s)$ is given by

$$G(s) = \begin{bmatrix} G_{11}(s) & G_{12}(s) & \cdots & G_{1r}(s) \\ \vdots & & & \\ G_{p1}(s) & G_{p2}(s) & \cdots & G_{pr}(s) \end{bmatrix}. \tag{3.44}$$

The TFM for the controller is a $r \times p$ matrix

$$\frac{1}{s}C(s) = \frac{1}{s} \begin{bmatrix} c_{11}(s) & c_{12}(s) & \cdots & c_{1p}(s) \\ \vdots & & & \\ c_{r1}(s) & c_{r2}(s) & \cdots & c_{rp}(s) \end{bmatrix}. \tag{3.45}$$

Then $G(s)C(s)$ will be a TFM of size $p \times p$, and the reference system will be a TFM of size $p \times p$

$$G_r(s) = \begin{bmatrix} G_{r1}(s) & 0 & \cdots & 0 \\ 0 & G_{r2}(s) & \cdots & 0 \\ \vdots & & \ddots & \\ 0 & 0 & \cdots & G_{rp} \end{bmatrix}. \tag{3.46}$$

3.2.1 Impulse response

Before writing out the impulse response, we need to define $\mathcal{C}_{\cdot i}$ and $\mathcal{B}_{j\cdot}$ as

$$\mathcal{C}_{\cdot i} = \begin{bmatrix} \mathcal{C}_{1i} \\ \mathcal{C}_{2i} \\ \vdots \\ \mathcal{C}_{ri} \end{bmatrix}_{3r \times 1} \tag{3.47}$$

$$\mathcal{B}_{j\cdot} = \begin{bmatrix} \mathcal{B}_{j1} & \mathcal{B}_{j2} & \cdots & \mathcal{B}_{jr} \end{bmatrix}_{n \times 3r}, \tag{3.48}$$

where $\mathcal{C}_{ij}$ is a vector with the PID coefficients and $\mathcal{B}_{ij}$ is a convolution matrix defined as before. Now it is possible to write

$$\mathcal{C}_{\cdot i}^T \mathcal{B}_{j\cdot}^T = \mathcal{C}_{1i}^T \mathcal{B}_{j1}^T + \mathcal{C}_{2i}^T \mathcal{B}_{j2}^T + \cdots + \mathcal{C}_{ri}^T \mathcal{B}_{jr}^T. \tag{3.49}$$

with $y_{b,ri}$ and $Y_{b,ri}(t)$ defined in the same way as before.

$$
\begin{aligned}
y_{I11} &= \mathscr{L}^{-1}\left\{ \frac{1}{a(s)} \begin{bmatrix} b_{11}(s) & b_{12}(s) & \cdots & b_{1r}(s) \end{bmatrix} \begin{bmatrix} c_{11}(s) \\ c_{21}(s) \\ \vdots \\ c_{r1}(s) \end{bmatrix} \right\} \\
&= \mathscr{L}^{-1}\left\{ \frac{b_{11}(s)c_{11}(s) + b_{12}(s)c_{21}(s) + \cdots + b_{1r}c_{r1}(s)}{a(s)} \right\} \\
&= \left[\mathcal{C}_{11}^T\mathcal{B}_{11}^T + \mathcal{C}_{21}^T\mathcal{B}_{12}^T + \cdots + \mathcal{C}_{r1}^T\mathcal{B}_{1r}^T\right] Y_b(t) = \mathcal{C}_{\cdot 1}^T\mathcal{B}_{1\cdot}^T Y_b(t) \\
y_{I12} &= \mathcal{C}_{\cdot 2}^T\mathcal{B}_{1\cdot}^T Y_b(t) \\
&\vdots \\
y_{I1p} &= \mathcal{C}_{\cdot p}^T\mathcal{B}_{1\cdot}^T Y_b(t) \\
y_{I21} &= \mathcal{C}_{\cdot 1}^T\mathcal{B}_{2\cdot}^T Y_b(t) \\
&\vdots \\
y_{I2p} &= \mathcal{C}_{\cdot p}^T\mathcal{B}_{2\cdot}^T Y_b(t) \\
&\vdots \\
y_{Ip1} &= \mathcal{C}_{\cdot 1}^T\mathcal{B}_{p\cdot}^T Y_b(t) \\
&\vdots \\
y_{Ipp} &= \mathcal{C}_{\cdot p}^T\mathcal{B}_{p\cdot}^T Y_b(t).
\end{aligned}
\tag{3.50}
$$

Both $y_b(t)$ and $Y_b(t)$ are defined in the same way as before. The impulse response for the reference system is given by

$$
\begin{aligned}
y_{Ir1} &= \mathcal{B}_{r1}^T Y_{b,r1}(t) \\
y_{Ir2} &= \mathcal{B}_{r2}^T Y_{b,r2}(t) \\
&\vdots \\
y_{Irp} &= \mathcal{B}_{rp}^T Y_{b,rp}(t)
\end{aligned}
\tag{3.51}
$$

and

$$
\begin{aligned}
\mathcal{B}_{r1}^T &= \begin{bmatrix} b_{0,r1} & \cdots & b_{n_{r1}-1,r1} \end{bmatrix} \\
\mathcal{B}_{r2}^T &= \begin{bmatrix} b_{0,r2} & \cdots & b_{n_{r2}-1,r2} \end{bmatrix} \\
&\vdots \\
\mathcal{B}_{rp}^T &= \begin{bmatrix} b_{0,rp} & \cdots & b_{n_{rp}-1,rp} \end{bmatrix}.
\end{aligned}
\tag{3.52}
$$

3.2.2 Impulse response cost function

Consider the cost function

$$\begin{aligned}
\mathcal{J} &= \int_0^\infty [(y_{I11}(t) - y_{Ir1}(t))^2 + \omega_{12}(y_{I12}(t) - 0)^2 + \cdots + \omega_{1p}(y_{I1p}(t) - 0)^2 \\
&+ \omega_{21}(y_{I21}(t) - 0)^2 + (y_{I22}(t) - y_{Ir2}(t))^2 + \cdots + \omega_{2p}(y_{I2p}(t) - 0)^2 \\
&\vdots \\
&+ \omega_{p1}(y_{Ip1}(t) - 0)^2 + \omega_{p2}(y_{Ip2}(t) - 0)^2 + \cdots + (y_{Ipp}(t) - y_{Irp}(t))^2]dt
\end{aligned} \tag{3.53}$$

where the ω_{ij} constants are weight coefficients controlling the decoupling as before. Again, to simplify the notation, the following matrices are intoduced

$$\mathcal{A}_{n\times n} = \int_0^\infty Y_b(t)Y_b^T(t)dt, \tag{3.54}$$

$$\mathcal{G}_{i(3r\times 3r)} = \begin{bmatrix} \mathcal{B}_{i1}^T \\ \mathcal{B}_{i2}^T \\ \vdots \\ \mathcal{B}_{ir}^T \end{bmatrix} \mathcal{A} \begin{bmatrix} \mathcal{B}_{i1} & \mathcal{B}_{i2} & \cdots & \mathcal{B}_{ir} \end{bmatrix} = \mathcal{B}_{i\cdot}^T \mathcal{A} \mathcal{B}_{i\cdot}, \tag{3.55}$$

$$\mathcal{H}_{i(n\times(n_{ri}-1))} = \int_0^\infty Y_b(t)Y_{b,ri}^T(t)dt, \tag{3.56}$$

$$\mathcal{D}_{i(3r\times 1)} = \mathcal{B}_{i\cdot}^T \mathcal{H}_i \mathcal{B}_{ri}, \tag{3.57}$$

$$\mathcal{A}_{ri((n_{ri}-1)\times(n_{ri}-1))} = \int_0^\infty Y_{b,ri}(t)Y_{b,ri}^T(t)dt, \tag{3.58}$$

and

$$\mathcal{M}_{i(1\times 1)} = \mathcal{B}_{ri}^T \mathcal{A}_{ri} \mathcal{B}_{ri}. \tag{3.59}$$

Parts of the cost function can be written as

$$\begin{aligned}
\int_0^\infty (y_{I,ii}(t) - y_{I,ri}(t))^2 dt &= \mathcal{C}_{\cdot i}^T \mathcal{G}_i \mathcal{C}_{\cdot i} - 2\mathcal{C}_{\cdot i}^T \mathcal{D}_i + \mathcal{M}_i \\
\int_0^\infty (y_{I,ij}(t) - 0)^2 dt &= \omega_{ij} \mathcal{C}_{\cdot j}^T \mathcal{G}_i \mathcal{C}_{\cdot j}
\end{aligned} \tag{3.60}$$

3.2.3 Impulse response minimization

To minimize the impulse response all partial derivatives of the cost function with respect to the PID controller coefficents are set to zero

$$\frac{\partial \mathcal{J}_I}{\partial \mathcal{C}_{\cdot i}} = 0. \tag{3.61}$$

The derivatives are then written out as

$$\begin{aligned}
\frac{\partial \mathcal{J}_I}{\partial \mathcal{C}_{\cdot 1}} &= 2\mathcal{G}_1\mathcal{C}_{\cdot 1} + 2\omega_{21}\mathcal{G}_2\mathcal{C}_{\cdot 1} + \cdots + 2\omega_{p1}\mathcal{G}_p\mathcal{C}_{\cdot 1} - 2\mathcal{D}_1 = 0 \\
\frac{\partial \mathcal{J}_I}{\partial \mathcal{C}_{\cdot 2}} &= 2\omega_{12}\mathcal{G}_1\mathcal{C}_{\cdot 2} + 2\mathcal{G}_2\mathcal{C}_{\cdot 2} + \cdots + 2\omega_{p2}\mathcal{G}_p\mathcal{C}_{\cdot 2} - 2\mathcal{D}_2 = 0 \\
&\vdots \\
\frac{\partial \mathcal{J}_I}{\partial \mathcal{C}_{\cdot p}} &= 2\omega_{1p}\mathcal{G}_1\mathcal{C}_{\cdot p} + 2\omega_{2p}\mathcal{G}_2\mathcal{C}_{\cdot p} + \cdots + 2\mathcal{G}_p\mathcal{C}_{\cdot p} - 2\mathcal{D}_p = 0,
\end{aligned} \tag{3.62}$$

resulting in the following linear system of equations

$$\begin{aligned}
\left(\mathcal{G}_1 + \omega_{21}\mathcal{G}_2 + \cdots + \omega_{p1}\mathcal{G}_p\right)\mathcal{C}_{\cdot 1} &= \mathcal{D}_1 \\
\left(\omega_{12}\mathcal{G}_1 + \mathcal{G}_2 + \cdots + \omega_{p2}\mathcal{G}_p\right)\mathcal{C}_{\cdot 2} &= \mathcal{D}_2 \\
&\vdots \\
\left(\omega_{1p}\mathcal{G}_1 + \omega_{2p}\mathcal{G}_2 + \cdots + \mathcal{G}_p\right)\mathcal{C}_{\cdot p} &= \mathcal{D}_p
\end{aligned} \tag{3.63}$$

that determines the PID controllers coefficents, for all the $r \cdot p$ PID controller in the MIMO PID controllers.

3.2.4 Step response

The step response for the elements transfer functions in the TFM $G(s)C(s)$ is defined in the same way as before as

$$\begin{aligned}
y_{S,ij} &= \int_0^t y_{I,ij}(u)du \\
&= \mathcal{C}_{\cdot j}^T\mathcal{B}_{i\cdot}^T \int_0^t Y_b(u)du \\
&= \hat{y}_{S,ij}(t) + \overline{y}_{S,ij}(t),
\end{aligned} \tag{3.64}$$

again with $\hat{y}_{S,ij}(t)$ and $\overline{y}_{S,ij}(t)$ as the transient and stationary parts, respectively. They are given by

$$\hat{y}_{S,ij}(t) = \mathcal{C}_{\cdot j}^T\mathcal{B}_{i\cdot}^T Y_b^{(-1)}(t) \tag{3.65}$$

and

$$\overline{y}_{S,ij}(t) = \mathcal{C}_{\cdot j}^T\mathcal{B}_{i\cdot}^T \left(\begin{bmatrix} \frac{1}{a_0} & 0 & \cdots & 0 \end{bmatrix}_{1\times n}\right)^T = \frac{b_{0,i\cdot}}{a_0}K_{I,\cdot j} \tag{3.66}$$

with $b_{0,i\cdot}$ and $K_{I,\cdot j}$ given by

$$K_{I,\cdot i} = \begin{bmatrix} K_{I,1i} & K_{I,2i} & \cdots & K_{I,ri} \end{bmatrix} \tag{3.67}$$

and

$$b_{0,i\cdot} = \begin{bmatrix} b_{0,i1} & b_{0,i2} & \cdots & b_{0,ir} \end{bmatrix}. \tag{3.68}$$

The matrices $\mathcal{A}^{(-1)}_{n\times n}$, $\mathcal{G}^{(-1)}_{i(3r\times 3r)}$, $\mathcal{H}^{(-1)}_{i(n\times(n_{ri}-1))}$, $\mathcal{D}^{(-1)}_{i(3r\times 1)}$, $\mathcal{A}^{(-1)}_{ri((n_{ri}-1)\times(n_{ri}-1))}$ and $\mathcal{M}^{(-1)}_{i(1\times 1)}$ are defined as

$$\mathcal{A}^{(-1)}_{n\times n} = \int_0^\infty Y_b^{(-1)}(t)Y_b^{(-1)}(t)^T dt, \tag{3.69}$$

$$\mathcal{G}^{(-1)}_{i(3r\times 3r)} = \begin{bmatrix} B_{i1}^T \\ B_{i2}^T \\ \vdots \\ B_{ir}^T \end{bmatrix} \mathcal{A}^{(-1)} \begin{bmatrix} B_{i1} & B_{i2} & \cdots & B_{ir} \end{bmatrix} = \mathcal{B}_{i\cdot}^T \mathcal{A}^{(-1)} \mathcal{B}_{i\cdot} \tag{3.70}$$

$$\mathcal{H}^{(-1)}_{i(n\times(n_{ri}-1))} = \int_0^\infty Y_b^{(-1)}(t)Y_{b,ri}^{(-1)}(t)^T dt, \tag{3.71}$$

$$\mathcal{D}^{(-1)}_{i(3r\times 1)} = \mathcal{B}_{i\cdot}^T \mathcal{H}_i^{(-1)} B_{ri}, \tag{3.72}$$

$$\mathcal{A}^{(-1)}_{ri((n_{ri}-1)\times(n_{ri}-1))} = \int_0^\infty Y_{b,ri}^{(-1)}(t)Y_{b,ri}^{(-1)}(t)^T dt, \tag{3.73}$$

and

$$\mathcal{M}^{(-1)}_{i(1\times 1)} = \mathcal{B}_{ri}^T \mathcal{A}_{ri}^{(-1)} \mathcal{B}_{ri}. \tag{3.74}$$

3.2.5 Step response cost function

Like before we begin by looking at the transient part of the cost function

$$\begin{aligned}
\mathcal{J}_S &= \int_0^\infty [(\hat{y}_{S11}(t) - \hat{y}_{Sr1}(t))^2 + \omega_{12}(\hat{y}_{S12}(t) - 0)^2 + \cdots + \omega_{1p}(\hat{y}_{S1p}(t) - 0)^2 \\
&+ \omega_{21}(\hat{y}_{S21}(t) - 0)^2 + (\hat{y}_{S22}(t) - \hat{y}_{Sr2}(t))^2 + \cdots + \omega_{2p}(\hat{y}_{S2p}(t) - 0)^2 \\
&\vdots \\
&+ \omega_{p1}(\hat{y}_{Sp1}(t) - 0)^2 + \omega_{p2}(\hat{y}_{Sp2}(t) - 0)^2 + \cdots + (\hat{y}_{Spp}(t) - \hat{y}_{Srp}(t))^2]dt,
\end{aligned} \tag{3.75}$$

with $\int_0^\infty (\hat{y}_{Sii} - \hat{y}_{S,ri})^2 dt$ given by

$$\int_0^\infty (\hat{y}_{Sii} - \hat{y}_{S,ri})^2 dt = \mathcal{C}_{\cdot i}^T \mathcal{G}_i^{(-1)} \mathcal{C}_{\cdot i} - 2\mathcal{C}_{\cdot i}^T \mathcal{D}_i^{(-1)} + \mathcal{M}_i^{(-1)} \tag{3.76}$$

and $\int_0^\infty \omega_{ij}(\hat{y}_{Sij}(t) - 0)^2 dt$ given by

$$\int_0^\infty \omega_{ij}(\hat{y}_{Sij}(t) - 0)^2 dt = \omega_{ij}\mathcal{C}_{\cdot j}^T \mathcal{G}_i^{(i)} \mathcal{C}_{\cdot j} \tag{3.77}$$

Again for the cost function to be finite the stationary part has to be the same for the controlled system and the reference system. This results in the constraints

$$\begin{aligned}\frac{b_{0,i\cdot}}{a_0}K_{I,\cdot i} &= \frac{b_{0,ri}}{a_{0,ri}}\\ \frac{b_{0,i\cdot}}{a_0}K_{I,\cdot j} &= 0, \quad i \neq j.\end{aligned} \tag{3.78}$$

Like for all the previous cost functions for step responses, we augment it with a Lagrangian function to include these constraints

$$\begin{aligned}\mathcal{J}_{S,\lambda} = \hat{\mathcal{J}}_S \quad &+ \quad \lambda_{11}\left(\frac{b_{0,1\cdot}}{a_0}K_{I\cdot 1} - \frac{b_{0,r1}}{a_{0,r1}}\right) + \lambda_{12}\left(\frac{b_{0,1\cdot}}{a_0}K_{I\cdot 2}\right) + \cdots + \lambda_{1p}\left(\frac{b_{0,1\cdot}}{a_0}K_{I\cdot p}\right)\\ &+ \quad \lambda_{21}\left(\frac{b_{0,2\cdot}}{a_0}K_{I\cdot 1}\right) + \lambda_{22}\left(\frac{b_{0,2\cdot}}{a_0}K_{I\cdot 2} - \frac{b_{0,r2}}{a_{0,r2}}\right) + \cdots + \lambda_{2p}\left(\frac{b_{0,2\cdot}}{a_0}K_{I\cdot p}\right)\\ &\quad \vdots\\ &+ \quad \lambda_{p1}\left(\frac{b_{0,p\cdot}}{a_0}K_{I\cdot 1}\right) + \lambda_{p2}\left(\frac{b_{0,p\cdot}}{a_0}K_{I\cdot 2}\right) + \cdots + \lambda_{pp}\left(\frac{b_{0,p\cdot}}{a_0}K_{I\cdot p} - \frac{b_{0,rp}}{a_{0,rp}}\right).\end{aligned} \tag{3.79}$$

3.2.6 Step response minimization

Now we minimize the step response cost function by finding all the partial derivatives and setting them equal to zero

$$\begin{aligned}\frac{\partial \mathcal{J}_{S,\lambda}}{\partial \mathcal{C}_{\cdot i}} &= 0\\ \frac{\partial \mathcal{J}_{S,\lambda}}{\partial \lambda_{\cdot i}} &= 0.\end{aligned} \tag{3.80}$$

Writing them out gives

$$\begin{aligned}\frac{\partial \mathcal{J}_{S,\lambda}}{\partial \mathcal{C}_{\cdot 1}} &= 2\mathcal{G}_1^{(-1)}\mathcal{C}_{\cdot 1} - 2\mathcal{D}_1^{(-1)} + 2\omega_{21}\mathcal{G}_2^{(-1)}\mathcal{C}_{\cdot 1} + \cdots + 2\omega_{p1}\mathcal{G}_p^{(-1)}\mathcal{C}_{\cdot 1}\\ &\quad + \frac{1}{a_0}\left(b_{0,\cdot 1}u_1^T + b_{0,\cdot 2}u_4^T + \cdots + b_{0,\cdot p}u_{3p-2}^T\right)^T\lambda_{\cdot 1} = 0\\ \frac{\partial \mathcal{J}_{S,\lambda}}{\partial \mathcal{C}_{\cdot 2}} &= 2\omega_{12}\mathcal{G}_1^{(-1)}\mathcal{C}_{\cdot 2} + 2\mathcal{G}_2^{(-1)}\mathcal{C}_{\cdot 2} - 2\mathcal{D}_2^{(-1)} + \cdots + 2\omega_{p2}\mathcal{G}_p^{(-1)}\mathcal{C}_{\cdot 2}\\ &\quad + \frac{1}{a_0}\left(b_{0,\cdot 1}u_1^T + b_{0,\cdot 2}u_4^T + \cdots + b_{0,\cdot p}u_{3p-2}^T\right)^T\lambda_{\cdot 2} = 0\\ &\quad \vdots\\ \frac{\partial \mathcal{J}_{S,\lambda}}{\partial \mathcal{C}_{\cdot p}} &= 2\omega_{1p}\mathcal{G}_1^{(-1)}\mathcal{C}_{\cdot p} + 2\omega_{2p}\mathcal{G}_2^{(-1)}\mathcal{C}_{\cdot p} + \cdots + 2\mathcal{G}_p^{(-1)}\mathcal{C}_{\cdot p} - 2\mathcal{D}_p^{(-1)}\\ &\quad + \frac{1}{a_0}\left(b_{0,\cdot 1}u_1^T + b_{0,\cdot 2}u_4^T + \cdots + b_{0,\cdot p}u_{3p-2}^T\right)^T\lambda_{\cdot p} = 0\end{aligned} \tag{3.81}$$

and for the λ_{ij}

$$
\begin{aligned}
\frac{\partial \mathcal{J}_{S,\lambda}}{\partial \lambda_{\cdot 1}} &= \frac{1}{a_0}\left(b_{0,\cdot 1}u_1^T + b_{0,\cdot 2}u_4^T + \cdots + b_{0,\cdot p}u_{3p-2}^T\right)\mathcal{C}_{\cdot 1} - \begin{bmatrix} \frac{b_{0,r1}}{a_{0,r1}} \\ 0 \\ \vdots \\ 0 \end{bmatrix} = 0 \\
\frac{\partial \mathcal{J}_{S,\lambda}}{\partial \lambda_{\cdot 2}} &= \frac{1}{a_0}\left(b_{0,\cdot 1}u_1^T + b_{0,\cdot 2}u_4^T + \cdots + b_{0,\cdot h}u_{3p-2}^T\right)\mathcal{C}_{\cdot 2} - \begin{bmatrix} 0 \\ \frac{b_{0,r2}}{a_{0,r2}} \\ 0 \\ \vdots \\ 0 \end{bmatrix} = 0 \\
&\vdots \\
\frac{\partial \mathcal{J}_{S,\lambda}}{\partial \lambda_{\cdot p}} &= \frac{1}{a_0}\left(b_{0,\cdot 1}u_1^T + b_{0,\cdot 2}u_4^T + \cdots + b_{0,\cdot p}u_{3p-2}^T\right)\mathcal{C}_{\cdot p} - \begin{bmatrix} 0 \\ \vdots \\ 0 \\ \frac{b_{0,rp}}{a_{0,rp}} \end{bmatrix} = 0.
\end{aligned}
\tag{3.82}
$$

Now u_k is a column vector of size $3r$ with the k-th element as 1. It is worth mentioning that the derivatives above are the same as for the square system $G(s)$, because the open loop system $G(s)C(s)$ is a square system. These derivatives can be rewritten in a compact matrix form as

$$
\begin{bmatrix}
\left[\mathcal{G}_1^{(-1)} + \omega_{21}\mathcal{G}_2^{(-1)} + \cdots + \omega_{p1}\mathcal{G}_p^{(-1)}\right]_{3r\times 3r} & \frac{1}{a_0}\left[b_{\cdot 1,0}u_1^T + b_{\cdot 2,0}u_4^T + \cdot + b_{\cdot p,0}u_{3p-2}^T\right]^T_{3r\times p} \\
\frac{1}{a_0}\left[b_{\cdot 1,0}u_1^T + b_{\cdot 2,0}u_4^T + \cdot + b_{\cdot p,0}u_{3p-2}\right]_{p\times 3r} & [0]_{p\times p}
\end{bmatrix}
\begin{bmatrix} [C_{\cdot 1}]_{3r\times 1} \\ [\lambda_{\cdot 1}]_{p\times 1} \end{bmatrix}
$$

$$
= \begin{bmatrix} \left[\mathcal{D}_1^{(-1)}\right]_{3r\times 1} \\ \begin{bmatrix} \frac{b_{0,r1}}{a_{0,r1}} \\ 0 \\ \vdots \\ 0 \end{bmatrix}_{p\times 1} \end{bmatrix}
$$

$$
\begin{bmatrix}
\omega_{12}\mathcal{G}_1^{(-1)} + \mathcal{G}_2^{(-1)} + \cdots + \omega_{p2}\mathcal{G}_p^{(-1)} & \frac{1}{a_0}\left[b_{\cdot 1,0}u_1^T + b_{\cdot 2,0}u_4^T + \cdot + b_{\cdot p,0}u_{3p-2}\right]^T \\
\frac{1}{a_0}\left[b_{\cdot 1,0}u_1^T + b_{\cdot 2,0}u_4^T + \cdot + b_{\cdot p,0}u_{3p-2}\right] & 0
\end{bmatrix}
\begin{bmatrix} C_{\cdot 2} \\ \lambda_{\cdot 2} \end{bmatrix}
= \begin{bmatrix} \mathcal{D}_2^{(-1)} \\ 0 \\ \frac{b_{0,r2}}{a_{0,r2}} \\ 0 \\ \vdots \\ 0 \end{bmatrix}
\tag{3.83}
$$

$$\vdots$$

$$
\begin{bmatrix}
\omega_{1p}\mathcal{G}_1^{(-1)} + \omega_{2p}\mathcal{G}_2^{(-1)} + \cdots + \mathcal{G}_p^{(-1)} & \frac{1}{a_0}\left[b_{\cdot 1,0}u_1^T + b_{\cdot 2,0}u_4^T + \cdot + b_{\cdot p,0}u_{3p-2}\right]^T \\
\frac{1}{a_0}\left[b_{\cdot 1,0}u_1^T + b_{\cdot 2,0}u_4^T + \cdot + b_{\cdot p,0}u_{3p-2}\right] & 0
\end{bmatrix}
\begin{bmatrix} C_{\cdot p} \\ \lambda_{\cdot p} \end{bmatrix}
= \begin{bmatrix} \mathcal{D}_p^{(-1)} \\ 0 \\ \vdots \\ \frac{b_{0,rp}}{a_{0,r3}} \end{bmatrix}
$$

3.2.7 More outputs than control inputs

It is only possible to find an optimized MIMO PID controller if the number of control inputs is higher or equal to the number of outputs, $r \geq p$. This is due to the fact that it is impossible to solve Equation (3.83) unless the matrix

$$\frac{1}{a_0}\left[b_{\cdot 1,0}u_1^T + b_{\cdot 2,0}u_4^T + \cdot + b_{\cdot p,0}u_{3p-2}\right]_{p\times 3r}$$

has rank p or higher. The matrix only has rank p or higher if $r \geq p$ since the matrix can only have rank as high as r. This is because the matrix has p row vectors and each row vector has $3r$ elements but only r elements are nonzero, and they are in the same positions for all the rows. The same elements are always nonzero for all the row vectors. For example if $p = 3$ and $r = 2$ then

$$\frac{1}{a_0}\left[b_{\cdot 1,0}u_1^T + b_{\cdot 2,0}u_4^T + \cdot + b_{\cdot p,0}u_{3p-2}\right]_{p\times 3r} = \begin{bmatrix} x & 0 & 0 & x & 0 & 0 \\ x & 0 & 0 & x & 0 & 0 \\ x & 0 & 0 & x & 0 & 0 \end{bmatrix}, \quad (3.84)$$

where x is a nonzero element. In this example, the matrix can never have a rank higher than two when $r = 2$. Another way to show this would be to look at how the inverse of a block matrix is found, see [39].

$$\begin{bmatrix} X & Z^T \\ Z & 0 \end{bmatrix}^{-1} = \begin{bmatrix} X^{-1} - X^{-1}Z^T(ZX^{-1}Z^T)^{-1}Z^TX^{-1} & -X^{-1}Z(-ZX^{-1}Z^T)^{-1} \\ (ZX^{-1}Z^T)^{-1}Z^TX^{-1} & (-ZX^{-1}Z^T)^{-1} \end{bmatrix} \quad (3.85)$$

This well known equation is only valid if X^{-1} and $(-ZX^{-1}Z^T)^{-1}$ exist. If Z is a $p \times 3r$ matrix and if X is a $3r \times 3r$ matrix, then $(-ZX^{-1}Z^T)$ will be a $p \times p$ matrix. This $p \times p$ matrix is only invertible if it has rank p. It will only have rank p if Z has rank p or higher. These are also necessary conditions for the overall matrix to be invertable.

Chapter 4

Examples

4.1 A system with 2 control inputs and 2 outputs

We begin by finding the optimized MIMO PID controller for an arbitrarily chosen 2 I/O system given by

$$\begin{aligned} G(s) &= \frac{1}{a(s)}\begin{bmatrix} s^2+3s+2 & s^2+10s+9 \\ s+10 & -5 \end{bmatrix} \\ &= \frac{1}{a(s)}\begin{bmatrix} (s+2)(s+1) & (s+9)(s+1) \\ s+10 & -5 \end{bmatrix} \end{aligned} \tag{4.1}$$

where

$$\begin{aligned} \frac{1}{a(s)} &= \frac{1}{s^5+15s^4+85s^3+225s^2+274s+120} \\ &= \frac{1}{(s+1)(s+2)(s+3)(s+4)(s+5)}. \end{aligned} \tag{4.2}$$

A reference system that meets our design requirements is chosen

$$G_r = \begin{bmatrix} \frac{0.723}{s+1.53} & 0 \\ 0 & \frac{0.5}{s+1} \end{bmatrix}, \tag{4.3}$$

which results in the closed loop reference system

$$G_{r,cl} = \begin{bmatrix} \frac{0.7230}{s^2+1.53s+0.723} & 0 \\ 0 & \frac{0.5}{s^2+1s+0.5} \end{bmatrix}. \tag{4.4}$$

The $\mathcal{B}_{ij}$ matrices for the system are

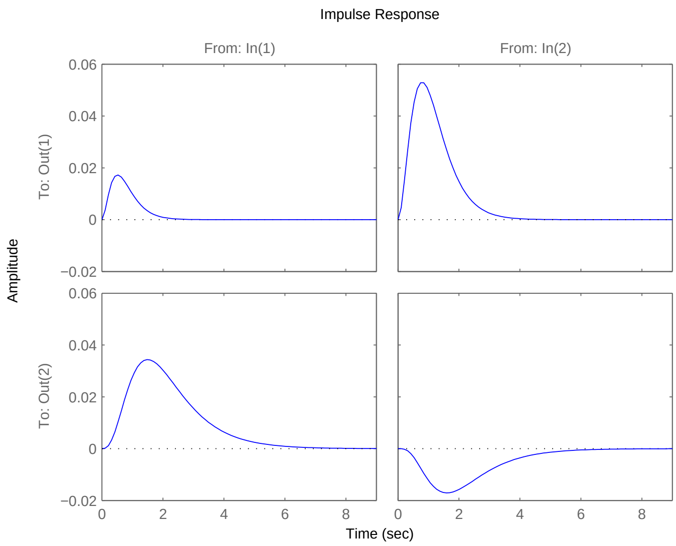

Figure 4.1: Impulse response for the open loop system $G(s)$.

$$\mathcal{B}_{11} = \begin{bmatrix} 2 & 0 & 0 \\ 3 & 2 & 0 \\ 1 & 3 & 2 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix} \qquad \mathcal{B}_{12} = \begin{bmatrix} 9 & 0 & 0 \\ 10 & 9 & 0 \\ 1 & 10 & 9 \\ 0 & 1 & 10 \\ 0 & 0 & 1 \end{bmatrix}$$
$$\mathcal{B}_{21} = \begin{bmatrix} 10 & 0 & 0 \\ 1 & 10 & 0 \\ 0 & 1 & 10 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \qquad \mathcal{B}_{22} = \begin{bmatrix} -5 & 0 & 0 \\ 0 & -5 & 0 \\ 0 & 0 & -5 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{4.5}$$

4.1.1 Impulse optimization

We begin by optimizing using the impulse response. The impulse response for the system $G(s)$ is shown in Figure 4.1, we want it to look more like the impulse response for the open loop reference system shown in Figure 4.2. Lets begin the optimization by using the Matlab function `lyap` to find the matrix $\mathcal{A}$

$$\mathcal{A} = \left[\begin{array}{cccc} 0.000019290123457 & 0 & -0.000013778659612 & \cdots \\ 0 & 0.000013778659612 & 0 & \cdots \\ -0.000013778659612 & 0 & 0.000052358906526 & \cdots \\ 0 & -0.000052358906526 & 0 & \cdots \\ 0.000052358906526 & 0 & -0.000675154320988 & \cdots \end{array} \right.$$
$$\left. \begin{array}{cc} 0 & 0.000052358906526 \\ -0.000052358906526 & 0 \\ 0 & -0.000675154320988 \\ 0.000675154320988 & 0 \\ 0 & 0.043041776895944 \end{array} \right] \tag{4.6}$$

We have two $\mathcal{D}_i$ matrices

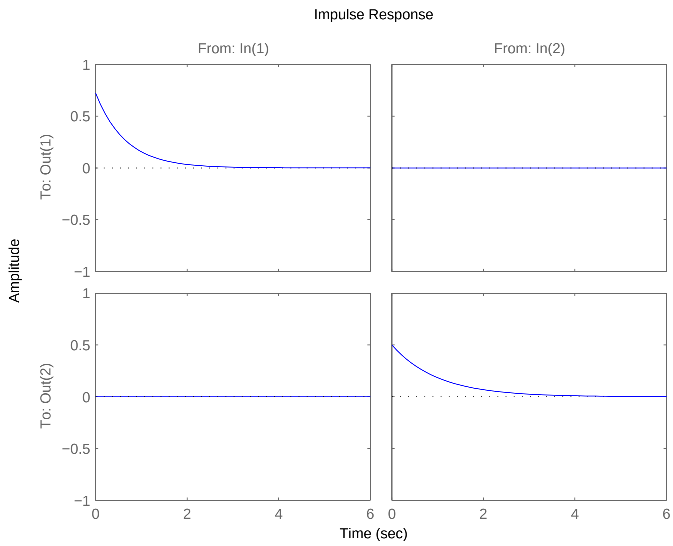

Figure 4.2: Impulse response for the open loop reference system.

$$\mathcal{D}_1 = \begin{bmatrix} 0.0044 \\ 0.0068 \\ 0.0103 \\ 0.0132 \\ 0.0202 \\ 0.0309 \end{bmatrix} \quad \text{and} \quad \mathcal{D}_2 = \begin{bmatrix} 0.0076 \\ 0.0076 \\ 0.0076 \\ -0.0035 \\ -0.0035 \\ -0.0035 \end{bmatrix}, \tag{4.7}$$

With $\omega_{ij} = 10$ this gives us the following two matrix equations

$$\begin{bmatrix} 0.0196 & 0 & -0.0153 & -0.0090 & -0.0002 & 0.0050 \\ 0 & 0.0153 & 0 & 0.0002 & -0.0050 & 0.0025 \\ -0.0153 & 0.0000 & 0.1057 & 0.0050 & -0.0025 & 0.0306 \\ -0.0090 & 0.0002 & 0.0050 & 0.0076 & 0 & -0.0095 \\ -0.0002 & -0.0050 & -0.0025 & -0.0000 & 0.0095 & 0 \\ 0.0050 & 0.0025 & 0.0306 & -0.0095 & 0 & 0.1157 \end{bmatrix} \begin{bmatrix} K_{I11} \\ K_{P11} \\ K_{D11} \\ K_{I21} \\ K_{P21} \\ K_{D21} \end{bmatrix} = \begin{bmatrix} 0.0044 \\ 0.0068 \\ 0.0103 \\ 0.0132 \\ 0.0202 \\ 0.0309 \end{bmatrix} \tag{4.8}$$

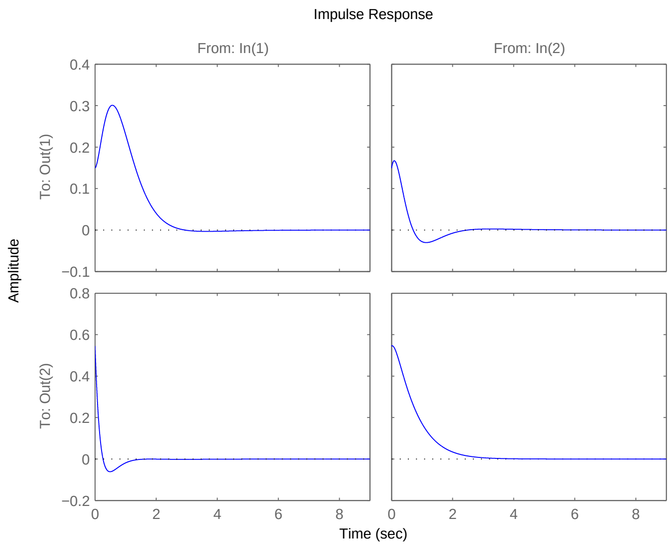

Figure 4.3: Impulse response for the open loop system with the controller in Equation 4.10.

and

$$\begin{bmatrix} 0.0039 & 0 & -0.0114 & 0.0056 & 0.0046 & -0.0185 \\ 0 & 0.0114 & 0 & -0.0046 & 0.0185 & 0.0507 \\ -0.0114 & -0.0000 & 0.4722 & -0.0185 & -0.0507 & 0.5655 \\ 0.0056 & -0.0046 & -0.0185 & 0.0279 & 0 & -0.0612 \\ 0.0046 & 0.0185 & -0.0507 & 0 & 0.0612 & 0 \\ -0.0185 & 0.0507 & 0.5655 & -0.0612 & 0 & 1.0278 \end{bmatrix} \begin{bmatrix} K_{I12} \\ K_{P12} \\ K_{D12} \\ K_{I22} \\ K_{P22} \\ K_{D22} \end{bmatrix} = \begin{bmatrix} 0.0076 \\ 0.0076 \\ 0.0076 \\ -0.0035 \\ -0.0035 \\ -0.0035 \end{bmatrix} \tag{4.9}$$

This results in the following optimized MIMO PID controller

$$\begin{aligned} \frac{1}{s}C(s) &= \frac{1}{s}\begin{bmatrix} 0.1489s^2 + 1.1957s + 2.8177 & 0.6580s^2 + 5.3578s + 4.0103 \\ 0.5450s^2 + 2.8461s + 5.6402 & -0.6153s^2 - 1.4299s - 0.9731 \end{bmatrix} \\ &= \frac{1}{s}\begin{bmatrix} 0.1489(s + 4.0142 + 1.6748i)(s + 4.0142 - 1.6748i) & 0.6580(s + 7.3090)(s + 0.8339) \\ 0.5450(s + 2.6109 + 1.8792i)(s + 2.6109 - 1.8792i) & -0.6153(s + 1.1619 + 0.4810i)(s + 1.1619 - 0.4810i) \end{bmatrix} \end{aligned} \tag{4.10}$$

The impulse response for the system with this controller in open loop can be seen in Figure 4.3. The impulse response with the controller looks more like the impulse response for the reference system then it did without the controller. Figure 4.4 shows shows how the closed loop system with the optimzed MIMO PID controller response to a unit step signal on reference inputs 1 and 2 at $t = 0s$ and $t = 20s$ respectively. The solid line are the controlled system outputs and the broken line is output for the reference system. The outputs for

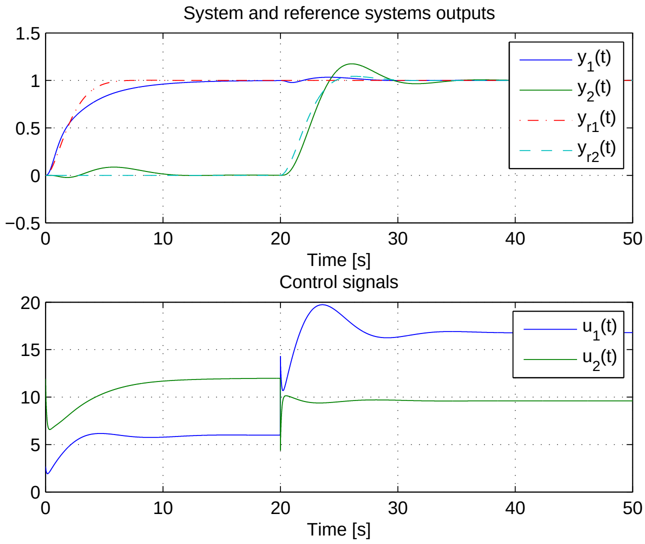

Figure 4.4: Step response for the closed loop system with the controller in Equation 4.10.

the controlled system follow the outputs for the reference system very closely and little coupling is noticable. The figure also shows the control signals and how they both respond at the same time to compensate for the coupling in the system.

A zero pole plot for each tranfer function element in $G(s)\frac{1}{s}C(s)$ is shown in Figure 4.5.

4.1.2 Step optimization

For the step optimization lets begin by looking at the step response for the open loop system $G(s)$, see Figure 4.6. It shows the step response for a unit step on input 1 at $t_1 = 0s$ and a unit step on input 2 at $t_2 = 20s$. There is an obvious coupling noticable, at time $t = 0s$ and $t = 20s$ both outputs react. The outputs also never reach the desired output value. Figure 4.7 shows the step response for the closed loop reference system with steps at the inputs at the same times as in Figure 4.6. From these figures we can see that $G_{r1}(s)$ has no overshoot and a risetime of $T_{r_1} = 3.4s$, and $G_{r_2}(s)$ has a 4% overshoot and a risetime of $T_{r_2} = 3.1s$. For the system $G(s)$ we have same four B_{ij} matrices as in (4.5)

The Matlab function `lyap` is used to find the matrix $\mathcal{A}^{(-1)}$

$$\mathcal{A}^{(-1)} = \begin{bmatrix} 0.0001137 & -0.0000347 & -0.0000193 & 0 & 0.0000138 \\ -0.0000347 & 0.0000193 & -0 & -0.0000138 & 0 \\ -0.0000193 & 0 & 0.0000138 & 0 & -0.0000524 \\ 0 & -0.0000138 & 0 & 0.0000524 & 0 \\ 0.0000138 & 0 & -0.0000524 & 0 & 0.0006752 \end{bmatrix}. \quad (4.11)$$

The $\mathcal{H}_i^{(-1)}$ matrices are also found using Matlab's lyap function, and with $\mathcal{H}_i^{(-1)}$

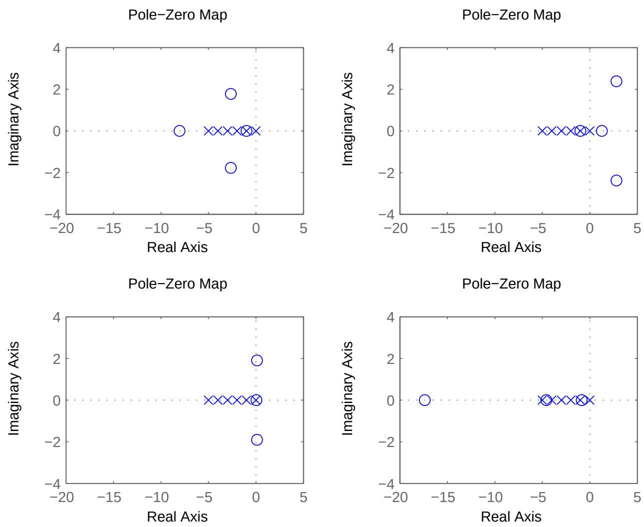

Figure 4.5: Zero pole plot for each transfer function element in $G(s)\frac{1}{s}C(s)$.

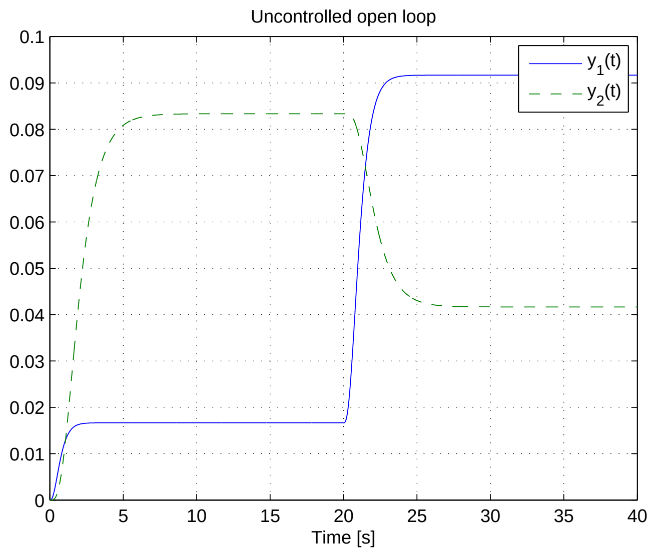

Figure 4.6: Uncontrolled step response for the system in Equation (4.1).

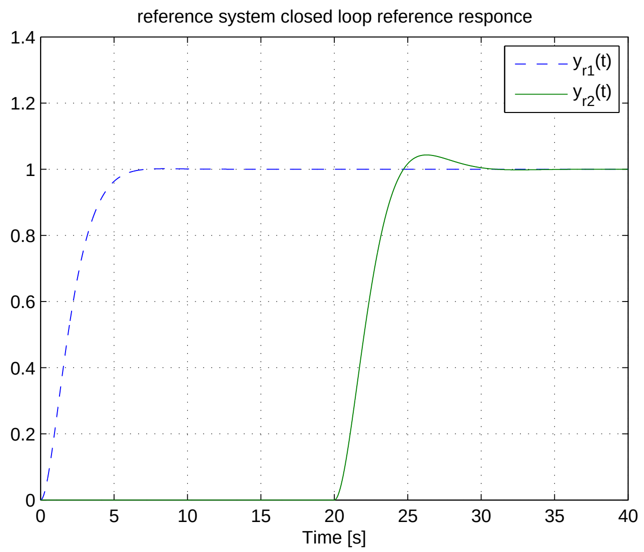

Figure 4.7: Step response for the closed loop reference system.

and all $\mathcal{B}_{ij}$ known, $\mathcal{D}_i^{(-1)}$ can be found to be

$$\mathcal{D}_1^{(-1)} = \begin{bmatrix} 0.0033 \\ -0.0029 \\ -0.0044 \\ 0.0175 \\ -0.0086 \\ -0.0132 \end{bmatrix} \quad \text{and} \quad \mathcal{D}_2^{(-1)} = \begin{bmatrix} 0.0340 \\ -0.0076 \\ -0.0076 \\ -0.0174 \\ 0.0035 \\ 0.0035 \end{bmatrix}. \tag{4.12}$$

With $\mathcal{A}^{(-1)}$ and $\mathcal{D}_i^{(-1)}$ known we set up the matrix equation in Equation (2.136). All ω_{ij} coefficients are chosen as $\omega_{ij} = 10$. For this example we have the two linear systems of equations

$$\begin{bmatrix} 0.1071 & -0.0349 & -0.0196 & -0.0543 & 0.0160 & 0.00960 & 0.0167 & 0.0833 \\ -0.0349 & 0.0196 & 0 & 0.0175 & -0.0090 & -0.0002 & 0 & 0 \\ -0.0196 & 0 & 0.0153 & 0.0090 & 0.0002 & -0.0050 & 0 & 0 \\ -0.0543 & 0.0175 & 0.0090 & 0.0330 & -0.0115 & -0.0076 & 0.0750 & -0.0417 \\ 0.0160 & -0.0090 & 0.0002 & -0.0115 & 0.0076 & 0 & 0 & 0 \\ 0.0090 & -0.0002 & -0.0050 & -0.0076 & 0 & 0.0095 & 0 & 0 \\ 0.0167 & 0 & 0 & 0.0750 & 0 & 0 & 0 & 0 \\ 0.0833 & 0 & 0 & -0.0417 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \mathcal{K}_{I11} \\ \mathcal{K}_{P11} \\ \mathcal{K}_{D11} \\ \mathcal{K}_{I21} \\ \mathcal{K}_{P21} \\ \mathcal{K}_{D21} \\ \lambda_{11} \\ \lambda_{21} \end{bmatrix} = \begin{bmatrix} 0.0033 \\ -0.0029 \\ -0.0044 \\ 0.0175 \\ -0.0086 \\ -0.0132 \\ 0.4725 \\ 0 \end{bmatrix} \tag{4.13}$$

and

$$\begin{bmatrix} 0.0122 & -0.0049 & -0.0039 & 0.0024 & -0.0023 & -0.0056 & 0.0167 & 0.0833 \\ -0.0049 & 0.0039 & 0 & -0.0067 & 0.0056 & 0.0046 & 0 & 0 \\ -0.0039 & 0 & 0.0114 & -0.0056 & -0.0046 & 0.0185 & 0 & 0 \\ 0.0024 & -0.0067 & -0.0056 & 0.0484 & -0.0290 & -0.0279 & 0.0750 & -0.0417 \\ -0.0023 & 0.0056 & -0.0046 & -0.0290 & 0.0279 & 0 & 0 & 0 \\ -0.0056 & 0.0046 & 0.0185 & -0.0279 & 0 & 0.0612 & 0 & 0 \\ 0.0167 & 0 & 0 & 0.0750 & 0 & 0 & 0 & 0 \\ 0.0833 & 0 & 0 & -0.0417 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \mathcal{K}_{I11} \\ \mathcal{K}_{P12} \\ \mathcal{K}_{D12} \\ \mathcal{K}_{I22} \\ \mathcal{K}_{P22} \\ \mathcal{K}_{D22} \\ \lambda_{22} \\ \lambda_{22} \end{bmatrix} = \begin{bmatrix} 0.0340 \\ -0.0076 \\ -0.0076 \\ -0.0174 \\ 0.0035 \\ 0.0035 \\ 0 \\ 0.5000 \end{bmatrix}. \tag{4.14}$$

It is now possible to find the $\mathcal{C}_{ij} = \begin{bmatrix} K_{Iij} & K_{Pij} & K_{Dij} \end{bmatrix}$ vectors,

$$\begin{aligned} \mathcal{C}_{11}^T &= \begin{bmatrix} 2.8353 & 1.1439 & 0.1574 \end{bmatrix} & \mathcal{C}_{12}^T &= \begin{bmatrix} 5.6706 & 2.8310 & 0.5557 \end{bmatrix} \\ \mathcal{C}_{21}^T &= \begin{bmatrix} 5.4000 & 1.2624 & 6.0961 \end{bmatrix} & \mathcal{C}_{22}^T &= \begin{bmatrix} -1.2000 & -0.8283 & -1.7045 \end{bmatrix}. \end{aligned} \tag{4.15}$$

This gives us the controller

$$\begin{aligned} \frac{1}{s}C(s) &= \frac{1}{s}\begin{bmatrix} 0.1574s^2 + 1.1439s + 2.8353 & 0.5557s^2 + 2.8310s + 5.6706 \\ 6.0961s^2 + 1.2624s + 6.0961 & -1.7045s^2 - 0.8283s - 1.2 \end{bmatrix} \qquad (4.16) \\ &= \frac{1}{s}\begin{bmatrix} 0.1574(s - 3.6344 - 2.1926i)(s - 3.6344 + 2.1926i) & 0.5557(s + 2.5470 - 1.9278i)(s + 2.5470 + 1.9278i) \\ 6.0961(s + 3.6604)(s + 1.1686) & -1.7045(s + 1.0289 - 0.6246i)(s + 1.0289 + 0.6246i) \end{bmatrix} \end{aligned}$$

The step response and the control signals for the controlled closed loop system with the controller above can be seen in Figure 4.8. The outputs of the controlled system are almost identical to the outputs for the reference system. There is a small coupling noticeable when the second step comes on input 2 which the controller compensates for and it disappears quickly.

The zero pole plot for each element transfer function in the open loop TFM $G(s)C(s)$ is shown in Figure 4.9. By zooming in on Figure 4.9, see Figure 4.10, we can see where the zeros in the optimized open loop MIMO PID controller and system are located. Note for example that the resulting zeros in element $(1,1)$ are generated by $G_{11}(s)c_{11}(s) + G_{12}(s)c_{21}(s) = 0$.

The step responses for the controlled closed loop system are shown in Figures 4.11 and 4.12, where $\omega_{ij} = 1$ and $\omega_{ij} = 100$, respectively. For $\omega_{ij} = 1$ the controller becomes

$$\begin{aligned} \frac{1}{s}C(s) &= \frac{1}{s}\begin{bmatrix} 0.1460s^2 + 1.1441s + 2.8353 & 0.5638s^2 + 2.8277s + 5.6706 \\ 2.0667s^2 + 6.0657s + 5.4000 & -1.0837s^2 - 1.5558s - 1.2 \end{bmatrix} \qquad (4.17) \\ &= \frac{1}{s}\begin{bmatrix} 0.1460(s + 3.9190 - 2.0161i)(s + 3.9190 + 2.0161i) & 0.5638(s + 2.5077 - 1.9414i)(s + 2.5077 + 1.9414i) \\ 2.0667(s - 1.4674 + 0.6778i)(s + 1.4674 - 0.6778i) & -1.0837(s + 0.7178 + 0.7695i)(s + 0.7178 - 0.7695i) \end{bmatrix}. \end{aligned}$$

For $\omega_{ij} = 100$ the controller is

$$\begin{aligned} \frac{1}{s}C(s) &= \frac{1}{s}\begin{bmatrix} 0.1428s^2 + 1.1461s + 2.8353 & 0.5227s^2 + 2.8381s + 5.6706 \\ 0.7129s^2 + 6.1165s + 5.4 & -0.6464s^2 - 1.808s - 1.2 \end{bmatrix} \qquad (4.18) \\ &= \frac{1}{s}\begin{bmatrix} 0.1428(s + 4.0118 - 1.9377i)(s + 4.0118 + 1.9377i) & 0.5227(s + 2.7147 - 1.8650i)(s + 2.7147 + 1.8650i) \\ 0.7129(s + 7.5809)(s + 0.9992) & -0.6464(s + 1.7137)(s + 1.0833) \end{bmatrix}. \end{aligned}$$

The closed loop outputs in Figures 4.8, 4.11 and 4.12 are all very similar. There is however a small difference between the figures. The controlled system outputs follow the reference system outputs a little bit closer with larger ω_{ij} values.

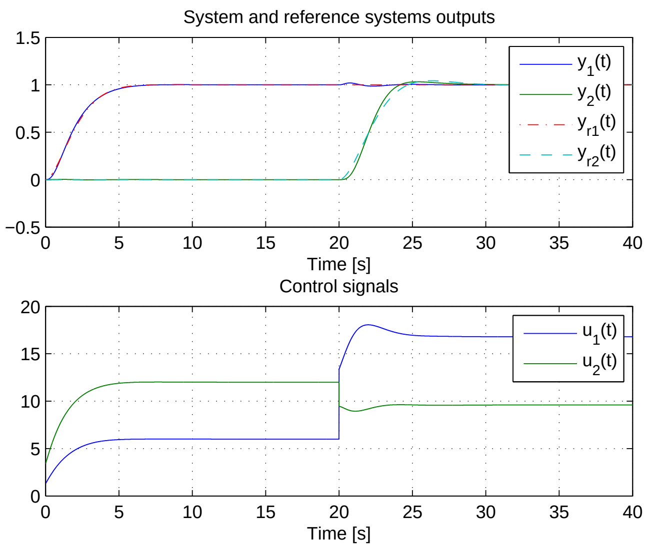

Figure 4.8: Step response and the control signals for the controlled closed loop system with $\omega_{ij} = 10$.

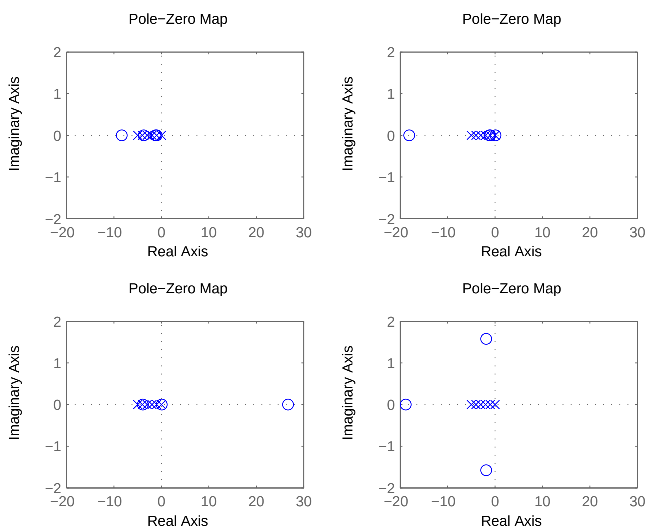

Figure 4.9: The zero pole plot for the element transfer functions in $G(s)C(s)$ for $\omega_{ij} = 10$.

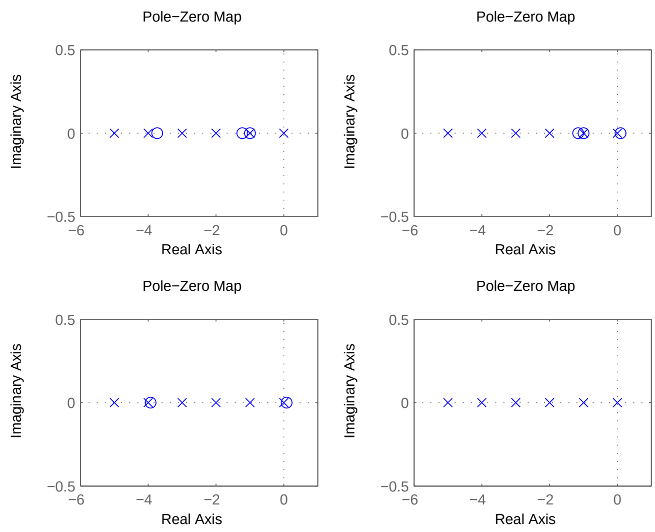

Figure 4.10: Zoomed in on Figure 4.9.

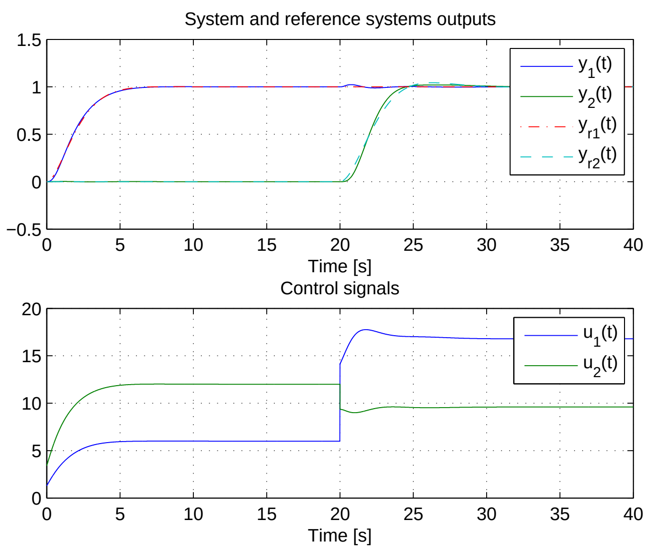

Figure 4.11: Step response and control signal for the controlled closed loop system with $\omega_{ij} = 1$.

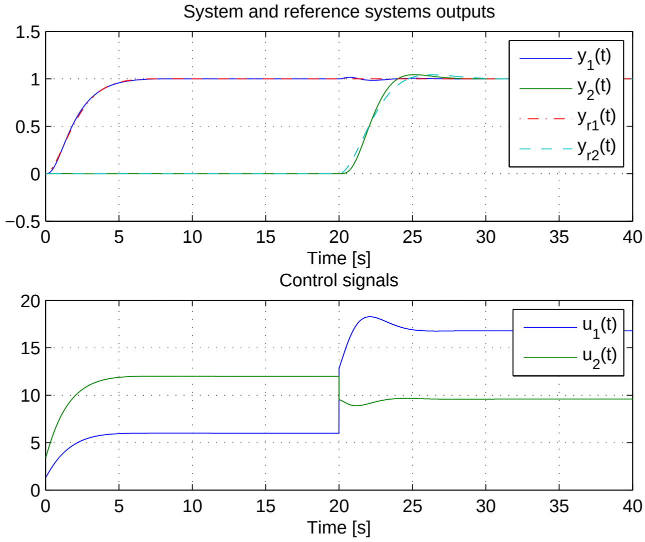

Figure 4.12: Step response and control signal for the controlled closed loop system with $\omega_{ij} = 100$.

We use the system closed loop step response to compare the impulse optimization and the step optimization. Then the step optimization is better, the controlled closed loop system step response follow the closed loop reference system step response better then in the impulse optimization. This is partly due to the fact that step optimization ensures the correct open loop DC gain, therefore the integrator has less work to do when following a reference step. In general, it is best to use impulse minimization if the system is most often subject to impulse type imputs and step minimization if the system is most often subject to step type inputs.

4.2 A system with 4 control inputs and 3 outputs

We now look at a system that does not have the same number of control inputs and outputs. We choose $G(s)$ again arbitrarily as

$$\begin{aligned} G(s) &= \frac{1}{a(s)} \begin{bmatrix} s^2+10s+9 & s^2+3s+2 & s^2+3s+2 & s^2+10s+9 \\ s+10 & -5 & s^2+3s+2 & s^2+10s+9 \\ s^2+5s+4 & s+2 & -5 & s^2+10s+9 \end{bmatrix} \\ &= \frac{1}{a(s)} \begin{bmatrix} (s+9)(s+1) & (s+2)(s+1) & (s+2)(s+1) & (s+9)(s+1) \\ s+10 & -5 & (s+2)(s+1) & (s+9)(s+1) \\ (s+4)(s+1) & s+2 & -5 & (s+9)(s+1) \end{bmatrix} \end{aligned} \tag{4.19}$$

with $\frac{1}{a(s)} = \frac{1}{s^5+15s^4+85s^3+225s^2+274s+120} = \frac{1}{(s+1)(s+2)(s+3)(s+4)(s+5)}$. This system has the open loop step response seen in Figure 4.13. The steps in Figure 4.13 come on at $t_1 = 0s$, $t_2 = 10s$, $t_3 = 20s$ and $t_4 = 30s$ on control inputs u_1, u_2, u_3 and u_4, respectively. Since the TFM $G(s)$ is a 3×4 matrix, the TFM for the optimized MIMO PID controller will be a 4×3 TFM

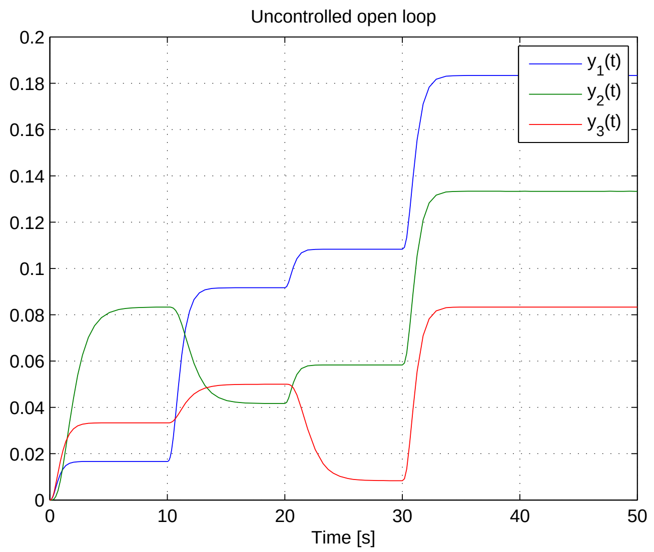

Figure 4.13: $G(s)$ in open loop without controller.

$$C(s) = \begin{bmatrix} c_{11}(s) & c_{12}(s) & c_{13}(s) \\ c_{21}(s) & c_{22}(s) & c_{23}(s) \\ c_{31}(s) & c_{32}(s) & c_{33}(s) \\ c_{41}(s) & c_{42}(s) & c_{43}(s) \end{bmatrix}. \tag{4.20}$$

This controller results in the closed loop system having 3 reference inputs and 3 outputs. The reference system will then be chosen as a diagonal TFM of size 3×3. A reference system that states our design requirements is chosen

$$G_r(s) = \begin{bmatrix} \frac{0.723}{s+1.53} & 0 & 0 \\ 0 & \frac{0.5}{s+1} & 0 \\ 0 & 0 & \frac{0.723}{s+1.53} \end{bmatrix}. \tag{4.21}$$

The closed loop reference systems are

$$G_{r,cl}(s) = \begin{bmatrix} \frac{0.723}{s^2+1.53s+0.723} & 0 & 0 \\ 0 & \frac{0.5}{s^2+s+0.5} & 0 \\ 0 & 0 & \frac{0.723}{s^2+1.53s+0.723} \end{bmatrix}. \tag{4.22}$$

The step response for the closed loop reference system can be seen in Figure 4.14, the steps come on at $t_1 = 0s$, $t_2 = 10s$ and $t_3 = 20s$ on inputs u_{r1}, u_{r2} and u_{r3}, respectively. From Figure 4.14 we can see that $G_{r1}(s)$ and $G_{r3}(s)$ have no overshoot and a risetime $T_{r1} = 3.4s$, $G_{r2}(s)$ has a 4% overshoot and the risetime $T_{r2} = 3.1s$. Our system $G(s)$ has twelve $\mathcal{B}_{ij}$ matrices, let us take a look at a few of them

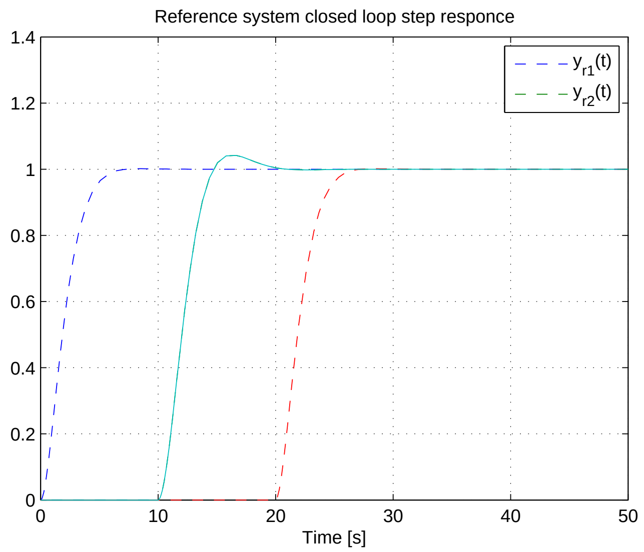

Figure 4.14: Step response for the closed loop reference system $G_{r,cl}(s)$.

$$\mathcal{B}_{11} = \begin{bmatrix} 2 & 0 & 0 \\ 3 & 2 & 0 \\ 1 & 3 & 2 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix} \qquad \mathcal{B}_{12} = \begin{bmatrix} 9 & 0 & 0 \\ 10 & 9 & 0 \\ 1 & 10 & 9 \\ 0 & 1 & 10 \\ 0 & 0 & 1 \end{bmatrix}$$
$$\mathcal{B}_{21} = \begin{bmatrix} 10 & 0 & 0 \\ 1 & 10 & 0 \\ 0 & 1 & 10 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \qquad \mathcal{B}_{22} = \begin{bmatrix} -5 & 0 & 0 \\ 0 & -5 & 0 \\ 0 & 0 & -5 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}. \tag{4.23}$$

The rest of the $\mathcal{B}_{ij}$ matrices can be written in a similar way. Using the Matlab function `lyap` we find the $A^{(-1)}$ matrix as before in (4.11)

$$\mathcal{A}^{(-1)} = 10^{-3} \begin{bmatrix} 0.1137 & -0.0347 & -0.0193 & 0 & 0.0138 \\ -0.0347 & 0.0193 & 0 & -0.0138 & 0 \\ -0.0193 & 0 & 0.0138 & 0 & -0.0524 \\ 0 & -0.0138 & 0 & 0.0524 & 0 \\ 0.0138 & 0 & -0.0524 & 0 & 0.6752 \end{bmatrix}. \tag{4.24}$$

The $\mathcal{H}_i^{(-1)}$ matrices are also found using the function `lyap` and with $\mathcal{B}_{ij}$ and $\mathcal{H}_i^{(-1)}$ known, $\mathcal{D}_i^{(-1)}$ is easily found

$$\mathcal{D}_1^{(-1)} = \begin{bmatrix} 0.0033 \\ -0.0029 \\ -0.0044 \\ 0.0175 \\ -0.0086 \\ -0.0132 \\ 0.0033 \\ -0.0029 \\ -0.0044 \\ 0.0175 \\ -0.0086 \\ -0.0132 \end{bmatrix} \quad \mathcal{D}_2^{(-1)} = \begin{bmatrix} 0.0340 \\ -0.0076 \\ -0.0076 \\ -0.0174 \\ 0.0035 \\ 0.0035 \\ 0.0042 \\ -0.0042 \\ -0.0042 \\ 0.0236 \\ -0.0139 \\ -0.0139 \end{bmatrix} \quad \mathcal{D}_3^{(-1)} = \begin{bmatrix} 0.0073 \\ -0.0045 \\ -0.0069 \\ 0.0044 \\ -0.0011 \\ -0.0017 \\ -0.0118 \\ 0.0016 \\ 0.0025 \\ 0.0175 \\ -0.0086 \\ -0.0132 \end{bmatrix}. \tag{4.25}$$

With these matrices known it is possible to set up the matrix equations in Equation (3.83) and to solve for all the $\mathcal{C}_{ij} = \begin{bmatrix} K_{I,ij} & K_{P,ij} & K_{D,ij} \end{bmatrix}$ vectors. With $\omega_{ij} = 10$ the vectors become

$$\begin{aligned}
\mathcal{C}_{11}^T &= \begin{bmatrix} 0.3176 & 1.5905 & 3.0727 \end{bmatrix} \\
\mathcal{C}_{12}^T &= \begin{bmatrix} 0.1074 & -0.1904 & -1.1585 \end{bmatrix} \\
\mathcal{C}_{13}^T &= \begin{bmatrix} -0.2797 & -0.2077 & 0.8362 \end{bmatrix} \\
\mathcal{C}_{21}^T &= \begin{bmatrix} 0.4749 & 2.9638 & 5.8063 \end{bmatrix} \\
\mathcal{C}_{22}^T &= \begin{bmatrix} -0.5677 & -1.9454 & -4.9477 \end{bmatrix} \\
\mathcal{C}_{23}^T &= \begin{bmatrix} 0.1534 & -0.5765 & 0.4778 \end{bmatrix} \\
\mathcal{C}_{31}^T &= \begin{bmatrix} 0.2330 & 2.8486 & 3.1725 \end{bmatrix} \\
\mathcal{C}_{32}^T &= \begin{bmatrix} 0.4855 & 5.9556 & 4.6167 \end{bmatrix} \\
\mathcal{C}_{33}^T &= \begin{bmatrix} -0.7781 & -9.5541 & -8.3398 \end{bmatrix} \\
\mathcal{C}_{41}^T &= \begin{bmatrix} -0.2618 & -1.1513 & -0.8934 \end{bmatrix} \\
\mathcal{C}_{42}^T &= \begin{bmatrix} -0.0286 & 0.3644 & 4.1792 \end{bmatrix} \\
\mathcal{C}_{43}^T &= \begin{bmatrix} 0.8605 & 3.3830 & 1.1896 \end{bmatrix}.
\end{aligned} \tag{4.26}$$

Figure 4.15 shows how well the closed loop system outputs follow the reference system with this controller, along with the closed loop system control signals. We see that the system is almost decoupled, only small oscillations occur at the outputs when a step comes on the other inputs.

4.3 Ferrosilicon Furnace

A model of a Ferrosilicon furnace was developed in [40]. The purpose of developing the model was to find a MIMO method of control for the furnace. In this control problem the goal is to control and decouple currents in three electrodes in the furnace. It is possible to increase or decrease the current in each electrode by moving it down or up. The currents are heavily coupled and cyclically coupled since it is really a three phase electrical system. Let us now see if it is possible to control and decouple the system using the optimized MIMO PID controller. After the model of the ferrosilicon furnace has been transformed into

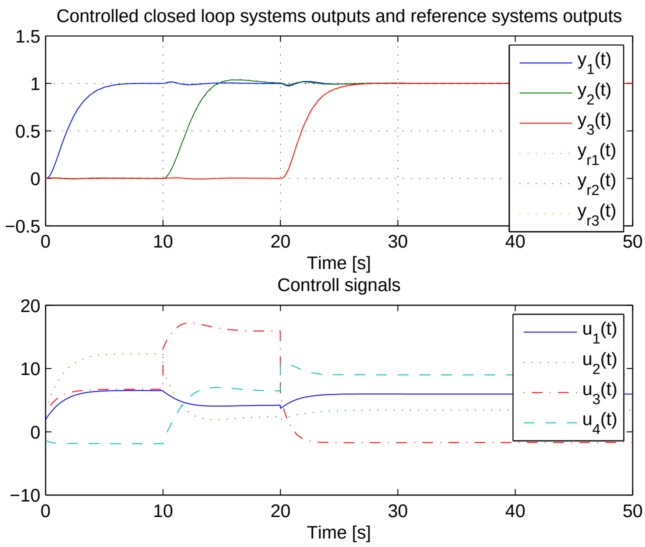

Figure 4.15: Step response and control signals for the closed loop system.

a continuous time, the TFM becomes

$$G(s) = \frac{1}{a_f(s)} \begin{bmatrix} -1.055s^2 - 0.2584s - 0.0009118 & -0.818s^2 - 0.1937s - 0.0009118 & \cdots \\ 0.203s^2 + 0.0878s - 0.0007294 & -1.055s^2 - 0.2584s - 0.0009118 & \cdots \\ -0.818s^2 - 0.1937s - 0.0009118 & 0.203s^2 + 0.0878s - 0.0007294 & \cdots \end{bmatrix}$$

$$\cdots \begin{matrix} 0.203s^2 + 0.0878s - 0.0007294 \\ -0.818s^2 - 0.1937s - 0.0009118 \\ -1.055s^2 - 0.2584s - 0.0009118 \end{matrix} \Bigg] \tag{4.27}$$

with $\frac{1}{a_f(s)} = \frac{1}{s^2+0.3835s+0.007112}$. The step response for the open loop ferrosilicon furnace with no controller is shown in Figure 4.16. It can easily been seen that the system is heavily coupled. For the ferrosilicon furnace, the reference system is chosen as

$$G_r(s) = \begin{bmatrix} \frac{1}{s+5} & 0 & 0 \\ 0 & \frac{1}{s+5} & 0 \\ 0 & 0 & \frac{1}{s+5} \end{bmatrix}, \tag{4.28}$$

resulting in the closed loop reference system

$$G_{r,cl}(s) = \begin{bmatrix} \frac{1}{s^2+5s+1} & 0 & 0 \\ 0 & \frac{1}{s^2+5s+1} & 0 \\ 0 & 0 & \frac{1}{s^2+5s+1} \end{bmatrix}. \tag{4.29}$$

The step response for the reference system can be seen in Figure 4.17. This reference system has no overshoot for any of the three outputs. It has the risetime $T_r = 13.6s$ for all three outputs. For this system $m_{ij} + 2 = 4 > n$, therefore a small trick is used. We add three dummy poles to the system for the optimization, if the dummy poles have a unity DC gain and are in the left half plane far away from the poles of the element transfer function, they will not

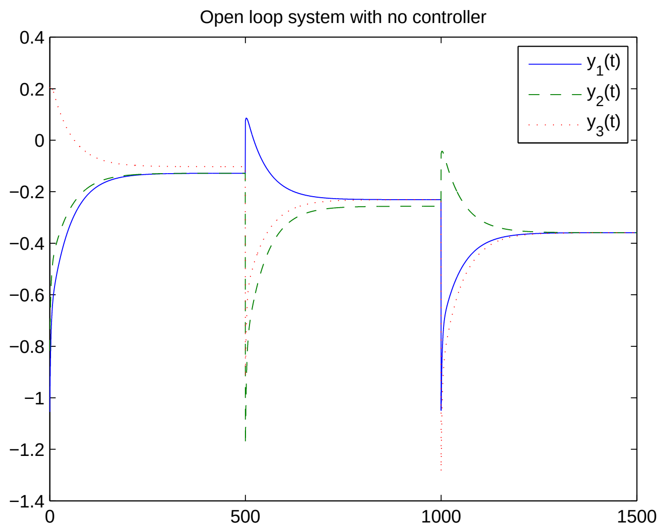

Figure 4.16: Step response for the open loop ferrosilicon furnace with no controller.

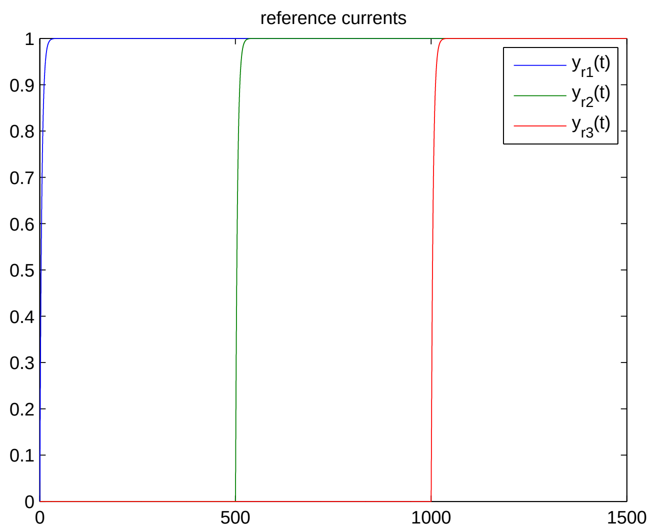

Figure 4.17: Step response for the closed loop reference system for the ferrosilicon furnace.

affect the system. For the ferrosilicon furnace three dummy poles are added to each element transfer function at -36.3922, which is 100 times the fastest pole in the system. Then $n = 5$ and it is possible to find the optimized MIMO PID controller. Then the TFM looks like this

$$\begin{aligned} G_d(s) &= \frac{1}{a_d(s)} \left[\begin{array}{cc} -5.085 \cdot 10^4 s^2 - 1.245 \cdot 10^4 s - 43.95 & -3.943 \cdot 10^4 s^2 - 9335s - 43.95 \\ 9784s^2 + 4232s - 35.16 & -5.085 \cdot 10^4 s^2 - 1.245 \cdot 10^4 s - 43.95 \\ -3.943 \cdot 10^4 s^2 - 9335s - 43.95 & 9784s^2 + 4232s - 35.16 \end{array} \right. \cdots \\ & \cdots \left. \begin{array}{c} 9784s^2 + 4232s - 35.16 \\ -3.943 \cdot 10^4 s^2 - 9335s - 43.95 \\ -5.085 \cdot 10^4 s^2 - 1.245 \cdot 10^4 s - 43.95 \end{array} \right] \end{aligned} \tag{4.30}$$

with $\frac{1}{a_d(s)} = \frac{1}{s^5+109.5602s^4+4015.0587s^3+49722.0717s^2+18510.3727s+342.7730}$. With this new system the $\mathcal{B}_{ij}$ matrices become

$$\begin{aligned} \mathcal{B}_{11} &= 10^4 \begin{bmatrix} -0.0044 & 0 & 0 \\ -1.2454 & -0.0044 & 0 \\ -5.0849 & -1.2454 & -0.0044 \\ 0 & -5.0849 & -1.2454 \\ 0 & 0 & -5.0849 \end{bmatrix} \\ \mathcal{B}_{12} &= 10^3 \begin{bmatrix} 0.0352 & 0 & 0 \\ 4.2318 & -0.0352 & 0 \\ 9.7841 & 4.2318 & -0.0352 \\ 0 & 9.7841 & 4.2318 \\ 0 & 0 & 9.7841 \end{bmatrix} \\ \mathcal{B}_{13} &= 10^4 \begin{bmatrix} -0.0044 & 0 & 0 \\ -0.9335 & -0.0044 & 0 \\ -3.9426 & -0.9335 & -0.0044 \\ 0 & -3.9426 & -0.9335 \\ 0 & 0 & -3.9426 \end{bmatrix}. \end{aligned} \tag{4.31}$$

Because of symmetry in the system the rest of the matrices are $\mathcal{B}_{21} = \mathcal{B}_{13}$, $\mathcal{B}_{22} = \mathcal{B}_{11}$, $\mathcal{B}_{23} = \mathcal{B}_{12}$, $\mathcal{B}_{31} = \mathcal{B}_{12}$, $\mathcal{B}_{32} = \mathcal{B}_{13}$ and $\mathcal{B}_{33} = \mathcal{B}_{11}$. Matlab is then used to find $\mathcal{A}^{(-1)}$, $\mathcal{D}_i^{(-1)}$ and to solve for the $\mathcal{C}_{ij}$ vectors.

$$\begin{aligned} \mathcal{C}_{11}^T &= \begin{bmatrix} -2.7857 & 0.0472 & -0.0797 \end{bmatrix} \\ \mathcal{C}_{12}^T &= \begin{bmatrix} 5.0143 & -0.0332 & 0.0584 \end{bmatrix} \\ \mathcal{C}_{13}^T &= \begin{bmatrix} -2.7857 & 0.0351 & -0.0570 \end{bmatrix} \\ \mathcal{C}_{21}^T &= \begin{bmatrix} -2.7857 & 0.0351 & -0.0570 \end{bmatrix} \\ \mathcal{C}_{22}^T &= \begin{bmatrix} -2.7857 & 0.0472 & -0.0797 \end{bmatrix} \\ \mathcal{C}_{23}^T &= \begin{bmatrix} 5.0143 & -0.0332 & 0.0584 \end{bmatrix} \\ \mathcal{C}_{31}^T &= \begin{bmatrix} 5.0143 & -0.0332 & 0.0584 \end{bmatrix} \\ \mathcal{C}_{32}^T &= \begin{bmatrix} -2.7857 & 0.0351 & -0.0570 \end{bmatrix} \\ \mathcal{C}_{33}^T &= \begin{bmatrix} -2.7857 & 0.0472 & -0.0797 \end{bmatrix} \end{aligned} \tag{4.32}$$

Using these optimized MIMO PID controllers results in the step response for

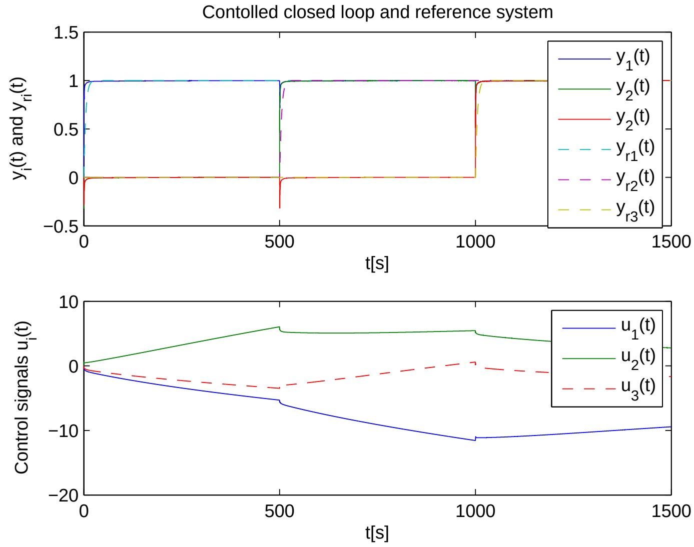

Figure 4.18: Step response and control signal for the closed loop ferrosilicon furnace with the optimized MIMO PID controller.

the closed loop ferrosilicon furnace seen in Figure 4.18. Because of how heavily the system is coupled, we don't see as good result as in the previous examples. The outputs are not able to follow the reference system as closely as we would want. When a step comes on a reference input 1, a large reaction is seen on outputs 2 and 3. Even though this is a larger reaction than we had hoped to see, the controller is able to compensate for it and outputs 2 and 3, return to the same value as reference inputs 2 and 3 respectively, very quickly. The same happens when a step comes a reference inputs 2 or 3 for corresponding outputs. Because of the large time scale in Figure 4.18 it is helpfull to zoom in, see Figure 4.19.

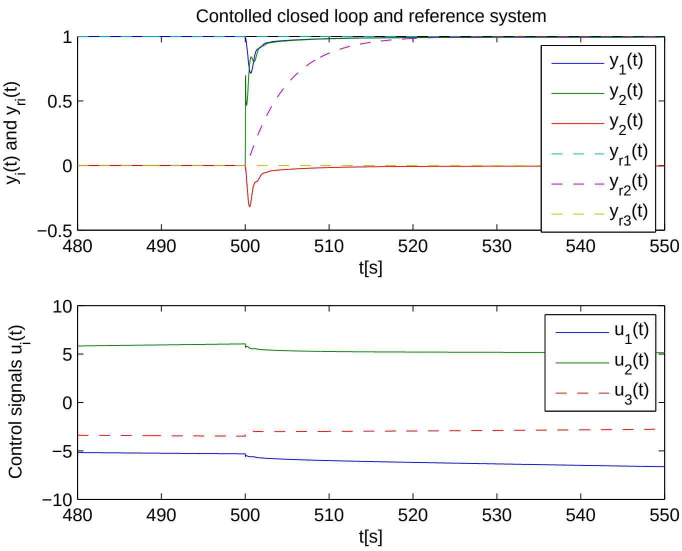

Figure 4.19: Zoomed in on Figure 4.18.

Chapter 5

Concluding remarks

5.1 Final remarks

An optimized MIMO PID controller has been derived, where the controllers track a chosen open loop reference system. It is not possible to fully decouple a MIMO system using this optimized MIMO PID controller. The controlled system can, however, come very close to being fully decoupled. The optimized MIMO PID controller was derived first for a system with two inputs and two outputs. It was then extended to the general case of p inputs and p outputs. Then the optimized MIMO PID controller was derived for systems that do not have the same number of inputs and outputs. Beginning with a system with three inputs and two outputs, it was then extended to the general case of r inputs and p outputs. It was further shown that the optimized MIMO PID controller can not control systems with more outputs than control inputs. In order to calculate the optimal MIMO PID controller it is possible to use two different methods to find the Gramian matrices $\mathcal{A}$ and $\mathcal{A}^{(-1)}$ used in the optimization. Both methods were presented but only one of them was used in the examples. At the end three examples were shown, demonstrating how well the optimized MIMO PID controller works. The first example is a system with 2 control inputs and 2 outputs, the second example is a system with 4 control inputs and 3 outputs. The final example is a real system, a ferrosilicon furnace, with three control inputs and three outputs. In the first two examples almost no coupling effect was noticed and the outputs follow the chosen reference system very closely. When the model of the ferrosilicon furnace in the third example is written in a state space form the D matrix is not zero. This direct effect on the outputs along with the high cyclic coupling effect causes the decoupling and control of the system to be harder to handle. Even though the coupling effects are high in the controlled system, the optimized MIMO PID controller is able to compensate for them quickly and the outputs all have the desired reference value as a final value.

5.2 Future work

The next steps in research on the optimized MIMO PID controller could be to figure out the structure in the matrices in Equations (2.136) and (3.83), and to find out the conditions under which the equation has a solution. The possible use of the optimized MIMO PID controller to stabilize an unstable system as well as to decouple it, is another exciting research topic, although this may have to be done in an inner state feedback style loop. Closed loop stability is very important for control systems and it would be interesting to know if the optimized MIMO PID controller can always guarantee closed loop stability. It would also be useful to derive the optimized MIMO PID controller for discrete time systems, since digitalized system dominate the industry, both in system identification as well as in controller setups. For the ferrosilicon furnace the next steps would be to use the full model including the movement of the electrods, time delay and the limits on the inputs signals, see [41] and [42], in the design of the controller. Finally, minimization of the Matlab code is an important practical issue.

Appendix A

Matlab code

A.1 Square systems

A.1.1 Impulse optimization

The code used to find the optimized MIMO PID controller for a square system using impulse optimization.

```
function [C Gc_dummy] = mimopidImp(Gc,Gr,W)
%C_ij = [K_Dij K_Pij K_Iij]
%Gc_dummy is the Gc system with the dummy poles if they where use
%if no dummy poles where use the it is just Gc


numGc = get(Gc,'num');
denGc = get(Gc,'den');
denGc = denGc{1,1};
numGr = get(Gr,'num');
denGr = get(Gr,'den');

[m mm] = size(numGc);
[mr mmr] = size(Gr);

%Check if the size of Gc and Gr are the same
if (m == mm) & (mr == mr) & (m == mr)

for i=1:m
   for j =1:m
     numGc{i,j} = lagavigur(numGc{i,j});
      numGr{i,j} = lagavigur(numGr{i,j});
    end
  end

   %check if dummy poles have to be added
   dp = 0;
   for i=1:m
```

```
     for j =1:m
         if (length(denGc)-length(numGc{i,j}) < 3)
             dp = 1; %ef dp set sem 1 þá þarf að bæta við dummy pól
         end
     end
end

if dp == 0
    disp('Dummy pole is not used')

    nota_numGc = numGc;
    nota_denGc = denGc;

elseif dp == 1
    disp('Dummy pole is used')

     polar = roots(denGc); %næ í póla kerfisins
     poltl = max(abs(polar)); %finn stærsta pólinn
     DumPole = 100*poltl;     %nota 100 sinnum stærsti pólinn sem dummy pól

 %Check how many dummy poles to use
 for i = 1:m
     for j=1:m
         n(i,j) = 3-(length(denGc)-length(numGc{i,j}));
     end
 end
 NrOfDumPole = max(max(n));
 denGc_dummy =  conv(denGc,poly(-ones(1,NrOfDumPole)*DumPole));
 nota_denGC = denGc_dummy;

 %num is scaled so the DC gain is correct
 for i=1:m
     for j=1:m
             numGc_dummy{i,j} = numGc{i,j}.*denGc_dummy(end)/denGc(end);
     end
 end

 nota_numGc = numGc_dummy;
 nota_denGc = denGc_dummy;
end

%B marices created
for i=1:m
     for j =1:m
        B{i,j} = convmtx(nota_numGc{i,j}(end:-1:1),3).';
        [s{i,j} ss{i,j}] = size(B{i,j});
        if s{i,j} < (length(nota_denGc)-1)
            B{i,j}(length(nota_denGc)-1,:)  = [0 0 0];
        end
     end
```

```
   end

   %Br and Hr matrices created
   for i=1:m
      Br{i} = numGr{i,i}(end:-1:1).';
      H{i} = H_nytt(roots(nota_denGc),roots(denGr{i,i}));

   % A martrix found
   Alyap = Aimp(nota_denGc);

      %BBdalk{i} = Bi.
      %BBlina{i} = B.i
      for i = 1:m
           BBdalk{i} = B{i,1}.';
           BBlinu{i} = B{i,1};
           for z = 2:m
               BBdalk{i} = [BBdalk{i}; B{i,z}.'];
               BBlinu{i} = [BBlinu{i}, B{i,z}];
           end
      end


        for j = 1:m
            G{j} = 0;
          for i = 1:m
                G{j} = G{j} +  W(i,j)*BBdalk{i}*Alyap*BBlinu{i};
          end
        end

      %D matrices created
      for i = 1:m
            D{i} = BBdalk{i}*H{i}*Br{i};
      end

      for i = 1:m
         C{i} = G{i}\D{i};
      end
else
    %Gc and Gr do not have the same size, stop runing
    disp('Gc and Gr are not of the same size')
    C = 0;
    Gc_dummy = Gc;
    return
end
%Gc_dummy is also return
Gc_dummy = tf(nota_numGc,nota_denGc);
end
```

A.1.2 Step optimization

The code used to find the optimized MIMO PID controller for a square systems using step optimization.

```
function [C Gc_dummy] = mimopid(Gc,Gr,W)
%C_ij = [K_Dij K_Pij K_Iij]
%Gc_dummy is the Gc system with the dummy poles if they where use
%if no dummy poles where use the it is just Gc
%

numGc = get(Gc,'num');
denGc = get(Gc,'den');
denGc = denGc{1,1};
numGr = get(Gr,'num');
denGr = get(Gr,'den');

[p pp] = size(numGc);
[pr ppr] = size(Gr);

%checking if the size of the system and the
%reference system are the same
if (p == pp) & (pr == pr) & (p == pr)

   for i=1:p
        for j =1:p
            numGc{i,j} = lagavigur(numGc{i,j});
            numGr{i,j} = lagavigur(numGr{i,j});
        end
    end

   %check if dummy poles have to be added
   dp = 0;
   for i=1:p
        for j =1:p
            if (length(denGc)-length(numGc{i,j}) < 3)
                dp = 1;
            end
        end
   end

   if dp == 0 % no dummy pole have to be added
       disp('þarf ekki að bæta við dummy pól')

       nota_numGc = numGc;
       nota_denGc = denGc;

   elseif dp == 1
       disp('þarf að bæta við dummy pól')

        polar = roots(denGc);
```

```
    poltl = max(abs(polar));
    DumPole = 100*poltl;

 %check how many dummy poles to add
 for i = 1:p
     for j=1:p
         n(i,j) = 3-(length(denGc)-length(numGc{i,j}));
     end
 end
 NrOfDumPole = max(max(n));
 denGc_dummy =  conv(denGc,poly(-ones(1,NrOfDumPole)*DumPole));
 nota_denGC = denGc_dummy;

 %correcting the numenator so the dc-value is the same as before
 for i=1:p
     for j=1:p
             numGc_dummy{i,j} = numGc{i,j}.*denGc_dummy(end)/denGc(end);
     end
 end

 nota_numGc = numGc_dummy;
 nota_denGc = denGc_dummy;

end

%B matrix created
for i=1:p
     for j =1:p
        B{i,j} = convmtx(nota_numGc{i,j}(end:-1:1),3).';
        [e{i,j} ee{i,j}] = size(B{i,j});
        if e{i,j} < (length(nota_denGc)-1)
            B{i,j}(length(nota_denGc)-1,:)  = [0 0 0];
        end
     end
end

%Br and Hr matrices are created
for i=1:p
   Br{i} = numGr{i,i}(end:-1:1).';
   Hr{i} = Hrfylki(denGr{i,i},nota_denGc);
end

% A matrix is found
Alyap = Afylki(nota_denGc);


    %BBdalk{i} = Bi.
    %BBlina{i} = B.i
   for i = 1:p
        BBdalk{i} = B{i,1}.';
```

```
            BBlinu{i} = B{i,1};
            for z = 2:p
                BBdalk{i} = [BBdalk{i}; B{i,z}.'];
                BBlinu{i} = [BBlinu{i}, B{i,z}];
            end
      end


%G{j} are all the omega_ij*G matrices added together
        for j = 1:p
            G{j} = 0;
          for i = 1:p
                G{j} = G{j} +  W(i,j)*BBdalk{i}*Alyap*BBlinu{i};
          end
        end

      %D matrix created
      for i = 1:p
            D{i} = BBdalk{i}*Hr{i}*Br{i};
      end

      %u_k vectors
      t = zeros(3*p,1);
      i = 1;
      while  i<3*p
          u{i} = t;
          u{i}(i) = 1;
          i = i+3;
      end

      for z = 1:p
          j = 1;
          r{z} = 0;
          for i = 1:p
              r{z} = r{z} + nota_numGc{z,i}(end)*u{j};
              j = j+3;
          end
      end


      DC_vector = r{1};
      for i = 2:p
          DC_vector = [DC_vector,r{i}];
      end
      DC_vector = 1/nota_denGc(end)*DC_vector;

      for i =1:p
          F{i} = [G{i} DC_vector; DC_vector.' zeros(p) ];
      end
```

```
      m = zeros(p,1);

      for i = 1:p
          v{i} = m;
          v{i}(i) = 1;
      end

      for i = 1:p
             E{i} = [D{i}; (numGr{i,i}(end)/denGr{i,i}(end))*v{i}];
      end
      %calculate the PID coefficents
      for i = 1:p
         CC{i} =F{i}\E{i};
      end

else
    %Gc og Gr are not of the same size, stop
    disp('Gc and Gr are not of the same size')
    C = 0;
    Gc_dummy = Gc;
    return
end

%útbý C fylki, sem er skilað
%prepare the C matrix to be returned
for i = 1:p
    w = 1;
    for j = 1:p
      C{i,j} = CC{i}(w+2:-1:w);
      w = w+3;
   end
end

Gc_dummy = tf(nota_numGc,nota_denGc);
end
```

A.2 Nonsquare system

The code used to find the optimized PID controller for a system with a nonsquare TFM using step optimization.

```
function [C Gc_dummy D G] = mimopid_nonsquare(Gc,Gr,W)
%C_ij = [K_Dij K_Pij K_Iij]
%Gc_dummy is the Gc system with the
%dummy poles if they where use if no
%dummy poles where use the it is just Gc
%Gc is a rxp TFM so Gr has to be a pxp TFM

numGc = get(Gc,'num');
```

```
denGc = get(Gc,'den');
denGc = denGc{1,1};
numGr = get(Gr,'num');
denGr = get(Gr,'den');

[p r] = size(numGc);
[Pref Rref] = size(Gr);

%checking if Gr is a pxp TFM
if (p == Pref) & (Rref == Pref)
for i=1:p
   for j =1:r
     numGc{i,j} = lagavigur(numGc{i,j});
    end
  end

  for i=1:Pref
   for j =1:Pref
     numGr{i,j} = lagavigur(numGr{i,j});
    end
  end

  %checking if dummy poles have to be added
  dp = 0;
  for i=1:p
   for j =1:r
     if (length(denGc)-length(numGc{i,j}) < 3)
      %if dp = 1, then dummy poles have to be added
      dp = 1;
    end
  end
end

% if dp = 0 then no dummy pole have to be added
if dp == 0
disp('þarf ekki að bæta við dummy pól')
  nota_numGc = numGc;
  nota_denGc = denGc;

elseif dp == 1 %dummy pole added
  disp('þarf að bæta við dummy pól')
  polar = roots(denGc);
  poltl = max(abs(polar));
  %use 100 times the fastest pole of the system
  DumPole = 100*poltl;

  %checking how many dummy poles to add
  for i = 1:p
   for j=1:r
     n(i,j) = 3-(length(denGc)-length(numGc{i,j}));
```

```
    end
  end

  NrOfDumPole = max(max(n));
    denGc_dummy =  conv(denGc,poly(-ones(1,NrOfDumPole)*DumPole));
    nota_denGC = denGc_dummy;

    %correcting the numerator so the dc-gain is correct
    for i=1:p
   for j=1:r
       numGc_dummy{i,j} = numGc{i,j}.*denGc_dummy(end)/denGc(end);
      end
    end

   nota_numGc = numGc_dummy;
   nota_denGc = denGc_dummy;
end

   for i=1:p
        for j =1:r
           B{i,j} = convmtx(nota_numGc{i,j}(end:-1:1),3).';
           [e{i,j} ee{i,j}] = size(B{i,j});
           %zeros added to the B matrix if needit
           if e{i,j} < (length(nota_denGc)-1)
               B{i,j}(length(nota_denGc)-1,:)  = [0 0 0];
           end
        end
   end

   %Br and Hr matrix created
   for i=1:Pref
      Br{i} = numGr{i,i}(end:-1:1).';
      Hr{i} = Hrfylki(denGr{i,i},nota_denGc);
   end

   % A matrix created
   Alyap = Afylki(nota_denGc);

  %BBdalk{i} = Bi.
  %BBlina{i} = B.i
  for i = 1:p
      BBdalk{i} = B{i,1}.';
      BBlinu{i} = B{i,1};

      for z = 2:r
          BBdalk{i} = [BBdalk{i}; B{i,z}.'];
          BBlinu{i} = [BBlinu{i}, B{i,z}];
        end
  end
```

```
  %G{j} are all the omega_ij*G marices added togeter
  for j = 1:p
      G{j} = 0;
      for i = 1:p
          G{j} = G{j} +  W(i,j)*BBdalk{i}*Alyap*BBlinu{i};
      end
  end

 %D matrix created
 for i = 1:p
     D{i} = BBdalk{i}*Ar{i}*Br{i};
 end

 %u_k vectors created
 t = zeros(3*r,1);
 i = 1;
 while  i<3*r
     u{i} = t;
     u{i}(i) = 1;
     i = i+3;
 end

for z = 1:p
    j = 1;
    Q{z} = 0;
    for i = 1:r
        Q{z} = Q{z} + nota_numGc{z,i}(end)*u{j};
        j = j+3;
    end
end

 W = Q{1};
 for i = 2:p
     W = [H,Q{i}];
 end

 W = 1/nota_denGc(end)*W;
 for i =1:p
     F{i} = [G{i} W; W.' zeros(p) ];
 end

 null_dalkur = zeros(p,1);

 for i = 1:p
     v{i} =  null_dalkur;
     v{i}(i) = 1;
 end

 for i = 1:p
     E{i} = [D{i}; (numGr{i,i}(end)/denGr{i,i}(end))*v{i}];
```

```
 end

 %calculate the PID coefficient
 for i = 1:p
    CC{i} =F{i}\E{i};
 end

%create the C matrix, that is returned
for i =1:p
    q =1;
    for j =1:r
        C{j,i} = CC{i}(q+2:-1:q);
        q = q+3;
    end
end

Gc_dummy = tf(nota_numGc,nota_denGc);

else
    disp('Gr ekki af réttri stærð')
    C = 1;
    Gc_dummy = 1;
    return
end
```

A.3 Code to find the matrices $\mathcal{A}^{(-1)}$ and $\mathcal{H}_i^{(-1)}$

Matlab's `lyap` function was use to find the matrix $\mathcal{A}^{(-1)}$.

```
function A = Afylki(a)
%a is the coefficient form a(s)
C = [zeros(1,length(a)-2)' eye(length(a)-2); -a(end:-1:2) ];
U = zeros(size(C));
U(1,1) = 1/a(end)^2;
A = lyap(C,U);
end
```

The function `lyap is also use to find the matrices` $\mathcal{H}_i^{(-1)}$.

```
function Ar = Arfylki(ar,a)
%a are the coefficient form a(s)
%ar are the coefficient form a_r(s)
C = [zeros(1,length(a)-2)' eye(length(a)-2); -a(end:-1:2) ] ;
Cr = [zeros(1,length(ar)-2)' eye(length(ar)-2); -ar(end:-1:2) ];
Ur = zeros(size(C,1),size(Cr,1));
Ur(1,1) = 1/(a(end)*ar(end));
Urr = zeros(size(Cr));
Ar = lyap(C,Cr',Ur);
end
```

A.4 Code to find the matrices $\mathcal{A}$ and $\mathcal{H}_i$

Matlab's `lyap` function was use to find the matrix $\mathcal{A}$.

```
function A = Aimp(a)
%a is the coefficient form a(s)
F = [zeros(1,length(a)-2)' eye(length(a)-2); -a(end:-1:2) ];
U = zeros(size(F));
U(end,end) = 1;
A = lyap(F,U);
end
```

The Matlab code to find the $\mathcal{H}_i$ matrices

```
function H = H_nytt(polar,polar_ref)

n = length(polar);
tol = 0.01;
[L d] = classifyRoots(polar,tol);
nr = length(polar_ref);
%flokka póla ref kerfisins
[Lr dr] = classifyRoots(polar_ref,tol);

[EE EEr] = constrEEandEErmatr(L,d,Lr,dr,n,nr);

[K,J] = cHmatrix_temp(n,L,d);

H = K.'*EEr;
end
```

Bibliography

[1] I.I. Ruiz-López, G.C. Rodríguez-Jimenes, M.A.García-Alvarado, *Robust MIMO PID Controllers Tuning Based on Complex/Real Ratio of the Characteristic Matrix Eigenvalues*, Chemical Engineering Science 61, 2006, pp. 4332 - 4340.

[2] M.J. Lengare, R.H. Chile, L.M. Waghmare, B. Parmar, *Auto Tuning of PID Controller for MIMO Processes*, World Academy of Science, Engineering and Technology 45, 2008, pp. 305 - 308.

[3] K. Tamura, H. Ohmori, *Auto-Tuning Method of Expanded PID Control for MIMO Systems*, SICE-ICASE International Joint Conference 2006, Bexco, Busan, Korea, Oct. 18 - 21, 2006, pp. 3270 - 3275.

[4] M. Ashraf, R. Mahmud, *Automatic Tuning of Digital PID Controllers for MIMO Processes*, Conference on Electrical and Computer Engineering, 2005, Saskatoon, Canada, May 1 - 4, 2005, pp. 2245 - 2248.

[5] A.N. Gündes, A.B. Özgüler, *PID Stabilization of MIMO Plants*, IEEE Transactions on Automatic Control, Vol. 52, No. 8, August 2007, pp. 1502 - 1508.

[6] A.N. Mete, A.N. Gündes, A.N. Palazoglu, *Reliable Decentralized PID Stabilization of MIMO Systems*, The 2006 American Control Conference, Minneapolis, Minnesota, USA, June 14 - 16, 2006, pp. 5306 - 5311.

[7] A.N. Gündes, A.N. Mete, A.N. Palazoglu, *Reliable decentralized PID controller synthesis for two-channel MIMO processes*, Automatica, Vol. 45, Feb. 2009, pp. 353-363.

[8] P.L. Falb, W.A. Wolovich, *Decoupling in the Design and Synthesis of Multivariable Control Systems*, IEEE Transactions on Automatic control, Vol. AC-12, No. 6, Dec. 1967, pp. 651 - 659.

[9] N. Otsuka, H. Inaba, *Block decoupling by incomplete state feedback for linear mulivariable systems*, International Journal of Systems Science, Vol. 22, No. 8, 1991, pp. 1419 - 1437.

[10] A.S. Hauksdóttir, M. Ierapetritou, *Simultaneous decoupling and pole placement without canceling invarient zeros*, 2001 Amercan Control Conference, Arlington, Virginia, USA, June 25 - 27, 2001, pp. 1675 - 1680.

[11] J. Stefanovski, *Suffcient Conditions for Linear Control System Decoupling by Static State Feedback*, IEEE Transactions on Automatic Control, Vol. 46, No. 6, june 2001, pp. 984 - 990.

[12] A.S. Hauksdóttir, *Analytic expressions of transfer function responses and choice of numerator coefficients (zeros)*, IEEE Transactions on Automatic Control, Vol. 41, No. 10, Oct. 1996, pp. 1482 - 1488

[13] A.S. Hauksdóttir, H. Hjaltadóttir, *Closed–form expressions of the transfer function responses*, The 2003 American Control Conference, Denver Colorado, June 4 - 6 2003, pp. 3234 - 3239.

[14] G. Herjólfsson, A.S. Hauksdóttir, S.Þ. Sigurðsson, *Closed Form Expressions of Linear Continuous- and Discrete Time Filter Responses*, NORSIG 2006, 7th Nordic Signal Processing Symposium, Reykjavík, Iceland, June 7-9, 2006.

[15] A.S. Hauksdóttir, S.Þ. Sigurðsson, S.Ö. Aðalgeirsson, G. Herjólfsson, *Closed form solutions of the Sylvester and the Lyapunov equations - closed form Gramians*, The 46th IEEE Conference on Decision and Control, New Orleans, Dec. 12 - 14 2007, pp. 2797 - 2802.

[16] A.S. Hauksdóttir, *Optimal zero locations of continuous-time systems with distinct poles tracking first-order step responses*, IEEE Transactions on Circuits and Systems-I: Fundamental Theory and Applications, Vol. 49, No. 5, May 2002, pp. 685 - 688.

[17] A.S. Hauksdóttir, *Optimal zero locations of continuous–time systems with distinct poles tracking reference step responses*, Dynamics of Continuous, Discrete, and Impulsive Systems, Series B: Applications & Algorithms, Vol. 11, 2004, pp. 353 - 361.

[18] G. Herjólfsson, B. Ævarsson, A.S. Hauksdóttir, S.Þ. Sigurðsson, *Zero Optimized Tracking of Continuous-Time Systems*, The 2005 Americal Control Conference, Portland, Oregon, June 8-10, 2005, pp. 1203 - 1208.

[19] A.S. Hauksdóttir, *Optimal zeros for model reduction of continuous-time systems*, The 2000 American Control Conference, Chicago, Illinois, USA, June 28-30, 2000, pp. 1889 - 1893.

[20] A.S. Hauksdóttir, *Optimal zeros for model reductions of discrete-time systems*, The 2001 American Control Conference, Arlington, Virginia, USA, June 25-27, 2001, pp. 2584 - 2589.

[21] G. Herjólfsson, B. Ævarsson, A.S. Hauksdóttir, S.Þ. Sigurðsson, *Closed Form $\mathcal{L}_2/\mathcal{H}_2$ Optimizing of Zeros for Model Reduction of Linear Continuous Time Systems*, International Journal of Control, Vol. 82, March 2009, pp. 555-570.

[22] G. Herjólfsson, A.S. Hauksdóttir, *Direct Computation of Optimal PID controller*, The 42nd IEEE Conference on Decision and Control, Maui, Hawaii, USA, Dec. 9 - 12, 2003, pp. 3234 - 3239.

[23] A.S. Hauksdóttir, *The Optimal impulse response tracking and disturbance rejecting controllers*, The 2004 American Control Conference, Boston, Massachusetts, June 30 - July 2, 2004, pp. 2075 - 2080.

[24] G. Herjólfsson, A.S. Hauksdóttir, *Direct computation of optimal discrete-time PID controllers*, The 2004 American Control Conference, Boston, Massachusetts, June 30 - July 2, 2004, pp. 46 - 51.

[25] A.S. Hauksdóttir, G. Herjólfsson, B. Ævarsson, S.Þ. Sigurðsson, *Zero Optimizing Continuous-Time Tracking and Disturbance Rejecting Controllers*, The 16th IFAC World Congress (International Federation of Automatic Control), Prague, July 4-8, 2005.

[26] A.S. Hauksdóttir, G. Herjólfsson, S.Þ. Sigurðsson, *Zero Optimizing Tracking and Disturbance Rejecting Controllers - The extended PID controller*, The 45th IEEE Conference on Decision and control (CDC), San Diego, California, USA, Dec. 13 - 15, 2006, pp. 8633 - 8638.

[27] A.S. Hauksdóttir, G. Herjólfsson, S.Þ. Sigurðsson, *Optimizing Zero Tracking and Disturbance Rejecting Controllers - The Generalized PID controller*, The 2007 American Control Confernce, New York City, USA, July 11-13, 2007, pp. 5790 - 5795.

[28] G. Herjólfsson, A.S. Hauksdóttir, S.Þ. Sigurðsson, *Closed Form $\mathcal{L}_2/\mathcal{H}_2$ Optimal PID controller*, to be submitted.

[29] A.S. Hauksdóttir, S.Þ. Sigurðsson, S.Ö. Aðalgeirsson, G. Herjólfsson, *Closed Form Expressions for Linear MIMO System Responses and Solutions of the Lyapunov Equation*, The 46th IEEE Conference on Decision and control (CDC), New Orleans, USA, Dec. 12-14, 2007, pp. 2797 - 2802.

[30] A.S. Hauksdóttir, S.Þ. Sigurðsson, S.Ö. Aðalgeirsson, H. Þorgilsson, G. Herjólfsson, *Closed Form Solutions of the Sylvester and the Lyapunov Equations - Closed Form Gramians*, The 2008 American Control Conference, Seattle, Washington, June 11-13, 2008, pp. 2585 - 2590.

[31] A.S. Hauksdóttir, S.Þ. Sigurðsson, *The continuous closed form controllability Gramian and it's inverse*, The 2009 American Control Conference, St. Louis, Missouri, USA, June 10-12, 2009, pp. 5345 - 5350.

[32] H. Þorgilsson, G. Herjólfsson, A.S. Hauksdóttir, S.Þ. Sigurðsson, *Scheduled Control of a Small Unmanned Underwater Vehicle Using Zero Optimization*, The 45th Conference on Decision and Control (CDC), San Diego California, USA, Dec. 13-15, 2006, pp. 5900 - 5905.

[33] B. Ævarsson, *Linear System Responses and Optimization Applications*, MS thesis, Department of Electrical and Computer Engineering, University of Iceland, Reykjavík, Iceland 2005.

[34] G. Herjólfsson, *Linear System Responses and Zero Optimizing Controllers*, MS thesis, Department of Electrical and Computer Engineering, University of Iceland, Reykjavík, Iceland, 2004.

[35] F.R. Gantmacher. *The Theory of Matrices*, Chelsea Publishing Company, New York, 1959.

[36] G. Herjólfsson, A.S. Hauksdóttir, S.Þ. Sigurðsson, *Closed Form Expressions of Linear Continuous and Discrete-Time Filter Responses*, Norsig 2006, 7th Nordic Signal Processing Symposium, 4 pages, Reykjavík Iceland, June 7 - 9, 2006.

[37] H. Þorgilsson, *Control of a Small Unmanned Underwater Vehicle Using Zero Optimized PID Controllers*, MS thesis, Department of Electrical and Computer Engineering, University of Iceland, Reykjavík, Iceland, 2006.

[38] A.S. Hauksdóttir, *Personal communication*, 2010.

[39] T. Kailath, *Linear Systems*, Prentice-Hall, 1980.

[40] A.S. Hauksdóttir, T. Söderström, Y.P. Thorfinnsson, A. Gestsson, *System Identification of a Three-Phase Submerged-Arc Ferosilicon Furnace*, IEEE Transactions on Control Systems Technology, Vol 3. No. 4, Dec. 1995, pp. 377 - 387.

[41] A.S. Hauksdóttir, A. Gestsson, A. Vésteinsson, *Submerged-arc ferrosilicon furnace similator: validation for different furnaces and operating ranges*, Control Engineering Practice, 1998, pp 1035 - 1042.

[42] A.S. Hauksdóttir, A. Gestsson, A. Vésteinsson, *Current control of a three-phase submerged arc ferrosilicon furnace*, Control Engineering Practice, Vol. 10, No. 4. pp. 457 - 463.