WILF-CLASSIFICATION OF MESH PATTERNS OF SHORT LENGTH

RESEARCH REPORT

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Abstract. This B.Sc. project deals with mesh pattern avoidance in permutations. A mesh pattern is a pair \( p = (\tau, R) \), where \( \tau \) is a permutation in \( S_k \) and \( R \subseteq [0, k] \times [0, k] \). The elements of \( R \) denote filled boxes in a mesh pattern. For a permutation \( \pi \) to contain a mesh pattern, it has to contain the underlying classical pattern \( \tau \), and no points in \( \pi \) can be located in the shaded boxes. In this paper we begin the study of Wilf-classifying mesh patterns by classifying 776 out of the 1024 mesh patterns of length 2. Two mesh patterns \( p \) and \( q \) are said to be equivalent if for any permutation \( \pi \), \( \pi \) avoids \( p \) if and only if \( \pi \) avoids \( q \). The paper introduces a new operation that preserves pattern equivalence and provides rules determining which additional boxes in a mesh pattern \( p \) can be shaded. This is useful to lower the number of patterns one needs to look at in the process of Wilf-classifying patterns. We also have some observations on the only non-trivial interval pattern of length 3.

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1. Introduction

Let $A$ be a finite, non-empty set. A one-to-one correspondence from $A$ to itself is a permutation. Let $A = \{1, 2, \ldots, n\}$ and denote a permutation as a word, $\pi = \pi_1\pi_2\cdots\pi_n$. For example, $1324$ is a permutation of the set $\{1, 2, 3, 4\}$ with $\pi_1 = 1$, $\pi_2 = 3$, $\pi_3 = 2$ and $\pi_4 = 4$. Let $S_n$ be the set of all permutations of length $n$.

A pattern is a permutation $p \in S_k$. The pattern $213 \in S_3$ can be drawn as follows, where the horizontal lines represent the values and the vertical ones denote the positions in the pattern.

$$231 = \begin{array}{c}
\text{\textbullet} \\
\text{\textbullet} \\
\text{\textbullet}
\end{array}$$

Since 2 is the first letter in $p$, it is located on the second horizontal line (counted from below) and on the first vertical line (counted from the left). The second letter in $p$ is 3 and therefore the second point is on the third horizontal line and the second vertical line. Similarly the point representing 1 is on the first horizontal line and the third vertical line. Permutations can be drawn in the same way.

We say that a pattern $p$ occurs in a permutation $\pi \in S_n$ if there is a subsequence of $\pi$ whose letters are in the same relative order of size as the letters of $p$. This sequence is called an occurrence of the pattern $p$ in the permutation $\pi$. If a pattern $p$ occurs in a permutation $\pi$ we say that $\pi$ contains the pattern $p$. For example, the permutation $25134$ contains the pattern $p = 312 = \begin{array}{c}
\text{\textbullet} \\
\text{\textbullet} \\
\text{\textbullet}
\end{array}$ with the subsequence $534$. Let us see this occurrence on a diagram, where the points corresponding to the occurrence of the pattern have been circled.

$$\pi = 25134 = \begin{array}{c}
\text{\textbullet} \\
\text{\textbullet} \\
\text{\textbullet} \\
\text{\textbullet}
\end{array}$$

A permutation that does not contain a pattern is said to avoid the pattern. For example, a permutation $\pi \in S_n$ avoids the pattern $p = 231$ if there do not exist $1 \leq i < j < k \leq n$ with $\pi(k) < \pi(i) < \pi(j)$. An example of a permutation that avoids $p$ is the following.

$$\pi = 51423 = \begin{array}{c}
\text{\textbullet} \\
\text{\textbullet} \\
\text{\textbullet} \\
\text{\textbullet} \\
\text{\textbullet}
\end{array}$$

Definition 1. Two patterns, $p$ and $q$, are said to be Wilf-equivalent if the same number of permutations of length $n$ avoid the patterns, for all $n$.

Definition 2. Wilf-classification is the process of sorting patterns into classes by Wilf-equivalence.

Classical patterns of length 3 were Wilf-classified by Simion and Schmidt in [8]. They showed that the number of permutations avoiding each of the
classical pattern of length 3, are the Catalan numbers. After that, classical patterns of more length have also been studied, but the Wilf-classification has not been finished.

**Vincular patterns** were first defined by Babson and Steingrímsson [2]. For a vincular pattern $p$ to occur in a permutation $\pi$, $p$ requires letters to be adjacent in $\pi$. For example the pattern $p = \begin{array}{c} \end{array}$ requires the letters corresponding to 1 and 3 in a permutation to be adjacent. Graphically this means that no points can be in the shaded area. We can see that $p$ occurs in the permutation

$$\pi = 53124 = \begin{array}{c} \end{array}$$

because $\pi$ includes the classical pattern 213 with the subsequence 324 and because the letters in the permutation that correspond to 1 and 3 in the pattern, i.e., 2 and 4, are adjacent in $\pi$.

The permutation $\pi = 52314$ avoids the pattern $p = \begin{array}{c} \end{array}$, since the only occurrence of the classical pattern 123 is the subsequence 234 and the letters 3 and 4 are not adjacent in the permutation. Let us see this graphically.

$$\pi = 52314 = \begin{array}{c} \end{array}$$

Vincular patterns of length 3 were Wilf-classified by Claesson in [3]. He proved that the number of permutations avoiding 8 of the vincular patterns of length 3, is given by the Bell numbers and the remaining 4 give the Catalan numbers.

A **bivincular pattern** is a pattern that puts constraints on positions and values in a permutation. Bivincular patterns where first introduced by Bousquet-Mélou, Claesson, Dukes and Kitaev [3]. The pattern $p = \begin{array}{c} \end{array}$ requires the letters corresponding to 1 and 3 in a permutation to be adjacent and it requires the letters corresponding to 2 and 3 in a permutation to be adjacent in size. The permutation

$$\pi = 14253 = \begin{array}{c} \end{array}$$

contains the pattern $p$ because it contains the classical pattern 132 with the subsequence 143 and letters 1 and 4 are adjacent and letters 3 and 4 are adjacent in size. On a diagram, we observe that there are no points in the shaded areas.

$$\pi = 14253 = \begin{array}{c} \end{array}$$

However the permutation $\pi = 12543$ contains one occurrence of the classical pattern 132 with the subsequence 253 but 5 and 3 are not adjacent in
values, so that is not an occurrence of $p$ and $\pi$ therefore avoids $p$.

\[
\pi = 12543 = \begin{array}{c}
\cdot
\end{array}
\]

Bivincular patterns of length 2 and 3 were Wilf-classified by Parviainen in [7].

Mesh patterns where first introduced by Brändén and Claesson [4]. A pair $(\tau, R)$, where $\tau$ is a permutation in $S_k$ and $R \subseteq [0, k] \times [0, k]$, where $[0, k]$ denotes the interval of the integers from 0 to $k$, is a mesh pattern.

**Definition 3.** Let $[i, j]$ denote the box whose corners have coordinates $(i, j), (i, j + 1), (i + 1, j + 1)$ and $(i + 1, j)$.

An example of a mesh pattern is the classical pattern 312 along with $R = \{[1, 2], [2, 1]\}$. We draw this as

\[
p = (312, \{[1, 2], [2, 1]\}) = \begin{array}{c}
\cdot
\end{array}
\]

Let us see a permutation that contains this pattern.

\[
\pi = 521643 = \begin{array}{c}
\cdot
\end{array}
\]

The permutation has an occurrence of the pattern $p$ with the subsequence 514, since it forms the classical pattern 312 and there are no points in the shaded areas.

Let’s now look at the permutation $\pi = 32145$. This permutation avoids the pattern $p = (123, \{[0, 1], [1, 0], [2, 2]\}) = \begin{array}{c}
\cdot
\end{array}$ because for all occurrences of the classical pattern 123 there is at least one point in at least one of the shaded boxes. For example, the subsequence 245 in $\pi$ is an occurrence of the classical pattern 123 but not of the pattern $p$ since the point representing 1 is in one of the shaded areas. Let us see this on a diagram.

\[
\pi = 32145 = \begin{array}{c}
\cdot
\end{array}
\]

In this project our goal is to start the Wilf-classification of mesh patterns.

2. Preliminaries

The number of mesh patterns increases very rapidly with length so to make the Wilf-classification a little easier we will use some operations to find Wilf-equivalence. Some of these operations are known, and others are new. The first known operations are the symmetries reverse, complement and inverse. For a given mesh pattern $(\tau, R)$ of length $n$, we define

\[
(\tau, R)^r = (\tau^r, R^r), \quad (\tau, R)^c = (\tau^c, R^c), \quad (\tau, R)^i = (\tau^i, R^i),
\]
where \( \tau^r \) is the usual reverse of the permutation \( \tau \), \( \tau^c \) the usual complement, \( \tau^i \) the usual inverse, and

\[
R^r = \{ [n - x, y] \mid [x, y] \in R \}, \\
R^c = \{ [x, n - y] \mid [x, y] \in R \}, \\
R^i = \{ [y, x] \mid [x, y] \in R \}.
\]

Hence, reverse is a reflection around the vertical center line, complement is a reflection around the horizontal center line and inverse is the reflection around the southwest to northeast diagonal. Following is an example of the use of these symmetries on the pattern \( p = (312, \{0, 1, 3, 2\}) \).

It is well-known that a permutation \( \pi \) avoids a mesh pattern \( p \) if and only if the permutation \( \pi^r \) avoids \( p^r \). That is, the reverse operation preserves Wilf-equivalence. The same applies for complement and inverse or any composition of these three operations.

**Definition 4.** We define the operation *up-shift* on a pattern \( p = (\tau, R) \), denoted \( p^\uparrow \). The point \((u,v)\) in a pattern \( \tau \), where \( u,v \in \{1, 2, \ldots, n\} \), becomes the point \((u, (v \mod n) + 1)\) in \( \tau^i \). Then for a box \([a,b]\), where \( a,b \in \{0, 1, \ldots, n\} \), in \( R \) we get \([a, (b + 1) \mod (n + 1)]\) in \( R^\uparrow \). Let the up-shift of \( p \) be \( p^\uparrow = (\tau^i, R^i) \). Up-shift can also be used on permutations.

**Example 5.** This example shows usage of up-shift on a mesh pattern of length 6.

**Observation 6.** Now, for a pattern \( p = \{\tau, R\} \) of length \( n \) where the top line is shaded, that is \( \{(1,n), (1,\) \(n), \ldots, (n,n)\} \subseteq R \), it holds that \( \pi \) avoids \( p \) if and only if \( \pi^\uparrow \) avoids \( p^\uparrow \). That is, up-shift preserves Wilf-equivalence for this kind of pattern.
Example 7. Here is another example that shows the usage of up-shift on a mesh pattern of length 6.

\[
\begin{array}{c}
\text{up-shift} \\
\end{array}
\]

According to Observation 6 these two patterns are Wilf-equivalent.

Given a permutation \( \pi \in S_n \), \( \pi^0 \) is the permutation \( 0 \pi \) and \( \pi \oplus 1 = (\pi_1 + 1 \mod (n+1))(\pi_2 + 1 \mod (n+1)) \cdots (\pi_n + 1 \mod (n+1)) \). For a given circular permutation \( \lambda \) of \([0, n] \), \( \lambda_0 \) is the permutation in \( S_n \) obtained by reading \( \lambda \) from 0, e.g. \( 14032_0 = 3214 \).

Definition 8. Úlfarsson defined a new operation in [11], called the toric operation. When toric is used on a classical pattern \( \tau \) of length \( n \) it is denoted \( \tau_t \) and \( \tau_t = ((\tau_0 \oplus 1)_0 \). Then for each box \([a, b]\), where \( a, b \in \{0, 1, \ldots, n\} \), in \( R \) we get \([a + (n + 1 - \ell) \mod (n + 1), b + 1 \mod (n + 1)]\) in \( R_t \), where \( \ell \) is the position of the letter \( n \) in the classical pattern \( \tau \). Now, by using the toric operation on a mesh pattern \( p = (\tau, R) \) we get \( p_t = (\tau_t, R_t) \). Toric can also be used on permutations.

Example 9. This example shows usage of the toric operation on a mesh pattern of length 8.

\[
\begin{array}{c}
\text{toric} \\
\end{array}
\]

Observation 10. For a pattern \( p = (\tau, R) \) of length \( n \) where the top line is shaded, that is \( \{(1, n), (1, n), \ldots, (n, n)\} \subseteq R \), it holds that \( \pi \) avoids \( p \) if and only if \( \pi_t \) avoids \( p_t \). That is, the toric operation preserves Wilf-equivalence for this kind of pattern.

Definition 11. Two mesh patterns \( p \) and \( q \) are said to be equivalent, denoted by \( \cong \), if for any permutation \( \pi \), \( \pi \) avoids \( p \) if and only if \( \pi \) avoids \( q \).

Obviously equivalent patterns are Wilf-equivalent.

Observation 12. Let \( R' \subseteq R \). Then any occurrence of \((\tau, R)\) in a permutation is an occurrence of \((\tau, R')\).

Example 13. Consider the mesh pattern \( p = (12, \emptyset) = \begin{array}{cc} & \end{array} \). Let \( u \) be the point \((1, 1)\) and \( v \) the point \((2, 2)\). We claim that the mesh pattern \( q = (12, \{0, 0\}) = \begin{array}{cc} & \end{array} \) is equivalent to \( p \). To prove this let \( \pi \) be a permutation that contains \( p \) and consider a particular occurrence of it. First we show
that if a permutation contains \( p \) it also contains \( q \). Let \( k \) be the number of points in the box \([0, 0]\) in \( \pi \). If \( k = 0 \), it is clear that \( \pi \) contains \( q \) as well. If \( k \geq 1 \), then we can choose the leftmost or the lowest point, call it \( d \), in the box \([0, 0]\) and think of that point as \( u \). Replacing \( u \) with \( d \), it is easy to see that the conditions of \( q \) are satisfied. This can be interpreted as shading the box \([0, 0]\). Note that any occurrence of \( q \) is an occurrence of \( p \) so the patterns are equivalent. This is a special case of Observation 12. Figure 1 shows this equivalence. In the former image in the figure, the pattern \( p \) can be found with the points \((7, 5)\) as \( u \) and \((8, 8)\) as \( v \). There are points in the area at the lower left of \( u \) and the circled point, \((2, 2)\), is the leftmost point in that area, and thus we denote that point as \( d \). In the latter image, point \( v \) is still \((8, 8)\) but now we think of \( u \) as \( d \) and hence, we can shade the area at the lower left of \( u \).

\[
\begin{align*}
\text{Figure 1. This picture shows two equivalent patterns in the permutation 72463158.}
\end{align*}
\]

This example generalizes to a new operation, introduced in Lemma 14, which preserves equivalence of mesh patterns.

**Lemma 14 (Shading Lemma).** Let \((\tau, R)\) be a mesh pattern of length \( n \) such that \( \tau(i) = j \) and \([i, j] \notin R\). If all of the following conditions are satisfied:

- The box \([i - 1, j - 1]\) is not in \( R \);
- At most one of the boxes \([i, j - 1]\), \([i - 1, j]\) is in \( R \);
- If the box \([\ell, j - 1]\) is in \( R \) (\( \ell \neq i - 1 \)) then the box \([\ell, j]\) is also in \( R \);
- If the box \([i - 1, \ell]\) is in \( R \) (\( \ell \neq j - 1 \)) then the box \([i, \ell]\) is also in \( R \);

then the patterns \((\tau, R)\) and \((\tau, R \cup \{[i, j]\})\) are equivalent. Analogous conditions determine if other boxes neighboring the point \((i, j)\) can be added to \( R \).

**Proof.** According to Observation 12 we know that if a permutation \( \pi \) contains the pattern \((\tau, R \cup \{[i, j]\})\), \( \pi \) also contains the pattern \((\tau, R)\), as \( R \subseteq R \cup \{[i, j]\}\).
Figure 2. If the condition of the lemma are satisfied the point \((i,j)\) can be moved to the top-most or the right-most point in the box \([i,j]\).

Now we have to show that if a permutation \(\pi\) contains the pattern \((\tau, R)\) it also contains the pattern \((\tau, R \cup \{(i,j)\})\). First we consider the case where \(R\) is empty. Consider an occurrence of the pattern \((\tau, R)\) in a permutation \(\pi\). Let \(k\) be the number of points in \([i,j]\) in \(\pi\). If \(k = 0\) we see that we can shade the box \([i,j]\) in \((\tau, R)\) since there are no points in the box \([i,j]\) in \(\pi\). Thus, if \(\pi\) contains \((\tau, R)\) it also contains \((\tau, R \cup \{(i,j)\})\) and hence, those patterns are equivalent. If \(k \geq 1\), then we can choose the rightmost or the topmost point, say \((u,v)\), in the box \([i,j]\) in \(\pi\) and think of that as point \((i,j)\). Thinking of \((u,v)\) as \((i,j)\), we can see that the box \([u,v]\) which lies inside the box \([i,j]\) will not include any other points. Therefore by choosing the rightmost or the topmost point \(\pi\) will contain both patterns. Hence, we can see that \((\tau, R)\) is equivalent to \((\tau, R \cup \{(i,j)\})\).

Now we consider the case where \([a,b] \in R\) for some \(a,b\). By choosing the rightmost or topmost point while shading the box \([i,j]\) we can imagine translating the line \(x = i\) to the line \(x = u\). By doing that, we extend the boxes \([i-1,j]\) for all \(j\). Given one or more shaded boxes \([i-1,\ell]\), with \(\ell \neq j-1, j\), we are allowed to extend the boxes if the box \([i,\ell]\) is also shaded. Since, the box \([i,\ell]\) does not contain any points extending the box \([i-1,\ell]\) into the area of the box \([i,\ell]\) does not add any points to the box \([i-1,\ell]\). Otherwise, there could be points in the box \([i,\ell]\) which would be included in the box \([i-1,\ell]\) after the extension. Similarly we see that in order to be able to extend the boxes \([\ell,j-1]\), where \(l \neq i-1, i\), we need to have the boxes \([\ell,j]\) shaded as well. Then we consider the boxes that are neighbors of \((i,j)\). If the box \([i-1,j-1] \notin R\) then we can choose the rightmost or the topmost point in the box \([i,j]\) and since the box \([i-1,j-1]\) is not shaded we can add points to the extended box. Assume \([i,j-1] \notin R\), then we can choose the rightmost or the topmost point in the box \([i,j]\) and the extended box \([i,j-1]\) can contain new points since this box is not shaded. The case
for the box $[i-1, j]$ is similar. Thus, if all conditions of the Shading Lemma are fulfilled, $\pi$ will contain both patterns $(\tau, R)$ and $(\tau, R \cup \{[i, j]\})$. Hence, the two patterns are equivalent. \hfill \Box

By using Lemma 14 the following equivalence can be found. The point that the arrow is pointing from is the point $(i, j)$ in the lemma.

**Example 15.**

![Shading Example](image)

First we can shade the box $[2, 1]$ because when choosing the leftmost or the topmost point in that box, both of the shaded boxes, $[1, 1]$ and $[2, 2]$, will not be extended. When shading the box $[1, 2]$ we choose the rightmost or the lowest point and the same applies as before. However, trying to shade the box $[2, 0]$ both boxes $[1, 1]$ and $[2, 1]$ will be extended into a non-shaded area and might therefore contain some points. The point $(3, 1)$ and the box $[2, 0]$ do not fulfill the condition of the lemma.

**Example 16.**

![Shading Example](image)

We can shade the box $[0, 5]$ by choosing the leftmost or the topmost point in the box, because none of the shaded boxes touching the lines $x = 0$ or $y = 5$ will be extended by this. We can also shade the box $[3, 3]$ by choosing the rightmost or the topmost point. This is because the shaded boxes touching the lines $x = 3$ and $y = 3$ will either not be extended, boxes $[0, 3]$ and $[3, 5]$, or are extended into a shaded area, box $[2, 5]$. However when trying to shade box $[1, 0]$, box $[4, 1]$ would have to be extended into a non-shaded area. Therefore box $[1, 0]$ cannot be shaded.

**Definition 17.** We define a set of operations called *switch operations*. Each operation performs the following. It takes a pattern $p$ and breaks it down into two parts by the largest letter $n$. The first part contains the part of the pattern $p$ that appears before $n$. The second part contains the part of $p$ that appears after $n$. Then one of the two operations, $id$, $r$, is used on each part, where $(\tau, R)^{id} = (\tau, R)$. After that these parts can be switched, i.e., part two will then appear before $n$ and part one after $n$. The operations will be denoted as, $S_{a,b,d}$. Where $a$ denotes the operation used on part one and $b$ denotes the operation used on part two. The letter $d$ is 1 if part one and two are switched and 0 otherwise. These switch operations can also be used on permutations.
Example 18. This example shows usage of the switch operation, $S_{r,r,1}$ on a mesh pattern of length 6.

![Diagram](image1)

Observation 19. Now, for a pattern $p = \{\tau, R\}$ of length $n$ where the top line is shaded, that is $\{(1, n), (1, n), \ldots, (n, n)\} \subseteq R$, it holds that $\pi$ avoids $p$ if and only if $\pi^{S_{a,b,d}}$ avoids $p^{S_{a,b,d}}$. That is, the switch operations preserve Wilf-equivalence for this kind of pattern.

Example 20. Here is another example showing usage of the switch operation, $S_{r,id,1}$, on a mesh pattern of length 6.

![Diagram](image2)

According to Observation 19 these two patterns are Wilf-equivalent.

Definition 21. A pattern class is a class containing patterns of the same length which are Wilf-equivalent.

The number of mesh patterns of length 1 is $2 \cdot 2^3 = 16$. By using the operations explained above, these 16 patterns can be sorted to 4 classes by Wilf-equivalence. We choose one pattern from each class to represent the class. Let us see those representative patterns on a diagram.

![Diagram](image3)

Occurrences of these patterns in a permutation are known. Each occurrence of the first pattern in a permutation $\pi$ is a left-to-right maxima in $\pi$. An occurrence of the second pattern in a permutation $\pi$ is a strong fixed point in $\pi$. Both left-to-right maxima and strong fixed points in connection to mesh patterns were introduced by Brändén and Claesson [4]. All permutations of length $n$ that contain the third pattern begin with $n$. There is only one permutation that contains the last pattern, namely 1.

The number of mesh patterns of length 2 is $2^{10} = 1024$. When using the operations described here above along with the computer algebra system Sage\(^1\) the mesh patterns of length 2 can be sorted into 64 classes by Wilf-equivalence. The code we used for that is in appendix C. Then by using the results of Wilf-classification of bivincular patterns done by Parviainen

\(^1\)www.sagemath.org
in [7] the number of classes will be decreased to 58. The classes which contain bivincular patterns can be found in section A.2. Next section contains classes for which we have found formulas for.

3. Pattern Classes

From the 58 pattern classes, we will present the proofs of 18 pattern classes. In Figure 3 we have the representative patterns for each of the proved pattern classes.

<table>
<thead>
<tr>
<th>Representative</th>
<th>Formula</th>
<th># of patterns</th>
</tr>
</thead>
<tbody>
<tr>
<td>![Pattern 1]</td>
<td>1</td>
<td>28</td>
</tr>
<tr>
<td>![Pattern 2]</td>
<td>$(n - 1)!$</td>
<td>40</td>
</tr>
<tr>
<td>![Pattern 3]</td>
<td>$a_n = n \cdot a_{n-1} - a_{n-2}$</td>
<td>32</td>
</tr>
<tr>
<td>![Pattern 4]</td>
<td>$a_n = (n - 1)a_{n-1} + (n - 2)a_{n-2}$</td>
<td>32</td>
</tr>
<tr>
<td>![Pattern 5]</td>
<td>$[x^n] \left(1 - \sum_{i=1}^{\infty} \frac{1}{i!} x^i\right)$</td>
<td>4</td>
</tr>
<tr>
<td>![Pattern 6]</td>
<td>$\sum_{i=1}^{n-1} \frac{(n-1)!}{i!}$</td>
<td>84</td>
</tr>
<tr>
<td>![Pattern 7]</td>
<td>$\frac{x^n}{n} \left(1 + \sum_{i=1}^{n} (i - 1)! \cdot x^i\right)$</td>
<td>60</td>
</tr>
<tr>
<td>![Pattern 8]</td>
<td>$n! - \sum_{i=1}^{n-1} \sum_{\ell=1}^{i} (i - \ell)! (n - i - \ell)! i!$</td>
<td>4</td>
</tr>
<tr>
<td>![Pattern 9]</td>
<td>$n! - \sum_{k=0}^{n-2} \sum_{j=0}^{k} j! (k - j)! (n - 2 - k)!$</td>
<td>4</td>
</tr>
<tr>
<td>![Pattern 10]</td>
<td>$n! - (n - 1)! + [x^n] \frac{F(x)}{1 + x F(x)}$</td>
<td>8</td>
</tr>
<tr>
<td>![Pattern 11]</td>
<td>$n! - \sum_{i=0}^{n-2} i! (n - 1 - i)!$</td>
<td>16</td>
</tr>
<tr>
<td>![Pattern 12]</td>
<td>$n! - \sum_{k=1}^{n-1} (k - 1)! (n - k - 1)$</td>
<td>24</td>
</tr>
</tbody>
</table>

Figure 3. The pattern classes for which we have formulas for the number of avoiding permutations.

3.1. Pattern class 1. This pattern class contains 4 patterns and is represented by the pattern

$$p = \begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix}$$

The other patterns in this pattern class can be found in Appendix A.1.
Proposition 22. The only permutation avoiding $p$ is $1^r = n(n-1) \cdots 21$ and therefore

$$|S_n(\begin{array}{c|c|c|c}1 & 2 & \cdots & n-1 \\ \hline n & n-1 & \cdots & 1 \end{array})| = 1$$

for all $n \geq 0$.

Proof. For all permutations in $S_n$, except for $1^r = n(n-1) \cdots 21$, we can choose the leftmost point $u$ such that there is at least one higher point to the right of $u$. Then we find the highest point to the right of $u$ and call it $v$. The points $u, v$ form the pattern $p$. This is true since there cannot be a point $k$ in the shaded area to the left of $u$ because then $k$ would be the leftmost point that has a higher point to the right of it. Also, there cannot be a point in the shaded area above $v$ because we chose $v$ to be the highest point to the right of $u$. \qed

3.2. Pattern class 2. This pattern class contains 16 patterns and is represented by the pattern

$$p = \begin{array}{c|c|c|c}1 & 2 & \cdots & n-1 \\ \hline n & n-1 & \cdots & 1 \end{array}$$

The other patterns in this pattern class can be found in Appendix A.1.

Proposition 23. The only permutation avoiding $p$ is $1^r$ and therefore

$$|S_n(\begin{array}{c|c|c|c}1 & 2 & \cdots & n-1 \\ \hline n & n-1 & \cdots & 1 \end{array})| = 1$$

for all $n \geq 0$.

Proof. Let $\pi$ be a permutation that avoids $p$. By 12, $\pi$ also avoids $q = \begin{array}{c|c|c|c}1 & 2 & \cdots & n-1 \\ \hline n & n-1 & \cdots & 1 \end{array}$. In Proposition 22 we showed that exactly one permutation of length $n$ avoids $q$, namely $1^r$. Therefore, it suffices to show that $1^r$ avoids $p$. Obviously, $1^r$ has no occurrence of the classical pattern 12, and hence $1^r$ avoids $p$. \qed

3.3. Pattern class 3. This pattern class contains 8 patterns and is represented by the pattern

$$p = \begin{array}{c|c|c|c}1 & 2 & \cdots & n-1 \\ \hline n & n-1 & \cdots & 1 \end{array}$$

The other patterns in this pattern class can be found in Appendix A.1.

Proposition 24. The only permutation avoiding $p$ is $1^r$ and therefore

$$|S_n(\begin{array}{c|c|c|c}1 & 2 & \cdots & n-1 \\ \hline n & n-1 & \cdots & 1 \end{array})| = 1$$

for all $n \geq 0$.

Proof. Let $\pi$ be a permutation that avoids $p$. By Observation 12 $\pi$ also avoids $q = \begin{array}{c|c|c|c}1 & 2 & \cdots & n-1 \\ \hline n & n-1 & \cdots & 1 \end{array}$. In Proposition 22 we showed that the only permutation of length $n$ that avoids $q$ is $1^r$. Therefore, we only need to show that $1^r$ avoids $p$. Since there is no occurrence of the classical pattern 12 in $1^r$, it avoids $p$. \qed
3.4. **Pattern class 4.** This pattern class contains 16 patterns and is represented by the pattern

\[ p = \begin{array}{c}
  \mathbb{1} \\
  \mathbb{1} \\
\end{array} \]

The other patterns in this pattern class can be found in Appendix A.1.

**Proposition 25.** All permutations that start with \( n \) avoid \( p \) and therefore

\[ |S_n \left( \begin{array}{c}
  \mathbb{1} \\
  \mathbb{1} \\
\end{array} \right) | = (n - 1)! \]

for all \( n \geq 1 \).

**Proof.** It is easy to see that if a permutation \( \pi \) starts with \( n \), then it avoids \( p \). Let \( \pi \) be a permutation that does not start with \( n \). Then let us choose the first letter of \( \pi \) and call it \( a \). Also, we let \( b \) be the first letter to the right of \( a \) that is greater than \( a \). Now, \( a \) and \( b \) form a non-inversion where there are no letters to the left of \( a \). Also, between \( a \) and \( b \) there are only letters lower than \( a \). Thus, \( \pi \) contains the pattern \( p \) and hence, all permutations avoiding the pattern \( p \) start with \( n \). \( \Box \)

3.5. **Pattern class 5.** This pattern class contains 8 patterns and is represented by the pattern

\[ p = \begin{array}{c}
  \mathbb{1} \\
  \mathbb{1} \\
\end{array} \]

The other patterns in this pattern class can be found in Appendix A.1.

**Proposition 26.** All permutations avoiding \( p \) must end with the letter 1 and therefore

\[ |S_n \left( \begin{array}{c}
  \mathbb{1} \\
  \mathbb{1} \\
\end{array} \right) | = (n - 1)! \]

for all \( n \geq 1 \).

**Proof.** A permutation \( \pi \) avoids the pattern \( p \) if and only if \( \pi \) contains no non-inversion or if there is at least one point in one of the shaded areas in \( p \). The only permutation not containing a non-inversion is \( 1^r \). Then, for a permutation \( \pi \) containing at least one non-inversion there are two cases. The first case is if \( \pi \) ends with the letter 1. It is easy to see that a permutation ending with 1 and having a non-inversion to the left of it avoids the pattern \( p \). The second case is if \( \pi \) does not end with the letter 1. Let \( a \) be the last letter of \( \pi \). Since \( a \neq 1 \) we can find a letter \( b = a - 1 \) to the left of \( a \). Then the letters \( a \) and \( b \) form a non-inversion where there are no letters to the right of \( a \). Furthermore, if there are some letters between \( a \) and \( b \) in the permutation they are greater than \( a \) and lower than \( b \). This shows that a permutation not ending with the letter 1 contains the pattern \( p \). \( \Box \)

3.6. **Pattern class 6.** This pattern class contains 8 patterns and is represented by the pattern

\[ p = \begin{array}{c}
  \mathbb{1} \\
  \mathbb{1} \\
\end{array} \]

The other patterns in this pattern class can be found in Appendix A.1.
Definition 27. Let $f(x)$ be a function and define $[x^n]f(x)$ as the coefficient of $x^n$ in the power series expansion of $f$.

Definition 28. Let $\text{des} (\pi)$ denote the number of descents in the permutation $\pi$.

Definition 29. The $n$-th Eulerian polynomial $E_n(x)$ can be defined by

$$\sum_{k=0}^{\infty} (k+1)^n x^k = \frac{E_n(x)}{(1-x)^{n+1}},$$

see for example [6]. It is well-known that the Eulerian numbers can also be defined recursively by

$$T_{n,k} = k \cdot T_{n-1,k} + (n-k+1) \cdot T_{n-1,k-1},$$

where $T_{1,1} = 1$.

Proposition 30. We have the following distribution for the number of descents in permutations of length $n$ avoiding $p$,

$$\sum_{\pi \in S_n(\pi)} x^{\text{des}(\pi)} = xE_{n-1}(x),$$

and therefore

$$|S_n(\pi)| = (n-1)!$$

for all $n \geq 1$.

Proof. By Lemma 14 we have

$$\begin{align*}
\begin{array}{cccc}
 & & & \\
 & & & \\
 & & & \\
 & & & \\
 & & & \\
 & & & \\
 & & & \\
\end{array}
\begin{array}{cccc}
 & & & \\
 & & & \\
 & & & \\
 & & & \\
 & & & \\
 & & & \\
 & & & \\
\end{array}
\begin{array}{cccc}
 & & & \\
 & & & \\
 & & & \\
 & & & \\
 & & & \\
 & & & \\
 & & & \\
\end{array}
\begin{array}{cccc}
 & & & \\
 & & & \\
 & & & \\
 & & & \\
 & & & \\
 & & & \\
 & & & \\
\end{array}
\begin{array}{cccc}
 & & & \\
 & & & \\
 & & & \\
 & & & \\
 & & & \\
 & & & \\
 & & & \\
\end{array}
\end{align*}
$$

Let $q = \begin{array}{cccc}
 & & & \\
 & & & \\
 & & & \\
 & & & \\
 \end{array}$. In order to construct all permutations of length $n$ we can take each permutation in $S_{n-1}$ and place $n$ in every possible position. Let $\pi'$ be a permutation obtained by adding the letter $n$ to $\pi$, where $\pi$ is a permutation in $S_{n-1}$ which contains the pattern $p$. Then there exist $i < j$ where the only letter lower than $\pi(j)$ in positions $1, 2, \ldots, j-1$ is $\pi(i)$. From this it follows that we can add the letter $n$ to any position in the permutation $\pi'$ for $\pi'$ to still satisfy these conditions and therefore contain the pattern $p$.

Now let $\pi$ be a permutation in $S_{n-1}$ that avoids $p$. Then for all $i < j$ with $\pi(i) < \pi(j)$ there are at least two letters in places $1, 2, \ldots, j-1$ lower than $\pi(j)$. In order to get a permutation of length $n$ that also avoids $p$ we can add the letter $n$ in all positions except between the letters $\pi(1)$ and $\pi(2)$. By adding the letter $n$ between the letters $\pi(1)$ and $\pi(2)$ we produce an ascent where the letter $n$ has only one letter to the left of it and therefore $\pi'$ would contain the pattern $p$. However by adding the letter $n$ in all other positions does not produce an ascent or produces an ascent that has more than one letter to the left of it and therefore $\pi'$ avoids the pattern $p$.

Since we are not allowed to add the letter $n$ between the letters $\pi(1)$ and $\pi(2)$, it follows that all permutations in $S_n(\pi)$ start with a descent.
We have the following distribution for the number of descents in permutations of length $n$ avoiding $p$,

$$\sum_{\pi \in S_n(\#\#)} x^{\text{des}(\pi)} = B_n(x).$$

We let $R_{n,k} = [x^k]B_n(x)$. In order to construct a permutation $\pi$ of length $n$ with $k$ descents, we have two choices. We can either take a permutation of length $n-1$ with $k$ descents and add $n$ to it without changing the number of descents or take a permutation of length $n-1$ with $k-1$ descents and increase the number of descents by one by adding $n$ to it. By adding $n$ inside a descent we do not change the number of descents. If we add $n$ in any of the other positions, that is not inside a descent, we increase the number of descents by one. Above we have shown that we can add the letter $n$ to a permutation of length $n-1$ that avoids $p$ in every position except between the first two letters of the permutation. We have also shown that every permutation avoiding $p$ begins with a descent. From this it follows that

$$R_{n,k} = (k - 1)R_{n-1,k} + (n - k + 1)R_{n-1,k-1}.$$  

We claim that $B_n(x) = xE_{n-1}(x)$ where $E_n(x)$ is the Eulerian polynomial, see Definition 29. So if we let $T_{n,k} = [x^k]E_n$ then $R_{n,k} = T_{n-1,k-1}$. We will prove this by induction. For $n = 2$ we have $B_2(x) = x^1$ and $E_1(x) = x^0$, so $R_{2,1} = 1$ and $T_{1,0} = 1$. Hence, the claim holds for the base case. Now we assume that $R_{N,k} = T_{N-1,k-1}$ holds for all $N < n$ and all $k$. By Definition 29, we have

$$R_{n,k} = (k - 1)R_{n-1,k} + (n - k + 1)R_{n-1,k-1}$$

(by induction hypothesis)

$$= (k - 1)T_{n-2,k-1} + (n - k + 1)T_{n-2,k-2}$$

$$= T_{n-1,k-1}.$$  

Thus, the claim holds for all $n$. Since $R_{n,k}$ and $T_{n,k}$ have the same recursion, they are equal. It is well-known that $\sum_{k=0}^{n-1} T_{n-1,k} = (n - 1)!$ and therefore the number of permutations of length $n$ avoiding $p$ is $(n - 1)!$. \hfill \Box

3.7. Pattern class 7. This pattern class contains 16 patterns and is represented by the pattern

$$p = \#\#$$

The other patterns in this pattern class can be found in Appendix A.1.

**Proposition 31.** We have the following generating function for the number of descents in permutations of length $n$ avoiding $p$,

$$\sum_{\pi \in S_n(\#\#)} x^{\text{des}(\pi)} = xE_{n-1}(x),$$

and therefore

$$|S_n(\#\#)| = (n - 1)!$$
for all $n \geq 0$.

Proof. By using Lemma 14 we have

$$\begin{array}{c}
\begin{array}{c}
\text{Pattern Class 7}
\end{array}
\end{array} \cong \begin{array}{c}
\begin{array}{c}
\text{Pattern Class 6}
\end{array}
\end{array}$$

Let $q = \begin{array}{c}
\begin{array}{c}
\text{Pattern Class 7}
\end{array}
\end{array}$. To construct all permutations of length $n$ we take each permutation $\pi$ in $S_{n-1}$ and place the letter 1 in every position in $\pi$ and add 1 to the remaining letters. Now let us take a permutation $\pi$ in $S_{n-1}$ which contains $p$, that is, there exist $1 \leq i < j \leq n$ such that there does not exist a letter $k$ such that $\pi(i) < k < \pi(j)$, and we do not have a letter $\ell$ such that $\ell > \pi(i)$. We can place 1 in any position in $\pi$ for these conditions to be satisfied for the new permutation, $\pi'$.

Now let us take a permutation $\pi$ in $S_{n-1}$ which avoids $p$, that is, for all $1 \leq i < j \leq n$ with $\pi(i) < \pi(j)$ we have a letter $k$ where $\pi(i) < k < \pi(j)$, or a letter $\ell$ such that $\ell > \pi(i)$. By placing 1 in front of 2 in $\pi$, $\pi'$ would contain an ascent with the letters 12, and therefore clearly contain the pattern. If 1 is placed in every other position in $\pi$, a non-inversion is created by the letter 1 and any letter greater than 2. Thus the conditions will be satisfied in $\pi'$ and $\pi'$ will avoid the pattern. By similar arguments as in Proposition 30, we find that pattern class 7 gives the same distribution for number of descents as pattern class 6. Hence the number of permutations of length $n$ avoiding the pattern $p$ is $(n-1)!$. □

3.8. Pattern class 8. This pattern class contains 16 patterns and is represented by the pattern

$$p = \begin{array}{c}
\begin{array}{c}
\text{Pattern Class 8}
\end{array}
\end{array}$$

The other patterns in this pattern class can be found in Appendix A.1.

Proposition 32. Permutations in $S_n$ avoid $p$ if and only if it holds that each ascent is the 23 of a 132 pattern or the 12 of a 312 pattern or both. Therefore,

$$|S_n \left(\begin{array}{c}
\begin{array}{c}
\text{Pattern Class 8}
\end{array}
\end{array}\right)\right| = a_n$$

where $a_n = n \cdot a_{n-1} - a_{n-2}$ and $a_{-1} = 0, a_0 = 1$.

Proof. The pattern $p$ can be modified using the complement symmetry. Then we obtain the following pattern.

$$\begin{array}{c}
\begin{array}{c}
\text{Pattern Class 8}
\end{array}
\end{array} \rightarrow \begin{array}{c}
\begin{array}{c}
\text{Pattern Class 8}
\end{array}
\end{array} = q.$$

Let $A_n$ denote $S_n \left(\begin{array}{c}
\begin{array}{c}
\text{Pattern Class 8}
\end{array}
\end{array}\right)$ and $a_n = |A_n|$. We recall Callan’s explanation of the number of permutations in $S_n$ where each descent is the 32 of a 132 pattern or the 21 of a 321 pattern, or both\(^2\).

Now, let $A_{n,k} = \{\pi \in A_n \mid \pi(n) = k\}$. For $k \in [2, n]$ we define a mapping $\varphi_k : A_{n,k} \rightarrow A_{n-1}$ such that for a permutation $\pi$, the mapping removes the

\(^2\)Personal communication.
last letter of a permutation and then subtracts 1 from each remaining letter
that is larger than \( k \). Then we can also define \( \varphi_k^{-1} : A_{n-1} \mapsto A_{n,k} \) to be the
mapping that appends \( k \) to the right end of a permutation and then adds 1
to all letters that are equal to \( k \) or larger.

This mapping is a bijection, since if \( \pi \in A_{n,k} \) then \( \varphi_k(\pi) \in A_{n-1} \) because
the mapping \( \varphi_k \) will not produce an occurrence of the pattern \( q \). This holds
since in \( \varphi_k(\pi) \), 1 must be in one of the positions \( \{1, 2, \ldots, n - 1\} \) (and
therefore be in the lower left shaded box, so we will not have an occurrence of
the pattern). Also, if \( \pi \in A_{n-1} \) then \( \varphi_k^{-1}(\pi) \in A_{n,k} \) since if \( \varphi_k^{-1} \) produces an
occurrence of the classical pattern 21 it cannot be an occurrence of \( q \) since it
also holds that the letter 1 must occur in one of the positions \( \{1, 2, \ldots, n - 1\} \)
in \( \pi \). Therefore, \(|A_{n,k}| = a_{n-1} \) for \( k \in [2, n] \).

Then for \( k = 1 \) we also have a mapping \( \varphi_1 : A_{n,1} \mapsto A_{n-1 \setminus A_{n-1,n-1}} \), such
that for a permutation \( \pi \) where the last letter is 1, the mapping removes that
letter and then subtracts 1 from each remaining entry in \( \pi \). The range of the
mapping is \( A_{n-1 \setminus A_{n-1,n-1}} \) because if it would be the case that \( \varphi(\pi)(n) = n - 1 \) then \( \pi \) would have ended with the descent \( n_1 \), which is an occurrence
of the pattern \( q \). This is a contradiction to \( \pi \) being in \( A_{n,1} \). Then we can
also define \( \varphi_1^{-1} : A_{n-1 \setminus A_{n-1,n-1}} \mapsto A_{n,1} \), which is a mapping that appends
the letter 1 to the end of a permutation and then adds 1 to all other letters
in the permutation.

Similar to the case where \( k \in [2, n] \), the mapping \( \varphi_1 \) is a bijection since
if \( \pi \in A_{n,1} \) then \( \varphi_1(\pi) \in A_{n-1 \setminus A_{n-1,n-1}} \). That is because the mapping
\( \varphi_1 \) will not produce an occurrence of the pattern \( q \). This holds since we
do not have an occurrence of \( q \) in \( \pi \) and thus, by removing the last letter
1 and subtracting 1 from each other letter in \( \pi \) we cannot construct an
occurrence of \( q \). Also, if \( \pi \in A_{n-1 \setminus A_{n-1,n-1}} \), then \( \varphi_1^{-1}(\pi) \in A_{n,1 \setminus A_{n-1,n-1}} \)
since the only possibility for \( \varphi_1^{-1}(\pi) \) to have an occurrence of the pattern \( q \)
is when \( n - 1 \) is the last letter in \( \pi \). There is no permutation ending with
\( n - 1 \) in \( A_{n,1 \setminus A_{n-1,n-1}} \) and thus, if there is no occurrence of the pattern
\( q \), then appending the letter 1 to the right of \( \pi \) and adding 1 to all other elements
in \( \pi \) will not construct an occurrence of \( q \). Then, we also have a mapping \( \varphi_{n-1} : A_{n-1,n-1} \mapsto A_{n-2} \), which is also a bijection by similar
arguments as above, and therefore, \(|A_{n-1,n-1}| = |A_{n-2}| \). Hence, we have
\(|A_{n,1}| = a_{n-1} - a_{n-2} \). This gives

\[
|A_{n,k}| = \sum_{k=1}^{n} |A_{n,k}|
\]

\[
= \sum_{k=2}^{n} |A_{n,k}| + |A_{n,1}|
\]

\[
= (n - 1) a_{n-1} + a_{n-1} - a_{n-2}
\]

\[
= n a_{n-1} - a_{n-2}.
\]

\(\square\)
3.9. **Pattern class** 9. This pattern class contains 16 patterns and is represented by the pattern

\[ p = \begin{array}{c}
\begin{array}{c}
\vdash
\end{array}
\end{array} \]

The other patterns in this pattern class can be found in Appendix A.1.

**Proposition 33.** Permutations in \( S_n \) avoid \( p \) if and only if it holds that each ascent is the 13 of a 213 pattern or the 23 of a 123 pattern or both. Therefore,

\[ \left| S_n \left( \begin{array}{c}
\begin{array}{c}
\vdash
\end{array}
\end{array} \right) \right| = a_n \]

where \( a_n = n \cdot a_{n-1} - a_{n-2} \) and \( a_{-1} = 0, a_0 = 1 \).

**Proof.** We modify the pattern \( p \) using Lemma 14 as follows

\[ \begin{array}{c}
\begin{array}{c}
\vdash
\end{array}
\end{array} \xrightarrow{L_{14}} \begin{array}{c}
\begin{array}{c}
\vdash
\end{array}
\end{array} \]

Then this statement can be proved using similar arguments as in Proposition 32, except here \( k \in [1, n] \setminus \{n - 1\} \).

\( \square \)

3.10. **Pattern class** 10. This pattern class contains 32 patterns and is represented by the pattern

\[ p = \begin{array}{c}
\begin{array}{c}
\vdash
\end{array}
\end{array} \]

The other patterns in this pattern class can be found in Appendix A.1.

**Proposition 34.** Permutations in \( S_n \) avoid \( p \) if and only if it holds that each ascent is the 12 of a 312 pattern or the 13 of a 213 pattern or both. Therefore,

\[ \left| S_n \left( \begin{array}{c}
\begin{array}{c}
\vdash
\end{array}
\end{array} \right) \right| = a_n \]

where \( a_n = (n-1)a_{n-1} + (n-2)a_{n-2} \) and \( a_0 = a_1 = 1 \).

**Proof.** The pattern \( p \) can be modified using Lemma 14 as follows

\[ \begin{array}{c}
\begin{array}{c}
\vdash
\end{array}
\end{array} \xrightarrow{L_{14}} \begin{array}{c}
\begin{array}{c}
\vdash
\end{array}
\end{array} \]

Let \( A_n \) denote \( S_n \left( \begin{array}{c}
\begin{array}{c}
\vdash
\end{array}
\end{array} \right) \) and \( a_n = |A_n| \). Now, let \( A_{n,k} = \{ \pi \in A(n) | \pi(n) = k \} \). For \( k \neq n \) we define a mapping \( \varphi_k : A_{n,k} \mapsto A_{n-1} \) such that for a permutation \( \pi \), the mapping removes \( k \) from \( \pi \) and then subtracts 1 from all letters larger than \( k \). Then we can also define \( \varphi_k^{-1} : A_{n-1} \mapsto A_{n,k} \) to be the mapping that appends \( k \) to the end of a permutation \( \pi \) and then adds 1 to all letters that are equal to \( k \) or larger. For \( k = n \) we also have a mapping \( \varphi_n : A_{n,n} \mapsto A_{n-1} \setminus A_{n-1,n-1} \) such that for a permutation \( \pi \) where the last letter is \( n \), the mapping removes that letter. The range of the mapping is \( A_{n-1} \setminus A_{n-1,n-1} \) because if \( \varphi(\pi)(n) = n - 1 \) then \( \pi \) would end with \( (n-1)n \) which is an occurrence of the pattern \( \begin{array}{c}
\begin{array}{c}
\vdash
\end{array}
\end{array} \). Then we can also define \( \varphi_1^{-1} = A_{n-1} \setminus A_{n-1,n-1} \mapsto A_{n,n} \) as the mapping that appends \( n \) to the end of a permutation.
The mappings above are bijections for similar reasons as the mappings for pattern class 8 in Proposition 32.

As explained above, in order to construct all permutations of length n avoiding the pattern $A_n$, we can append n at the end of all permutations of length $n - 1$ except for those ending with the letter $n - 1$. Hence,

$$a_n = na_{n-1} - a_{n-1, n-1}.$$ 

From the mappings we get $a_{n, n} = a_{n-1} - a_{n-1, n-1}$ which gives $a_{n-1, n-1} = a_{n-1} - a_{n, n}$. We also have $a_{n, n} = (n-2)a_{n-2}$ since for producing a permutation from $A_{n, n}$ we take a permutation in $A_{n-2}$ and append a letter from the set $\{1, 2, \ldots, n-2\}$ to it. Lastly we append n at the end of the permutation. Therefore,

$$a_n = na_{n-1} - (a_{n-1} - a_{n, n})$$
$$= na_{n-1} - a_{n-1} + (n-2)a_{n-2}$$
$$= (n-1)a_{n-1} + (n-2)a_{n-2},$$

which is what we wanted to prove. \[\square\]

3.11. **Pattern class** 11. This pattern class contains 4 patterns and is represented by the pattern

$$p = \begin{array}{c}
  \downarrow \\
  \downarrow \\
  \downarrow \\
  \downarrow \\
\end{array}$$

The other patterns in this pattern class can be found in Appendix A.1.

**Definition 35.** If a permutation sends letters $1, 2, \ldots, j$ to $1, 2, \ldots, j$, where $0 < j < n$, we say that $\{1, 2, \ldots, j\}$ is an invariant set. Permutations that do not contain an invariant set are called connected permutations.

According to Aguiar and Scottile in [1] the number of permutations of length n with no global descents is given by

$$1 - \frac{1}{\sum_n n!x^n},$$

which is the same as the number of connected permutations of length n.

**Proposition 36.** The permutations avoiding the pattern $p$ are the connected permutations, thus

$$|S_n \left( \begin{array}{c}
  \downarrow \\
  \downarrow \\
  \downarrow \\
  \downarrow \\
\end{array} \right) | = [x^n] \left( 1 - \frac{1}{\sum_n n!x^n} \right).$$

**Proof.** We will show the contrapositive, i.e., that the pattern $p$ occurs in a permutation $\pi$ if and only if $\pi$ has an invariant set.

In the pattern $p = \begin{array}{c}
  \downarrow \\
  \downarrow \\
  \downarrow \\
  \downarrow \\
\end{array}$, let us call the points in the pattern $v$ and $w$, respectively. If $p$ occurs in a permutation $\pi$, then there is no letter to the left of $w$ that is greater than $v$. Also, there can be no smaller letter than $v$ to the right of $w$. Hence, if there exist letters $a_1, a_2, \ldots, a_k$ to the left of $w$, then $a_1, a_2, \ldots, a_k$ are all smaller than $v$. 


On one hand, if \( \{a_1, a_2, \ldots, a_k\} = \emptyset \), then \( v \) is both the leftmost and the smallest letter in \( \pi \). Then \( \{v\} \) is an invariant set. On the other hand, if \( \{a_1, a_2, \ldots, a_k\} \neq \emptyset \) then the points \( a_1, a_2, \ldots, a_k \) are in the first \( k \) positions in \( \pi \) and hence \( \{a_1, a_2, \ldots, a_k, v\} \) is an invariant set.

If \( \pi \) has an invariant set, we know that the lowest \( k \) letters in the permutation are in the first \( k \) positions. Let us choose the highest letter in the invariant set, and call it \( v \). Then we choose \( w \) in position \( k+1 \). Now, \( v \) and \( w \) form the pattern \( p \).

Marked mesh patterns are a generalization of mesh patterns and were first introduced by Úlfarsson in [12]. A marked mesh pattern is a pair \((\tau, C)\), where \( \tau \) is a permutation in \( S_k \) and \( C \) is a set of pairs \((C, \boxempty j)\) where \( C \) is a subset of the square \([0, k] \times [0, k] \), \( j \) is a non-negative integer and \( \boxempty \) is one of the symbols \( \leq, =, \geq \).

An example of a marked mesh pattern \( p = (\tau, C) \) is the classical pattern 312 along with \( C = \{(\{0,2],[1,2]\}, \geq 3), (\{2,1\}, = 0)\} \). This pattern is drawn as

An occurrence of a marked mesh pattern \( p = (\tau, C) \) is an occurrence of the classical pattern \( \tau \). The occurrence must also satisfy that for each pair \((C, \boxempty j) \in C \) the number of points, \( a \), inside the area which is formed by the elements in \( C \), must fulfill the condition \( a \boxempty j \).

3.12. **Pattern class** 12. The pattern class contains 60 patterns and is represented by the pattern

\[
p = \begin{array}{c}
\begin{array}{c}
\geq 3 \\
\end{array}
\end{array}
\]

The other patterns in this pattern class can be found in Appendix A.1.

**Proposition 37.** The number of permutations that avoid the pattern

\[
\left| S_n \left( \begin{array}{c}
\begin{array}{c}
\geq 3 \\
\end{array}
\end{array} \right) \right|
\]

is \( \lceil \frac{x^n}{n} \rceil \log \left(1 + \sum_{n \geq 1} (n-1)!x^n\right) \).

**Proof.** We have the following pattern equivalence

\[
\begin{array}{c}
\begin{array}{c}
\geq 3 \\
\end{array}
\end{array} \cong \begin{array}{c}
\begin{array}{c}
\geq 1 \\
\end{array}
\end{array} = q
\]

Now we define two sequences,

\[
a_n = \left| S_n \left( \begin{array}{c}
\begin{array}{c}
\geq 3 \\
\end{array}
\end{array} \right) \right| \quad \text{and} \quad b_n = \left| S_n \left( \begin{array}{c}
\begin{array}{c}
\geq 1 \\
\end{array}
\end{array} \right) \right|,
\]

and let us show that

\[
a_n = b_n + b_{n-1}.
\]
We define a map,

$$\varphi : S_n(\mathcal{A}_b) \cup S_{n-1}(\mathcal{A}_b) \rightarrow S_n(q).$$

The mapping $\varphi$ maps $\pi \in S_n(\mathcal{A}_b)$ to itself in $S_n(q)$, we know that $\pi$ avoids $\mathcal{A}_b$, and therefore it also avoids $q$. For $\pi \in S_{n-1}(\mathcal{A}_b)$ the mapping $\varphi$ appends the letter $n$ to the right of $\pi$, and we obtain a permutation of length $n$ with the letter $n$ in the $n$-th position. The mapping is well-defined because the letter $n$ cannot be in position $n$ in $\pi \in S_n(\mathcal{A}_b)$ because that would be an occurrence of $\mathcal{A}_b$.

Then let us define the inverse map

$$\varphi^{-1} : S_n(q) \rightarrow S_n(\mathcal{A}_b) \cup S_{n-1}(\mathcal{A}_b).$$

For $\pi \in S_n(q)$ that ends with the letter $n$, we remove $n$ and then $\varphi^{-1}(\pi)$ is in $S_{n-1}(\mathcal{A}_b)$. This is the range of the mapping of $\pi$, because the letter $n$ is in position $n$ in $\pi$ and therefore there must be a point in at least one of the shaded boxes (since the $\geq 1$ condition is satisfied). Hence, by removing the letter $n$ and obtaining $\varphi^{-1}(\pi)$, we see that $\varphi^{-1}(\pi)$ avoids $\mathcal{A}_b$.

For $\pi \in S_n(q)$ that does not end with the letter $n$, $\pi$ maps to itself in $S_n(\mathcal{A}_b)$. The permutation $\pi$ avoids $\mathcal{A}_b$ as well for the following reasons. For a letter $u$ in $\pi$ that appears to the left of the letter $n$ there must be points in at least one of the shaded areas for $\pi$ to avoid $q$. Also for $v$ in $\pi$ that appears to the right of $n$ then there is at least one point, $n$, in the upper shaded box. Therefore, $\pi$ avoids $\mathcal{A}_b$, and hence $\varphi^{-1}(\pi) \in S_n(\mathcal{A}_b)$.

It is not hard to verify that the inverse map is also well-defined since each pattern in the set $S_n(q)$ maps to a unique element in $S_n(\mathcal{A}_b) \cup S_{n-1}(\mathcal{A}_b)$. This proves 1.

We recall Stanley’s observation in [9] of the generating function for $b_n$. That is,

$$G(x) = \frac{F(x)}{1 + xF(x)},$$

where $F(x) = \sum_{n \geq 0} n!x^n$. We define the generating function for $a_n$ to be

$$D(x) = \sum_{n \geq 1} a_nx^n.$$

Since $a_n = b_n + b_{n-1}$ we obtain

$$D(x) = G(x) + xG(x) - 1 = \frac{\sum_{n \geq 0} n!x^n}{1 + x \sum_{n \geq 0} n!x^n} + x \left( \frac{\sum_{n \geq 0} n!x^n}{1 + x \sum_{n \geq 0} n!x^n} \right) - 1 = \frac{\sum_{n \geq 0} n!x^n - 1}{1 + x \sum_{n \geq 0} n!x^n} = \frac{\sum_{n \geq 1} n!x^n}{1 + \sum_{n \geq 1} (n - 1)!x^n}.$$
So now we have $D(x) = xA'(x)$ where $A(x) = \log(1 + \sum_{n \geq 1} (n-1)!x^n)$ which implies that $A(x)$ is the logarithmic generating function of $a_n$, i.e.

$$A(x) = \sum_{n \geq 1} \frac{a_n}{n} x^n = \log(1 + \sum_{n \geq 1} (n-1)!x^n).$$

3.13. Pattern class 13. This pattern class contains 8 patterns and is represented by the pattern

$p = \overline{\overline{\overline{\overline{\overline{\overline{\overline{x}}}}}}}$

The other patterns in this pattern class can be found in Appendix A.1.

**Proposition 38.** A permutation $\pi$ of length $n$ contains $p$ if and only if $\pi$ begins with 1 and ends with a permutation of length $n-1$ that contains a strong fixed point. Thus,

$$|S_n\left(\overline{\overline{\overline{\overline{\overline{\overline{\overline{x}}}}}}}\right)| = n! - (n-1)! + [x^n] \frac{F(x)}{1 + xF(x)},$$

where $F(x) = \sum_{n \geq 1} (n-1)!x^{n-1}$.

**Proof.** Let $p$ be the pattern $\overline{\overline{\overline{\overline{\overline{\overline{\overline{x}}}}}}}$ and $q$ be the pattern $\overline{\overline{\overline{\overline{\overline{\overline{x}}}}}}$. A permutation $\pi$ contains $q$ if and only if it contains a strong fixed point. Then, by appending the letter 1 in front of each permutation of length $n-1$ containing the pattern $q$, and adding 1 to the remaining letters, we clearly obtain all permutations containing the pattern $p$.

Now, we know that the number of permutations of length $n-1$ that have no strong fixed points is $^3$

$$[x^n] \frac{F(x)}{1 + xF(x)}.$$

Hence, the number of permutations that contain a strong fixed point is

$$(n-1)! - [x^n] \frac{\sum_{n \geq 1} (n-1)!x^{n-1}}{1 + x \sum_{n \geq 1} (n-1)!x^{n-1}},$$

which is equal to the number of permutations of length $n$ containing the pattern $p$. □

3.14. Pattern class 14. This pattern class contains 24 patterns and is represented by the pattern

$p = \overline{\overline{\overline{\overline{\overline{\overline{\overline{x}}}}}}}$

The other patterns in this pattern class can be found in Appendix A.1.

**Proposition 39.** The number of permutations of length $n$ that start with $k$ and contain the pattern $p$ is $(k-1)!(n-k-1)!$ and therefore

$$|S_n\left(\overline{\overline{\overline{\overline{\overline{\overline{\overline{x}}}}}}}\right)| = n! - \sum_{k=1}^{n-1} (k-1)!(n-k-1)!.\footnote{http://oeis.org/A006932}$$
Proof. Let \( k \) be the first letter in a permutation \( \pi \). Thus, \( k \) is the former point in the pattern \( p \). For \( \pi \) to contain the pattern \( p \), the latter point of \( p \) must be \( k + 1 \). Also, the remaining letters of \( \pi \) must be in the two non-shaded boxes in \( p \). The letters from 1 to \( k - 1 \) must be placed in the lower non-shaded box, and the letters from \( k + 2 \) to \( n \) must be placed in the upper one. Hence, the number of letters in the lower non-shaded box is \( k - 1 \) and the number of letters in the upper box is \( n - (k + 1) \). It follows that the number of permutations that start with \( k \) and contain the pattern \( p \) is 
\[(k - 1)! (n - k - 1)! .\]

□

3.15. Pattern class 15. This pattern class contains 16 patterns and is represented by the pattern
\[ p = \begin{array}{c} \text{Pattern} \end{array} \]

The other patterns in this pattern class can be found in Appendix A.1.

Proposition 40. The number of permutations of length \( n \) which have \( n \) in position \( i \), where \( i = 0 \) is the rightmost position and contain the pattern \( p \) is \( i! (n - 1 - i)! \). Therefore
\[ |S_n \left( \begin{array}{c} \text{Pattern} \end{array} \right) | = n! - \sum_{i=0}^{n-2} i! (n - 1 - i)! .\]

Proof. By using the inverse, up-shift, and complement operations we obtain
\[ \begin{array}{c} \text{Pattern} \rightarrow \text{Pattern} \rightarrow \text{Pattern} \rightarrow \text{Pattern} \rightarrow \text{Pattern} \end{array} \]

Let \( q \) be the pattern \( \begin{array}{c} \text{Pattern} \end{array} \)

Let \( n \) be in position \( i \) where \( i = 0 \) is the rightmost position. Thus, \( n \) is the latter point in the pattern \( p \). For a permutation \( \pi \) to contain \( p \), the letters \( n - 1, n - 2, \ldots, n - i \) must be placed to the right of the letter \( n \), that is, in the rightmost non-shaded box. The remaining \( n - 1 - i \) letters are placed in the other non-shaded area. Thus, the number of permutations that have \( n \) in position \( i \) and contain the pattern \( p \) is
\[ i! (n - 1 - i)! .\]

□

3.16. Pattern class 16. This pattern class contains 4 patterns and is represented by the pattern
\[ p = \begin{array}{c} \text{Pattern} \end{array} \]

The other patterns in this pattern class can be found in Appendix A.1.

Observation 41. After some consideration, one can see that a permutation can either contain exactly one occurrence of the pattern \( p \) or none at all.

Proposition 42. The number of permutations containing the pattern \( p \), with \( i \) as the height of the former point of the pattern, counted from above, and \( \ell \) as the distance between the two points, is \( (i - \ell)! (n - i - \ell)! \ell! .\)
Hence,
\[ \left| S_n \left( \begin{array}{c} \text{ } \\
\text{ } \\
\text{ } \\
\text{ } \\
\text{ } \\
\text{ } \\
\text{ } \\
\text{ } \\
\text{ } \\
\text{ } \\
\end{array} \right) \right| = n! - \sum_{n=1}^{n-1} \sum_{i=1}^{i} (i - \ell)! (n - i - \ell)! n!, \]
for all \( n \geq 0 \).

**Proof.** Let \( p \) be the pattern \( \begin{array}{c} \text{ } \\
\text{ } \\
\text{ } \\
\text{ } \\
\text{ } \\
\text{ } \\
\text{ } \\
\text{ } \\
\text{ } \\
\end{array} \). After some consideration, one can see that a permutation can either contain exactly one occurrence of the pattern \( p \) or none at all. To construct a permutation containing the pattern, the following conditions must be satisfied. First of all we find the two letters in the permutation corresponding to the two points in the pattern, call them \( a \) and \( b \), respectively. We let \( i \) be the height of the letter \( a \) counted from above and \( \ell \) be the distance between \( a \) and \( b \). Then, for a permutation of length \( n \), we have \( a = n - i \) and \( b = n - i + \ell \).

Second of all, we see that the letters greater than \( b \) must be placed to the left of \( a \). This can be done \( (i - \ell)! \) ways. Furthermore, all the \( n - (i + 1) \) \( n - i - 1 \) letters lower than \( a \) must be placed between \( a \) and \( b \), and they can be ordered in \( (n - i - 1)! \) ways. At last, the letters between \( a \) and \( b \) in size can be placed either between \( a \) and \( b \) or to the right of \( b \). Thus, the number of ways to place these \( n - i - \ell - (n - i) - 1 = \ell - 1 \) letters is
\[
\sum_{j=0}^{\ell-1} (\ell - 1)_j j! (\ell - 1 - j) = \sum_{j=0}^{\ell-1} (\ell - 1)! = (\ell - 1)! \sum_{j=0}^{\ell-1} 1 = \ell!.
\]

\[ \square \]

3.17. **Pattern class 17.** This pattern class contains 4 patterns and is represented by the pattern

\[ p = \begin{array}{c} \text{ } \\
\text{ } \\
\text{ } \\
\text{ } \\
\text{ } \\
\text{ } \\
\text{ } \\
\end{array} \]

The other patterns in this pattern class can be found in Appendix A.1.

**Observation 43.** A permutation can either contain exactly one occurrence of the pattern \( p \) or none at all.

**Proposition 44.** The number of permutations of length \( n \) containing the pattern \( p \) with \( k \) as the height of the left point of the pattern, counted from above, and \( j \) the distance between the two points is
\[ j!(k - j)!(n - k)! \]
and therefore
\[ \left| S_n \left( \begin{array}{c} \text{ } \\
\text{ } \\
\text{ } \\
\text{ } \\
\text{ } \\
\text{ } \\
\text{ } \\
\text{ } \\
\text{ } \\
\text{ } \\
\end{array} \right) \right| = n! - \sum_{k=0}^{n-2} \sum_{j=0}^{k} j!(k - j)!(n - 2 - k)! \]
for all \( n \geq 0 \).
Proof. To construct a permutation containing the pattern, the following conditions must be satisfied. First of all we find the letters corresponding to the two points in the pattern $p$, let us call them $a$ and $b$, respectively. Let $k$ be the height of the letter $a$ counted from above and $j$ be the distance between $a$ and $b$. Then, in a permutation of length $n$, we have $a = n - k$ and $b = n - k + j$.

Second of all, the letters greater than $b$ must be to the left of $a$. These $n - (n - k + j) = k - j$ letters can be arranged in $(k - j)!$ different ways. In addition, the $n - k - 1$ letters lower than $a$ must be placed between $a$ and $b$, which can be done in $(n - k - 1)!$ different ways. At last, the $n - k + j - (n - k) - 1 = j - 1$ letters between $a$ and $b$ in size must be placed to the right of $b$, which can be done in $(j - 1)!$ ways. Hence, the number of permutations containing the pattern $p$ is

$$\sum_{k=1}^{n-1} \sum_{j=1}^{k} (n - k - 1)!(k - j)!(j - 1)! = \sum_{k=0}^{n-2} \sum_{j=0}^{k} j!(k - j)!(n - 2 - k)!.$$  

3.18. Pattern class 18. This pattern class contains 84 patterns and is represented by the pattern

$$p = \begin{array}{c} \text{\Large $a$} \\
\end{array} \begin{array}{c} \text{\Large $b$} \\
\end{array}$$

The other patterns in this pattern class can be found in Appendix A.1.

Proposition 45. The number of permutations of length $n$ which have $n$ in position $i$ counted from the right and contain the pattern $p$ is $\frac{(n-1)!}{i}$ and therefore

$$|S_n \begin{array}{c} \text{\Large $a$} \\
\end{array} \begin{array}{c} \text{\Large $b$} \\
\end{array} | = n! - \sum_{i=1}^{n-1} \frac{(n - 1)!}{i}.$$  

Proof. By using Lemma 14 we find the following equivalence

$$\begin{array}{c} \text{\Large $a$} \\
\end{array} \begin{array}{c} \text{\Large $b$} \\
\end{array} \begin{array}{c} \text{\Large $c$} \\
\end{array} \begin{array}{c} \text{\Large $d$} \\
\end{array} \begin{array}{c} \text{\Large $e$} \\
\end{array} \begin{array}{c} \text{\Large $f$} \\
\end{array} \begin{array}{c} \text{\Large $g$} \\
\end{array}$$

Let $q$ be the pattern $\begin{array}{c} \text{\Large $c$} \\
\end{array} \begin{array}{c} \text{\Large $d$} \\
\end{array}$.

To find an occurrence of the pattern $p$ in a permutation of length $n$, we must use the first letter in the permutation and the letter $n$. Then it must also hold that all the letters to the right of $n$ must be greater than the first letter. Thus, the number of permutations containing the pattern $p$ depends on the position of the letter $n$.

Recall that $i$ is the position of the letter $n$ counted from the right and let $k$ be the size of the first letter. Obviously, the letter $n$ cannot be in the last position counted from the right, and thus, $1 \leq k \leq n - 1$ and $1 \leq i \leq n - 1$.

Now, we choose $i - 1$ letters greater than $k$ to fill the positions to the right of $n$, which can be done in $\binom{n - (k+1)}{i - 1}$ ways. Then these letters can be arranged in $(i - 1)!$ ways. The remaining $n - i - 1$ letters will be placed between $n$ and $k$, which can be done in $(n - i - 1)!$ ways. This must hold
for each $1 \leq k \leq n - 1$. Therefore, the number of permutations containing the pattern $p$ is

$$\sum_{k=1}^{n-1} \binom{n-k-1}{i-1} (i-1)! (n-i-1)!.$$ 

We have

$$\sum_{k=1}^{n-1} \binom{n-k-1}{i-1} (i-1)! (n-i-1)! = i!(n-i-1)! \sum_{k=1}^{n-1} \binom{n-k-1}{i-1} = i!(n-i-1)! \frac{(n-1)!}{i!(n-1-i)!} = (n-1)!,$$

which implies

$$\sum_{k=1}^{n-1} \binom{n-k-1}{i-1} (i-1)! (n-i-1)! = \frac{(n-1)!}{i}. \quad \square$$

Now, there are 40 pattern classes left to find formula for. There are 248 patterns in these classes combined. In Figure 4 the representative patterns for each of the unproved classes. Then the other patterns can be found in Appendix A.3.
Figure 4. The pattern classes which we have not found formulas for.

4. Applications of the Reciprocity Theorem

The reciprocity theorem, proved by Brändén and Claesson [4], states that if \( p = (\pi, R) \) is a mesh pattern and \( p^* = (\pi, R^c) \), where \( R^c = \lbrack 0, |\pi| \rbrack^2 \setminus R \), then

\[
p = \sum_{\sigma \in S} \lambda(\sigma)\sigma, \quad \text{where } \lambda(\sigma) = (-1)^{|\sigma| - |\pi|} p^*(\sigma).
\]

We define \( S = \bigcup_{n=0}^{+\infty} S_n \). Here the pattern \( p \) is considered as a function and \( p(w) \) is the number of occurrences of \( p \) in \( w \). Similarly for \( \sigma \in S \).

**Definition 46.** \( S_i \oplus j \) is a set of permutations where \( j \) has been appended to all permutations of length \( i \).

**Proposition 47.** Let \( p = \begin{array}{c} \text{\#} \end{array} \) be a pattern representing pattern class 6, see Proposition 30. Then

\[
p = S_1 \oplus 2 - 2 \cdot S_2 \oplus 3 + 3 \cdot S_3 \oplus 4 - 4 \cdot S_4 \oplus 5 + \cdots
\]

\[
= \sum (-1)^{i+1} i \cdot S_i \oplus (i + 1).
\]

**Proof.** Here, \( p^* = \begin{array}{c} \text{\#} \end{array} \), so

\[
\lambda(\pi) = \begin{cases} n - 1 & \text{if } \pi(n) = n, \\ 0 & \text{otherwise}, \end{cases}
\]

and the formula now follows from the reciprocity theorem. \( \Box \)
Proposition 48. Let \( p = \begin{array}{|c|}
\end{array} \) be a pattern representing pattern class 29, see A.3 in the appendix. Then
\[
p = \sum_{\sigma \in S} (-1)^{|\sigma|}(\lambda(\pi) - 1)\pi.
\]

Definition 49. Let \( \lambda \) be the number of left-to-right minima.

Proof. Here, \( p^* = \begin{array}{|c|} \end{array} \) so \( \lambda(\pi) \) is the number of pairs \( 1 \leq u < v \leq n \) such that \( u \) is a left-to-right minimum and \( v \) is the letter \( n \) at the end of a permutation \( \pi \) of length \( n \).

\( \square \)

Definition 50. Let \( \lambda \) be the number of right-to-left maxima and then let \( \lambda(\pi) = \lambda(\pi) \) be the number of pairs of left-to-right minima and right-to-left maxima in a permutation \( \pi \).

Proposition 51. Let \( p = \begin{array}{|c|} \end{array} \) be the pattern representing pattern class 64, see A.3 in the appendix. Then
\[
p = \sum_{\sigma \in S} (-1)^{|\sigma|} \lambda(\pi)\pi.
\]

Proof. Here \( p^* = \begin{array}{|c|} \end{array} \) so \( \lambda(\pi) \) is the number of pairs \( 1 \leq u < v \leq n \) in \( \pi \) such that \( u \) is a left-to-right minima and \( v \) is a right-to-left maxima. That is, \( \lambda(\pi) = \lambda(\pi) \).

\( \square \)

5. Open Questions

Definition 52. A permutation \( \pi = \pi_1\pi_2\cdots\pi_n \in S_n \) is called simsun if the restriction of \( \pi \) to \( \{1, 2, \ldots, k\} \), for all \( 3 \geq k \geq n \), has no double descents, see [10].

Observation 53. Claesson and Brändén [4], and Úlfarsson [12] simultaneously and independently showed that a permutation is a simsun permutation if and only if it avoids the pattern
\[
\begin{array}{|c|}
\hline
\hline
\end{array}
\]

It is known that the number of simsun permutations of length \( n \) is \( E_{n+1} \), where \( E_n \) is the \( n \)-th Euler number [10].

Now, interval patterns were first defined by Woo and Yong in [14]. Then in an unpublished article called Interval Avoidance for Length Three Patterns\(^4\), Lankham and Woo showed that all interval patterns of length three are counted by the Catalan-numbers, except for one, the following pattern.

\[
p = \begin{array}{|c|}
\hline
\hline
\end{array}
\]

Observation 54. It is easy to see from Observation 12 that the following statement holds

\[ S_n \left( \begin{array}{c}
\vdots \\
\vdots \\
\end{array} \right) \subseteq S_n \left( \begin{array}{c}
\vdots \\
\vdots \\
\end{array} \right). \]

This gives grounds for using some known results for simsun permutations, such as bijections to binary trees, 1-2-trees \[13\], etc. Stanley provided a bijection between flip equivalence classes of binary trees and simsun permutations in \[9\].

Our idea of how to count the permutations containing \( p \) was to use observation 54 and and Stanley’s bijection and find how the trees corresponding to simsun permutations that contain \( p \) are different from the trees corresponding to simsun permutations not containing \( p \). We have found a way to describe the binary trees containing \( p \). In these trees one can find three nodes which labels are in decrasing order from left to right, let \( a \) be the leftmost node, \( b \) be the middle node and \( c \) be the rightmost node, and every node between \( a \) and \( b \) and \( b \) and \( c \) is lower than \( c \) or higher than \( a \). That is, the nodes to the left of \( a \) and the nodes to the right of \( c \) do not matter.

Example 55. The permutation \( \pi = 9 \ 3 \ 4 \ 6 \ 1 \ 5 \ 7 \ 8 \ 2 \ 11 \ 10 \) is simsun and it contains the pattern \( p \). The following figure shows the binary tree corresponding to \( \pi \) and the occurrence of \( p \) is marked in red. Here \( a = 6 \),

\[ b = 5 \] and \( c = 2 \). There is only one node between \( a \) and \( b \) and it is labelled with 1 which is lower than \( c \). Then between \( b \) and \( c \) there are two nodes and they are labelled with 7 and 8 which both are higher than \( a \).

The number of permutations avoiding the patterns in the next two conjectures is the same as in Proposition 36 for \( S_1, \ldots, S_8 \). Hence, we think that there might be more hidden operations that preserve Wilf-equivalence.
**Conjecture 56.** The following pattern is the representative pattern for pattern class 32, see Appendix A.3.

\[ p = \begin{array}{ccc}
1 & 1 & 1 \\
1 & 1 & 1 \\
2 & 2 & 2
\end{array} \]

The number of permutations of length \( n \) avoiding the pattern \( p \) is the same as the number of connected permutations of length \( n \). Thus, as in Proposition 36

\[
\left| S_n \left( \begin{array}{ccc} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 2 & 2 & 2 \end{array} \right) \right| = [x^n] \left( 1 - \frac{1}{\sum_n n!x^n} \right).
\]

It would be interesting to find a bijection between permutations that avoid the pattern \( p \) and the connected permutations.

**Conjecture 57.** The following pattern is the representative pattern for pattern class 54, see Appendix A.3.

\[ p = \begin{array}{ccc}
1 & 1 & 1 \\
1 & 1 & 1 \\
2 & 2 & 2
\end{array} \]

The number of permutations of length \( n \) avoiding the pattern is the same as the number of connected permutations of length \( n \). Thus, as in Proposition 36

\[
\left| S_n \left( \begin{array}{ccc} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 2 & 2 & 2 \end{array} \right) \right| = [x^n] \left( 1 - \frac{1}{\sum_n n!x^n} \right).
\]

It would be interesting to find a bijection between permutations that avoid the pattern \( p \) and the connected permutations.

**Conjecture 58.** The following pattern is the representative pattern for pattern class 29, see Appendix A.3.

\[ p = \begin{array}{ccc}
1 & 1 & 1 \\
1 & 1 & 1 \\
2 & 2 & 2
\end{array} \]

The number of permutations of length \( n \) avoiding the pattern is the same as the absolute value of the \( n \)-th line in column 0 of a triangular matrix given by the formula

\[
T(n,k) = \sum_{j=0}^{n-k} T(n-k,j) \cdot T(j+k-1,k-1)
\]

for \( n \geq k > 0 \) with \( T(0,0) = 1 \) and \( T(n,0) = -\sum_{j=1}^{n} T(n,j) \) for \( n > 0 \). It would be interesting to find a bijection between permutations that avoid the pattern \( p \) and the triangular matrix.

In some researches, permutations that avoid two classical patterns have been studied. It would be interesting to look at two mesh patterns and the permutations that avoid both of them.

Furthermore, it would also be interesting to extend Lemma 14 and find what conditions must be fulfilled to be able to shade two boxes at a time.

\[ ^5 \text{http://oeis.org/A101900} \]
6. Acknowledgements

First and foremost we would like to thank Henning Úlfarsson for all his help and time spent with us on the project. We would also like to thank Einar Steingrimsson for the support for Permutation Patterns 2011 as for the helpful criticism and comments. Furthermore, Sergey Kitaev, Anders Claesson and Anna Ingólfsdóttir for the useful remarks. David Callan for a helpful explanation on a particular pattern and Hjalti Magnússon for all the technical help with typesetting. At last, we thank our families for putting up with us all this time.

Appendix A. All the Mesh Patterns of Length 2

A.1. Patterns with formulas. Below, all the patterns we have found a formula for will be stated along with the formula for the number of permutations that avoid them. The sequence that follows the formula was found using Sage and it gives the number of permutations in $S_1, \ldots, S_8$, avoiding each of the patterns below. The patterns that belong to the same pattern class are listed together.

1. There is exactly one permutation that avoids each of the patterns for all $n \geq 1$, see Proposition 22.

   \begin{center}
   \includegraphics[width=0.1\textwidth]{pattern1}
   \end{center}

2. There is exactly one permutation that avoids each of the patterns for all $n \geq 1$, see Proposition 23.

   \begin{center}
   \includegraphics[width=0.1\textwidth]{pattern2}
   \end{center}

3. There is exactly one permutation that avoids each of the patterns for all $n \geq 1$, see Proposition 24.

   \begin{center}
   \includegraphics[width=0.1\textwidth]{pattern3}
   \end{center}

4. The number of permutations avoiding each of the patterns

   \begin{center}
   \includegraphics[width=0.1\textwidth]{pattern4}
   \end{center}
is \((n - 1)!\) for all \(n \geq 1\), see Proposition 25. The first terms in the sequence are 1, 1, 2, 6, 24, 120, 720, 5040.

5. The number of permutations avoiding each of the patterns

\[
\begin{array}{cccccc}
\begin{array}{ccc}
\cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot \\
\end{array} & \begin{array}{ccc}
\cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot \\
\end{array} & \begin{array}{ccc}
\cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot \\
\end{array} & \begin{array}{ccc}
\cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot \\
\end{array} & \begin{array}{ccc}
\cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot \\
\end{array} & \begin{array}{ccc}
\cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot \\
\end{array}
\end{array}
\]

is \((n - 1)!\) for all \(n \geq 1\), see Proposition 26. The first terms in the sequence are 1, 1, 2, 6, 24, 120, 720, 5040.

6. The number of permutations avoiding each of the patterns

\[
\begin{array}{cccccc}
\begin{array}{ccc}
\cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot \\
\end{array} & \begin{array}{ccc}
\cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot \\
\end{array} & \begin{array}{ccc}
\cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot \\
\end{array} & \begin{array}{ccc}
\cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot \\
\end{array} & \begin{array}{ccc}
\cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot \\
\end{array} & \begin{array}{ccc}
\cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot \\
\end{array}
\end{array}
\]

is \((n - 1)!\) for all \(n \geq 1\), see Proposition 30. The first terms in the sequence are 1, 1, 2, 6, 24, 120, 720, 5040.

7. The number of permutations avoiding each of the patterns

\[
\begin{array}{cccccc}
\begin{array}{ccc}
\cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot \\
\end{array} & \begin{array}{ccc}
\cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot \\
\end{array} & \begin{array}{ccc}
\cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot \\
\end{array} & \begin{array}{ccc}
\cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot \\
\end{array} & \begin{array}{ccc}
\cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot \\
\end{array} & \begin{array}{ccc}
\cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot \\
\end{array}
\end{array}
\]

is \((n - 1)!\) for all \(n \geq 1\), see Proposition 31. The first terms in the sequence are 1, 1, 2, 6, 24, 120, 720, 5040.

8. The number of permutations avoiding each of the patterns

\[
\begin{array}{cccccc}
\begin{array}{ccc}
\cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot \\
\end{array} & \begin{array}{ccc}
\cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot \\
\end{array} & \begin{array}{ccc}
\cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot \\
\end{array} & \begin{array}{ccc}
\cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot \\
\end{array} & \begin{array}{ccc}
\cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot \\
\end{array} & \begin{array}{ccc}
\cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot \\
\end{array}
\end{array}
\]

is given by the recursive formula \(a_n = n \cdot a_{n-1} - a_{n-2}\) where \(a_{-1} = 0, a_0 = 1\), see Proposition 32. The first terms in the sequence are 1, 1, 2, 7, 33, 191, 1304, 10241.

9. The number of permutations avoiding each of the patterns

\[
\begin{array}{cccccc}
\begin{array}{ccc}
\cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot \\
\end{array} & \begin{array}{ccc}
\cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot \\
\end{array} & \begin{array}{ccc}
\cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot \\
\end{array} & \begin{array}{ccc}
\cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot \\
\end{array} & \begin{array}{ccc}
\cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot \\
\end{array} & \begin{array}{ccc}
\cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot \\
\end{array}
\end{array}
\]
is given by the recursive formula \( a_n = n \cdot a_{n-1} - a_{n-2} \) where \( a_{-1} = 0, a_0 = 1 \), see Proposition 33. The first terms in the sequence are 1, 1, 2, 7, 33, 191, 1304, 10241.

10. The number of permutations avoiding each of the patterns

is given by the recursive formula \( a_n = (n - 1)a_{n-1} + (n - 2)a_{n-2} \) where \( a_0 = a_1 = 1 \), see Proposition 34. The first terms in the sequence are 1, 1, 3, 11, 53, 309, 2119, 16687.

11. The number of permutations avoiding each of the patterns

is given by the generating function

\[
1 - \frac{1}{\sum n!x^n},
\]

see Proposition 36. The first terms in the sequence are 1, 1, 3, 13, 71, 461, 3447, 29093.

12. The number of permutations avoiding each of the patterns
is given by
\[ \left[ \frac{x^n}{n} \right] \left( 1 + \sum_{n \geq 1} (n - 1)!x^n \right), \]
see Proposition 37. The first terms in the sequence are 1, 1, 4, 17, 91, 574, 4173, 34353.

13. The number of permutations avoiding each of the patterns

is given by
\[ n! - (n - 1)! + \left[ x^n \right] \frac{F(x)}{1 + xF(x)} \]
where \( F(x) = \sum_{n \geq 1} (n - 1)!x^{n-1} \), see Proposition 38. The first terms in the sequence are 1, 1, 5, 21, 110, 677, 4817, 38956.

14. The number of permutations avoiding each of the patterns
is given by the formula

\[ n! - \sum_{k=1}^{n-1} (k-1)!(n-k-1)! , \]

see Proposition 39. The first terms in the sequence are 1, 1, 4, 19, 104, 656, 4728, 38508.

15. The number of permutations avoiding each of the patterns

is given by the formula

\[ n! - \sum_{i=0}^{n-2} i!(n-1-i)! , \]

see Proposition 40. The first terms in the sequence are 1, 1, 3, 14, 80, 528, 3948, 33072.

16. The number of permutations avoiding each of the patterns

is given by the formula

\[ n! - \sum_{i=1}^{n-1} \sum_{\ell=1}^{i} (i-\ell)!(n-i-\ell)\ell! \]

for all \( n \geq 0 \), see Proposition 42. The first terms in the sequence are 1, 1, 2, 8, 47, 332, 2644, 23296.

17. The number of permutations avoiding each of the patterns

is given by the formula \( n! - \sum_{k=0}^{n-2} \sum_{j=0}^{k} j!(k-j)!(n-2-k)! \) for all \( n \geq 0 \), see Proposition 44. The first terms in the sequence are 1, 1, 3, 15, 89, 594, 4434, 36892.
18. The number of permutations avoiding each of the patterns is given by the formula

\[ n! - \sum_{i=1}^{n-1} \frac{(n-1)!}{i}, \]

see Proposition 45. The first terms in the sequence are 1, 1, 3, 13, 70, 446, 3276, 27252.
A.2. **Bivincular pattern classes.** Robert Parviainen Wilf-classified bivincular patterns of length two and three in [7]. Using Lemma 14, we found Wilf-equivalences between some mesh patterns and bivincular patterns of length two. Below we list the formulas from Robert’s paper for the number of permutations avoiding each of the seven pattern classes which include bivincular patterns.

19. There is exactly one permutation that avoids each of the patterns for all $n \geq 0$
20. The number of permutations avoiding each of the patterns
is \( n! \) for all \( n \geq 0 \) and the first terms in the sequence are 1, 1, 2, 6, 24, 120, 720.

21. The number of permutations avoiding each of the patterns
is given by the formula

\[ a_n = \sum_{i=0}^{n} (-1)^i(n - i + 1) \frac{n!}{i!}. \]

The first terms in the sequence are 1, 1, 3, 11, 53, 309, 2119.

22. The number of permutations avoiding each of the patterns

\[ \frac{n!}{2} \]

for all \( n \geq 0 \). The first terms in the sequence are 1, 1, 3, 12, 60, 360, 2520.
23. The number of permutations avoiding each of the patterns

is given by the formula $n! - (n-1)!$. The first terms in the sequence are 1, 1, 4, 18, 96, 600, 4320.

24. The number of permutations avoiding each of the patterns

is given by the formula $n! - (n-2)!$ for all $n > 1$. The first terms in the sequence are 1, 1, 5, 22, 114, 696, 4920.

25. The number of permutations avoiding each of the patterns
is given by the formula \( n! - \delta_{2n} \), where \( \delta_{ij} \) is the Kronecker delta symbol.

The first terms in the sequence are 1, 1, 6, 24, 120, 720, 5040.

A.3. **Classes we do not have formulas for.** By using Sage we compute the number of permutations in \( S_1, \ldots, S_8 \), avoiding each of the patterns below.

26. The number of permutations avoiding each of the patterns

is 1, 1, 3, 12, 62, 387, 2819, 23409.

27. The number of permutations avoiding each of the patterns

is 1, 1, 3, 12, 62, 385, 2789, 23040.

28. The number of permutations avoiding each of the patterns

is 1, 1, 3, 12, 61, 376, 2715, 22416.

29. The number of permutations avoiding each of the patterns

is 1, 1, 2, 5, 17, 71, 357, 2101.
30. The number of permutations avoiding each of the patterns

\[ \begin{array}{cccccccc}
| & | & | & | & | & | & | \\
| & | & | & | & | & | \\
| & | & | & | & | & | \\
| & | & | & | & | & | \\
| & | & | & | & | & | \\
| & | & | & | & | & | \\
| & | & | & | & | & | \\
| & | & | & | & | & | \\
\end{array} \]

is 1, 1, 3, 11, 51, 287, 1901, 14489.

31. The number of permutations avoiding each of the patterns

\[ \begin{array}{cccccccc}
| & | & | & | & | & | \\
| & | & | & | & | & | \\
| & | & | & | & | & | \\
| & | & | & | & | & | \\
| & | & | & | & | & | \\
| & | & | & | & | & | \\
| & | & | & | & | & | \\
| & | & | & | & | & | \\
\end{array} \]

is 1, 1, 3, 15, 85, 549, 4043, 33559.

32. The number of permutations avoiding each of the patterns

\[ \begin{array}{cccccccc}
| & | & | & | & | & | \\
| & | & | & | & | & | \\
| & | & | & | & | & | \\
| & | & | & | & | & | \\
| & | & | & | & | & | \\
| & | & | & | & | & | \\
| & | & | & | & | & | \\
| & | & | & | & | & | \\
\end{array} \]

is 1, 1, 3, 1, 71, 461, 3447, 29093.

33. The number of permutations avoiding each of the patterns

\[ \begin{array}{cccccccc}
| & | & | & | & | & | \\
| & | & | & | & | & | \\
| & | & | & | & | & | \\
| & | & | & | & | & | \\
| & | & | & | & | & | \\
| & | & | & | & | & | \\
| & | & | & | & | & | \\
| & | & | & | & | & | \\
\end{array} \]

is 1, 1, 2, 8, 43, 277, 2070, 17567.

34. The number of permutations avoiding each of the patterns

\[ \begin{array}{cccccccc}
| & | & | & | & | & | \\
| & | & | & | & | & | \\
| & | & | & | & | & | \\
| & | & | & | & | & | \\
| & | & | & | & | & | \\
| & | & | & | & | & | \\
| & | & | & | & | & | \\
| & | & | & | & | & | \\
\end{array} \]

is 1, 1, 3, 13, 69, 437, 3209, 26751.

35. The number of permutations avoiding each of the patterns

\[ \begin{array}{cccccccc}
| & | & | & | & | & | \\
| & | & | & | & | & | \\
| & | & | & | & | & | \\
| & | & | & | & | & | \\
| & | & | & | & | & | \\
| & | & | & | & | & | \\
| & | & | & | & | & | \\
| & | & | & | & | & | \\
\end{array} \]

is 1, 1, 3, 15, 85, 549, 4043, 33559.

36. The number of permutations avoiding each of the patterns

\[ \begin{array}{cccccccc}
| & | & | & | & | & | \\
| & | & | & | & | & | \\
| & | & | & | & | & | \\
| & | & | & | & | & | \\
| & | & | & | & | & | \\
| & | & | & | & | & | \\
| & | & | & | & | & | \\
| & | & | & | & | & | \\
\end{array} \]

is 1, 1, 2, 8, 43, 277, 2070, 17567.

37. The number of permutations avoiding each of the patterns

\[ \begin{array}{cccccccc}
| & | & | & | & | & | \\
| & | & | & | & | & | \\
| & | & | & | & | & | \\
| & | & | & | & | & | \\
| & | & | & | & | & | \\
| & | & | & | & | & | \\
| & | & | & | & | & | \\
| & | & | & | & | & | \\
\end{array} \]

is 1, 1, 2, 9, 52, 341, 2540, 21360.
38. The number of permutations avoiding each of the patterns

is 1, 1, 3, 12, 64, 412, 3074, 25946.

39. The number of permutations avoiding each of the patterns

is 1, 1, 2, 9, 54, 370, 2849, 24483.

40. The number of permutations avoiding each of the patterns

is 1, 1, 2, 9, 54, 370, 2849, 24483.

41. The number of permutations avoiding each of the patterns

is 1, 1, 3, 12, 64, 412, 3074, 25946.

42. The number of permutations avoiding each of the patterns

is 1, 1, 2, 8, 44, 290, 2204, 18930.

43. The number of permutations avoiding each of the patterns

is 1, 1, 3, 13, 70, 448, 3307, 27618.

44. The number of permutations avoiding each of the patterns

is 1, 1, 3, 14, 76, 480, 3491, 28792.

45. The number of permutations avoiding each of the patterns

is 1, 1, 3, 11, 55, 337, 2437, 20211.
46. The number of permutations avoiding each of the patterns

\[
\begin{array}{cccc}
\begin{array}{cc}
\bullet & \bullet \\
\bullet & \bullet \\
\end{array} & 
\begin{array}{cc}
\bullet & \bullet \\
\bullet & \bullet \\
\end{array} & 
\begin{array}{cc}
\bullet & \bullet \\
\bullet & \bullet \\
\end{array} & 
\begin{array}{cc}
\bullet & \bullet \\
\bullet & \bullet \\
\end{array}
\end{array}
\]

is 1, 1, 4, 15, 80, 493, 3565, 29279.

47. The number of permutations avoiding each of the patterns

\[
\begin{array}{cccc}
\begin{array}{cc}
\bullet & \bullet \\
\bullet & \bullet \\
\end{array} & 
\begin{array}{cc}
\bullet & \bullet \\
\bullet & \bullet \\
\end{array} & 
\begin{array}{cc}
\bullet & \bullet \\
\bullet & \bullet \\
\end{array} & 
\begin{array}{cc}
\bullet & \bullet \\
\bullet & \bullet \\
\end{array}
\end{array}
\]

is 1, 1, 2, 8, 42, 265, 1956, 16482.

48. The number of permutations avoiding each of the patterns

\[
\begin{array}{cccc}
\begin{array}{cc}
\bullet & \bullet \\
\bullet & \bullet \\
\end{array} & 
\begin{array}{cc}
\bullet & \bullet \\
\bullet & \bullet \\
\end{array} & 
\begin{array}{cc}
\bullet & \bullet \\
\bullet & \bullet \\
\end{array} & 
\begin{array}{cc}
\bullet & \bullet \\
\bullet & \bullet \\
\end{array}
\end{array}
\]

is 1, 1, 4, 16, 83, 512, 3671, 29983.

49. The number of permutations avoiding each of the patterns

\[
\begin{array}{cccc}
\begin{array}{cc}
\bullet & \bullet \\
\bullet & \bullet \\
\end{array} & 
\begin{array}{cc}
\bullet & \bullet \\
\bullet & \bullet \\
\end{array} & 
\begin{array}{cc}
\bullet & \bullet \\
\bullet & \bullet \\
\end{array} & 
\begin{array}{cc}
\bullet & \bullet \\
\bullet & \bullet \\
\end{array}
\end{array}
\]

is 1, 1, 2, 8, 44, 290, 2204, 18930.

50. The number of permutations avoiding each of the patterns

\[
\begin{array}{cccc}
\begin{array}{cc}
\bullet & \bullet \\
\bullet & \bullet \\
\end{array} & 
\begin{array}{cc}
\bullet & \bullet \\
\bullet & \bullet \\
\end{array} & 
\begin{array}{cc}
\bullet & \bullet \\
\bullet & \bullet \\
\end{array} & 
\begin{array}{cc}
\bullet & \bullet \\
\bullet & \bullet \\
\end{array}
\end{array}
\]

is 1, 1, 2, 9, 52, 341, 2540, 21360.

51. The number of permutations avoiding each of the patterns

\[
\begin{array}{cccc}
\begin{array}{cc}
\bullet & \bullet \\
\bullet & \bullet \\
\end{array} & 
\begin{array}{cc}
\bullet & \bullet \\
\bullet & \bullet \\
\end{array} & 
\begin{array}{cc}
\bullet & \bullet \\
\bullet & \bullet \\
\end{array} & 
\begin{array}{cc}
\bullet & \bullet \\
\bullet & \bullet \\
\end{array}
\end{array}
\]

is 1, 1, 2, 8, 41, 251, 1809, 14986.

52. The number of permutations avoiding each of the patterns

\[
\begin{array}{cccc}
\begin{array}{cc}
\bullet & \bullet \\
\bullet & \bullet \\
\end{array} & 
\begin{array}{cc}
\bullet & \bullet \\
\bullet & \bullet \\
\end{array} & 
\begin{array}{cc}
\bullet & \bullet \\
\bullet & \bullet \\
\end{array} & 
\begin{array}{cc}
\bullet & \bullet \\
\bullet & \bullet \\
\end{array}
\end{array}
\]

is 1, 1, 3, 11, 56, 349, 2560, 21453.

53. The number of permutations avoiding each of the patterns

\[
\begin{array}{cccc}
\begin{array}{cc}
\bullet & \bullet \\
\bullet & \bullet \\
\end{array} & 
\begin{array}{cc}
\bullet & \bullet \\
\bullet & \bullet \\
\end{array} & 
\begin{array}{cc}
\bullet & \bullet \\
\bullet & \bullet \\
\end{array} & 
\begin{array}{cc}
\bullet & \bullet \\
\bullet & \bullet \\
\end{array}
\end{array}
\]

is 1, 1, 4, 18, 99, 631, 4592, 37675.
54. The number of permutations avoiding each of the patterns

\[
\begin{array}{cccc}
\begin{array}{c}
\vdots \\
\end{array} & \begin{array}{c}
\vdots \\
\end{array} & \begin{array}{c}
\vdots \\
\end{array} & \begin{array}{c}
\vdots \\
\end{array}
\end{array}
\]

is 1, 1, 3, 13, 71, 461, 3447, 29093.

55. The number of permutations avoiding each of the patterns

\[
\begin{array}{cccc}
\begin{array}{c}
\vdots \\
\end{array} & \begin{array}{c}
\vdots \\
\end{array} & \begin{array}{c}
\vdots \\
\end{array} & \begin{array}{c}
\vdots \\
\end{array}
\end{array}
\]

is 1, 1, 2, 7, 35, 217, 1586, 13287.

56. The number of permutations avoiding each of the patterns

\[
\begin{array}{cccc}
\begin{array}{c}
\vdots \\
\end{array} & \begin{array}{c}
\vdots \\
\end{array} & \begin{array}{c}
\vdots \\
\end{array} & \begin{array}{c}
\vdots \\
\end{array}
\end{array}
\]

is 1, 1, 2, 9, 54, 370, 2849, 24483.

57. The number of permutations avoiding each of the patterns

\[
\begin{array}{cccc}
\begin{array}{c}
\vdots \\
\end{array} & \begin{array}{c}
\vdots \\
\end{array} & \begin{array}{c}
\vdots \\
\end{array} & \begin{array}{c}
\vdots \\
\end{array}
\end{array}
\]

is 1, 1, 2, 8, 42, 265, 1956, 16482.

58. The number of permutations avoiding both of the patterns

\[
\begin{array}{cccc}
\begin{array}{c}
\vdots \\
\end{array} & \begin{array}{c}
\vdots \\
\end{array} & \begin{array}{c}
\vdots \\
\end{array} & \begin{array}{c}
\vdots \\
\end{array}
\end{array}
\]

is 1, 1, 2, 7, 35, 218, 1598, 13398.

59. The number of permutations avoiding both of the patterns

\[
\begin{array}{cccc}
\begin{array}{c}
\vdots \\
\end{array} & \begin{array}{c}
\vdots \\
\end{array} & \begin{array}{c}
\vdots \\
\end{array} & \begin{array}{c}
\vdots \\
\end{array}
\end{array}
\]

is 1, 1, 4, 15, 80, 501, 3666, 30467.

60. The number of permutations avoiding both of the patterns

\[
\begin{array}{cccc}
\begin{array}{c}
\vdots \\
\end{array} & \begin{array}{c}
\vdots \\
\end{array} & \begin{array}{c}
\vdots \\
\end{array} & \begin{array}{c}
\vdots \\
\end{array}
\end{array}
\]

is 1, 1, 5, 20, 106, 657, 4707, 38267.

61. The number of permutations avoiding both of the patterns

\[
\begin{array}{cccc}
\begin{array}{c}
\vdots \\
\end{array} & \begin{array}{c}
\vdots \\
\end{array} & \begin{array}{c}
\vdots \\
\end{array} & \begin{array}{c}
\vdots \\
\end{array}
\end{array}
\]

is 1, 1, 3, 11, 53, 315, 2217, 17990.
62. The number of permutations avoiding both of the patterns

\[
\begin{align*}
\begin{array}{ccc}
& & \\
& & \\
& & \\
\end{array}
\end{align*}
\]

is 1, 1, 3, 10, 50, 290, 2018, 16023.

63. The number of permutations avoiding both of the patterns

\[
\begin{align*}
\begin{array}{ccc}
& & \\
& & \\
& & \\
\end{array}
\end{align*}
\]

is 1, 1, 3, 16, 94, 613, 4507, 37203.

64. The number of permutations avoiding both of the patterns

\[
\begin{align*}
\begin{array}{ccc}
& & \\
& & \\
& & \\
\end{array}
\end{align*}
\]

is 1, 1, 4, 20, 107, 664, 4755, 38621.

65. The number of permutations avoiding both of the patterns

\[
\begin{align*}
\begin{array}{ccc}
& & \\
& & \\
& & \\
\end{array}
\end{align*}
\]

is 1, 1, 5, 21, 109, 673, 4797, 38845.

**Appendix B. Glossary**

<table>
<thead>
<tr>
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In this section we will list the code we used to sort the 1024 patterns into the 65 pattern classes.

```python
# REVERSE
def rev(mpatt):
    """
The input, mpatt, is a mesh pattern of the form (p,R).
The output is the reverse of mpatt.
"""
    patt = mpatt[0]
    M = mpatt[1]
    m = len(patt)
    npatt = Permutation(patt).reverse()
    nM = Set([(m-x,y) for (x,y) in M])
    return (npatt,nM)

# INVERSE
def inve(mpatt):
    """
The input, mpatt is a mesh pattern of the form (p,R).
The output is the inverse of mpatt.
"""
    patt = mpatt[0]
    M = mpatt[1]
    m = len(patt)
    npatt = Permutation(patt).inverse()
    nM = Set([(y,x) for (x,y) in M])
    return (npatt,nM)

# COMPLEMENT
def compl(mpatt):
    """
The input, mpatt, is a pattern of the form (p,R).
The output is the complement of mpatt.
"""
    patt = mpatt[0]
    M = mpatt[1]
```
\[ m = \text{len}(\text{patt}) \]
\[ \text{npatt} = \text{Permutation}(\text{patt}).\text{complement}() \]
\[ \text{nM} = \text{Set}([[x, m-y] \ for \ (x, y) \ in \ M]) \]
\[ \text{return} \ (\text{npatt}, \text{nM}) \]

Listing 1. The code finds reverse, inverse and complement for a given mesh pattern.

```python
# SHADING LEMMA
def shlem(p, R, box):
    '':
    The input is p, R and box.
    p and R represent a mesh pattern. p is of the form [i, j] and is the underlying classical pattern of the mesh pattern. R is a set of coordinates which represent boxes that are shaded in the mesh pattern and are of the form ((a,b),(c,d),...). box is of the form (a,b) and is a coordinate representing a box we want to shade.
    Output is False if the box can not be shaded and True if it can be shaded according to the Shading Lemma.
    '':
    n = len(p)
    if box in R:
        return False
    i = box[0]
    j = box[1]
    listi = list()
    points = [(k, p[k-1]) for k in [1..n]]
    pbox = [(i, j), (i+1, j), (i, j+1), (i+1, j+1)]
    for a in [0..3]:
        if pbox[a] in points:
            listi.append(pbox[a])
    if len(listi) == 0:
        return False
    for L in listi:
        A = True
        if L == (i, j):
            if (i-1, j-1) in R:
                A = False
            elif (i, j-1) in R and (i-1, j) in R:
                A = False
        else:
            hor = [0..i-2]
            hor.extend([i+1..n])
            vert = [0..j-2]
            vert.extend([j+1..n])
            for k in hor:
```
if (k, j-1) in R and (k, j) not in R:
    A = False
for k in vert:
    if (i-1, k) in R and (i, k) not in R:
        A = False
if L == (i+1, j+1):
    if (i+1, j+1) in R:
        A = False
    elif (i, j+1) in R and (i+1, j) in R:
        A = False
else:
    hor = [0..i-1]
    hor.extend([i+2..n])
    vert = [0..j-1]
    vert.extend([j+2..n])
    for k in hor:
        if (k, j+1) in R and (k, j) not in R:
            A = False
    for k in vert:
        if (i+1, k) in R and (i, k) not in R:
            A = False
if L == (i+1, j):
    if (i+1, j-1) in R:
        A = False
    elif (i, j-1) in R and (i+1, j) in R:
        A = False
else:
    hor = [0..i-1]
    hor.extend([i+2..n])
    vert = [0..j-2]
    vert.extend([j+1..n])
    for k in hor:
        if (k, j-1) in R and (k, j) not in R:
            A = False
    for k in vert:
        if (i+1, k) in R and (i, k) not in R:
            A = False
if L == (i, j+1):
    if (i-1, j+1) in R:
        A = False
    elif (i, j+1) in R and (i-1, j) in R:
        A = False
else:
    hor = [0..i-2]
    hor.extend([i+1..n])
    vert = [0..j-1]
    vert.extend([j+2..n])
    for k in hor:
if (k, j+1) in R and (k, j) not in R:
    A = False
for k in vert:
    if (i-1, k) in R and (i, k) not in R:
        A = False
if A == True:
    return True
return False

Listing 2. Implementation of the Shading Lemma

#TORIC
'''
The first three functions are straightforward and
are a support to the function def toric_symm(meshp)
'''

def read_from_0(perm0):
    l = len(perm0)
    ind0 = perm0.index(0)
    start = [perm0[i] for i in [ind0+1..l-1]]
    end = [perm0[i] for i in [0..ind0-1]]
    start.extend(end)
    return start

def addmod(perm0, i):
    l = len(perm0)-1
    perm = list(perm0)
    for j in [0..l]:
        h = (perm0[j] + i).mod(l+1)
        perm[j] = h
    return perm

def toric_shift(perm):
    perm0 = [0]
    perm0.extend(perm)
    return Permutation(read_from_0(addmod(perm0, 1)))

def toric_symm(meshp):
    '''
    meshp is a mesh pattern of the form (p, R).
    The output is a new meshp such that boxes (i, j)
    become (i-L, j+1) modulo n+1 where n is the size
    of p and the underlying pattern is shifted by
    the usual toric symmetry.
    Note that this symmetry only makes sense if the
    top line of R is filled in.
    '''
    p = meshp[0]
R = meshp[1]
newR = []
n = len(p)
L = Permutation(p).inverse()(n)
for r in R:
i = r[0]; j = r[1]
newR.append(((i-L).mod(n+1),(j+1).mod(n+1)))
return (toric_shift(p),newR)

Listing 3. The Toric operation

#is classical
def is_classical(patt):
    '''
The input, patt, is a mesh pattern of the form (p,R).
The output is True if patt is classical and False otherwise.
    '''
    if len(patt[1]) == 0:
        return True
    else:
        return False

Listing 4. The code finds whether a pattern is classical or not

def horizontal_line(R,n,j):
    '''
The input is R, a set of boxes of the form (a,b), n, the length of a permutation, and j, a particular horizontal line.
    '''
    for k in [0..n]:
        if (k,j) not in R:
            return False
    return True

def vertical_line(R,n,i):
    '''
The input is R, a set of boxes of the form (a,b), n, the length of a permutation and i, a particular vertical line.
The output is True if the boxes in R form a vertical line, and False otherwise.
    '''
    for k in [0..n]:
        if (i,k) not in R:
            return False
    return True
Listing 5. The code finds whether a pattern contains a particular vertical or a horizontal shaded line

```python
def is_vincular(patt):
    
    The input, patt, is a mesh pattern of the form (p,R). 
The output is True if patt is vincular and False otherwise.
    
    n = len(patt[0])
    R = patt[1]
    checkR = list(R)
    while len(checkR) > 0:
        box = checkR[0]
        i = box[0]
        if vertical_line(R, n, i) == False:
            return False
        else:
            for k in [0..n]:
                if (i, k) in checkR:
                    checkR.remove((i, k))
    return True
```

Listing 6. The code finds whether a pattern is vincular or not

```python
def is_bivincular(patt):
    
    The input, patt, is a mesh pattern of the form (p,R). 
The output is True if patt is bivincular and False otherwise.
    
    n = len(patt[0])
    R = patt[1]
    checkR = list(R)
    while len(checkR) > 0:
        box = checkR[0]
        i = box[0]
        j = box[1]
        if vertical_line(R, n, i) == False:
            if horizontal_line(R, n, j) == False:
                return False
            else:
                for k in [0..n]:
                    if (k, j) in checkR:
                        checkR.remove((k, j))
        else:
            for k in [0..n]:
                if (i, k) in checkR:
```
checkR.remove((i,k))

Listing 7. The code finds whether a pattern is bivincular or not.

```python
#TEX
def tex(p,R):
    """
The input, p, is a mesh pattern of the form (p,R) and R is a set containing the shaded boxes in the pattern of the form (a,b).
The output is the latex code to draw the mesh pattern.
"""
    n = len(p)
    listi = list()
    w = len(R) - 1
    string1 = ""
    for k in [1..n-1):
        a = ""
        a = str(k) + "/" + str(p[k-1]) + ","
        string1 = string1 + a
    string1 += str(n) + "/" + str(p[n-1])
    string2 = ""
    for m in [0..len(R)-2]:
        b = ""
        b = str(R[m][0]) + "/" + str(R[m][1]) + ","
        string2 = string2 + b
    string2 += str(R[w][0]) + "/" + str(R[w][1])
    return "\pattern{scale = 1}\{" + str(n) + "}\{" + string1 + "}\{" + string2 + "}"
```

Listing 8. The code is used to draw mesh patterns in \LaTeX.

References


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