RH-18-2011

Dissertation for the degree of doctor of philosophy

Disc Formulas in Complex Analysis

Benedikt Steinar Magnússon

UNIVERSITY OF ICELAND

School of Engineering and Natural Sciences
Faculty of Physical Sciences
Reykjavík, December 2011

A dissertation presented to the University of Iceland School of Engineering and Natural Sciences in candidacy for the degree of doctor of philosophy

Doctoral committee

Prof. Ragnar Sigurðsson
Faculty of Physical Sciences, University of Iceland

Prof. Jón Ingólfur Magnússon
Faculty of Physical Sciences, University of Iceland

Prof. Lárus Thorlacius
NORDITA, Nordic Institute for Theoretical Physics, and
Faculty of Physical Sciences, University of Iceland

Opponents

Prof. Evgeny Poletsky
Syracuse University

Prof. Ahmed Zeriahi
Université Paul Sabatier

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Printed in Iceland by Háskólaprent
Science Institute Report RH-18-2011
ISBN 978-9935-9053-2-1

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Abstract

The main theme of this thesis are disc functionals in complex analysis, that is real valued functions from a set of analytic discs in a given manifold. The fundamental example is the Poisson disc functional of a given upper semicontinuous function, whose properties have been well studied in the last two decades. The main result proves that its envelope, which is a function on the manifold, is equal to the largest plurisubharmonic function dominated by the given function.

Our main goal is to generalize the theory of disc functionals and specifically the Poisson functional to the theory of quasiplurisubharmonic functions. We shall see how that case sheds new light on the connection between different disc functionals and the theory of disc functionals.

We start by studying the Poisson disc functional and we prove that its envelope is plurisubharmonic when the function in question is the difference of an upper semicontinuous function and a plurisubharmonic function. This leads us to the theory of quasiplurisubharmonic functions, or ω-plurisubharmonic functions, because this result is equivalent to the corresponding problem for ω-plurisubharmonic functions when the current ω has a global potential. The main work is then to generalize this result for those ω which do not have a global potential.

Ágrip (in Icelandic)

Meginstef þessarar ritgerðar er skífufelli í tvinnfallagreiningu. Það eru raungild föll á mengi af skífum í gefinni tvinnvíðáttu. Mikilvægasta dæmið um slíkt felli er Poisson-skífufellið fyrir gefið fall sem er hálfsamfellt að ofan. Eiginleikar Poisson-fellisins hafa verið vel rannsakaðir undanfarna tvo áratugi og helsta niðurstaðan segir að hjúpur þess, sem er fall á víðáttunni, sé jafn stærsta fjölundirþýða fallinu sem er yfirgnæft af gefna fallinu.

Markmiðið er að alhæfa fræðin um skífufelli, og sérstaklega Poisson skífufellið, fyrir hálffjölundirþýð föll. Það hefur í för með sér að hægt er að tengja saman ólík skífufelli og bæta þannig heildarmyndina sem við höfum af skífufellum.

Við byrjum á að skoða Poisson skífufellið og sanna að hjúpur þess er fjölundirþýður þegar fallið sem er gefið er mismunur tveggja falla, annars vegar falls sem er hálfsamfellt að ofan og hins vegar falls sem er fjölundirþýtt. Þessi niðurstaða vísar veginn að hálffjölundirþýðu föllunum, því hún er jafngild tilsvarandi niðurstöðu fyrir hálffjölundirþýð föll, eða ω-fjölundirþýð föll eins og þau eru líka kölluð, í því tilviki þegar straumurinn ω hefur víðfemt mætti. Aðalvinnan liggur svo í því að alhæfa þessa niðurstöðu fyrir þau tilvik þar sem straumurinn ω hefur ekki víðfemt mætti.

Acknowledgements

First of all I would like to thank my advisor prof. Ragnar Sigurðsson for his guidance and support. I wish to thank him for motivating me and pointing me the way into the field of quasiplurisubharmonic functions. His overview and knowledge has been invaluable to me.

I would like to thank my old fellow students and friends at the office, Baldvin, Erling, Olli and Siggi, and also everyone else who is or has been at the office.

My thanks go also to all my mathematical friends that I have met at KAUS, the yearly workshop for graduate students in complex analysis in Scandinavia, and other conferences.

I'm grateful for the opportunity to stay at the Mittag-Leffler institute in Stockholm in the spring of 2008, and also for the hospitality at Université Paul Sabatier, Toulouse, in 2009 and 2010.

This work was done at the Science Institute of the University of Iceland and supported by the Research Fund of the University of Iceland and a teaching assistant grant from the Faculty of Physical Sciences, University of Iceland.

Finally I want to thank my wife Hrafnhildur Karla and my three children Magnús Indriði, Jón Illugi and Sigrún Melkorka for their love, patience and support through the years.

1

Introduction

1.1 Introduction

The importance of analytic discs in complex analysis of several variables is undeniable. They play a crucial role when studying pseudoconvexity [24], Kobayashi metrics [22], CR-manifolds [2, Chapter VIII] and characterizing solutions of the Dirichlet problem for plurisubharmonic functions [1, 3]. In some sense, analytic discs are to complex analysis what line segments are to real analysis. To emphasize this connection consider the following problem. Given a real valued function φ on an open subset $X \subset \mathbb{R}^n$ we wish to find the largest convex function dominated by φ. It is not hard to show that

$$\sup\{u(x); u \text{ convex }, u \leq \varphi\} = \inf_l \{\alpha\, \varphi(l(1)) + (1-\alpha)\varphi(l(0))\}, \tag{1.1}$$

where the infimum is taken over all line segments $l : [0, 1] \to X$ such that there is an $\alpha \in [0, 1]$ with $x = \alpha\, l(1) + (1 - \alpha)\, l(0)$.

Lets clarify this a bit, since the methods we use in several complex variables are somewhat similar. In figure 1.1 we see how the graph of the expression in the right hand side of (1.1) has its endpoints on the graph of φ. By the definition of convexity, the graph of u must lie below this line segment. This fact justifies that the left hand side of (1.1) is less than or equal to the right

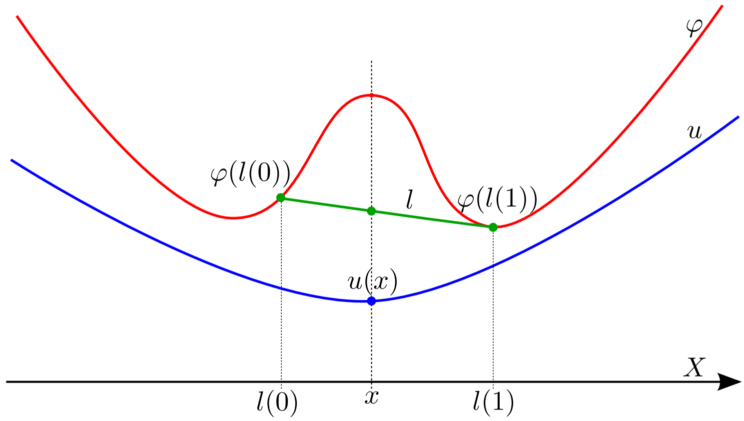

Figure 1.1: Visualization of the problem for one real dimension

hand side.

To prove the equality it suffices to show that the right hand side in (1.1) is not greater than φ and that it defines a convex function. Then it is in the family on the left hand side and we have an equality.

The corresponding problem for pluripotential theory in several complex variables involves finding the largest plurisubharmonic function dominated by a function φ on an open set $X \subset \mathbb{C}^n$. Poletsky [32–34] and Bu and Schachermayer [5] showed independently that when the function φ is upper semicontinuous then

$$\sup\{u(x); u \in \mathcal{PSH}(X), u \leq \varphi\} = \inf\left\{\int_{\mathbb{T}} \varphi \circ f \, d\sigma; f \in \mathcal{A}_X, f(0) = x\right\}. \quad (1.2)$$

Here, $\mathcal{A}_X$ is the family of all *closed analytic discs in* X, that is analytic functions from a neighbourhood of the closed unit disc into X, and σ is the arc length measure on the unit circle $\mathbb{T} = \{t \in \mathbb{C}; |t| = 1\}$ normalized to 1.

Formulas of this form are referred to as *disc formulas*, more specifically we call mappings from $\mathcal{A}_X$ to $\mathbb{R} \cup \{\infty, -\infty\}$ a *disc functional*, and the integral on the right hand side of (1.2) is called the *Poisson disc functional*. The envelope of a disc functional at a point x is then given by the infimum over all discs sending zero to x.

The goal of this thesis is to give a comprehensive overview of disc formulas in complex analysis and extend the theory of them to quasiplurisubharmonic functions.

In Chapter 3 we consider disc formulas for plurisubharmonic functions. Lárusson and Sigurdsson [25, 26], and Rosay [35] extended Poletsky's result to the case when X is a complex manifold, and Edigarian [10] showed that the function φ can be plurisuperharmonic. We will extend these results, as shown in [30], and prove that φ can in fact be the difference of an upper semicontinuous function and a plurisubharmonic function (Section 3.2). This will enable us to combine the Poisson disc functional in (1.2) with another disc formula for the Riesz disc functional (Section 3.4). This turns out to be a special case of the results for quasiplurisubharmonic functions presented later on. Recently Drnovšek and Forstnerič [8] extended Poletsky's result to locally irreducible complex spaces.

Recently [4, 11, 13–15] the classical pluripotential theory has been generalized to compact manifolds, for example to study Monge-Ampére equations and construct specific metrics. This is the theory of quasiplurisubharmonic function and in Chapter 4 we will consider disc formulas for them. These functions are also called ω-plurisubharmonic functions. Although it is common to consider ω to be a smooth, closed and positive current on a Kähler manifold, it suffices for our work to assume $\omega = \omega_1 - \omega_2$ is the difference of two closed, positive $(1,1)$-currents on a complex manifold X. We will describe the result here informally, for precise definitions see Sections 2.2 and 4.1.

In the case of ω-plurisubharmonic functions we first see how to formulate the problem correctly by defining the pullback $f^*\omega$ of ω by an analytic disc f. Its Riesz potential $R_{f^*\omega}$ then enables us to incorporate ω into the Poisson disc functional. We will prove in two steps that if $\varphi = \varphi_1 - \varphi_2$ is the difference of an ω_1-upper semicontinuous function φ_1 in $L^1_{\mathrm{loc}}(X)$ and a plurisubharmonic function φ_2, then for every $x \in X \setminus \operatorname{sing}(\omega)$,

$$\begin{aligned}\sup\{u(x); u \in \mathcal{PSH}(X,\omega), u \le \varphi\}& \\ = \inf\{-R_{f^*\omega}(0) + \int_{\mathbb{T}} \varphi \circ f \, d\sigma; f \in \mathcal{A}_X, f(0) = x\}&. \quad (1.3)\end{aligned}$$

This result was first introduced by the author in [29] and then generalized to the above form in [30].

1.2 Outline

In Chapter 2 we give the necessary background in pluripotential theory (Section 2.1), it includes upper semicontinuity, subharmonic functions, and plurisubharmonic functions. It also includes the pluripotential theory of quasiplurisubharmonic functions and their basic properties (Section 2.2), many of whom correspond to similar properties of plurisubharmonic functions. Section 2.3 introduces disc functionals and envelopes of disc functionals, and studies their general properties.

In Chapter 3 we study the Poletsky disc functional of a given function φ. We use a method of Bu and Schachermayer (Section 3.1) to prove that its envelope is plurisubharmonic when looking at sets in $\mathbb{C}^n$ and φ is an upper semicontinuous function. In Section 3.2 we generalize this by showing that φ can in fact be the difference of an upper semicontinuous function φ_1 and a plurisubharmonic function φ_2. This is done by using convolution but it requires some preciseness because we have to look at the sum of functions which can take the values $-\infty$ and $+\infty$. We see in Section 3.3 how this result can be extended to every complex manifold by using a theorem of Lárusson and Sigurdsson [26, Theorem 1.2]. This is a general theorem they used to extend Poletsky's results to manifolds, but the theorem states that the problem on manifolds can be reduced to domains of holomorphy in $\mathbb{C}^n$. Finally in Section 3.4 we show how the function φ_2 enables us to unify the Poisson disc functional and the Riesz functional, which is another disc functional studied by Poletsky [33, 34], Lárusson and Sigurdsson [25, 26] and Edigarian [10].

Chapter 4 starts by motivating the theory of disc functionals for quasi-plurisubharmonic functions 4.1 and there we define the generalization of the Poisson disc functional for ω-plurisubharmonic functions. The first step in proving equality (1.3) is in Section 4.2, where we look at the case when the current ω has a global potential. To prove the equality in this case we use the results from Chapter 3. In Section 4.3 we see how the general case can be reduced to sets with global potentials. This is done by presenting a general theorem which states that an envelope of a disc functional is quasiplurisubharmonic if it satisfies the following conditions. The disc functional has two mild continuity properties and the corresponding envelope on manifolds with global potentials is quasiplurisubharmonic.

2

Background

Harmonic functions, that is twice differentiable functions in the kernel of the Laplacian, $\Delta = \sum_{j=1}^{n} \frac{\partial^2}{\partial x_j^2}$, are of great importance in potential theory in n real-variables. In $\mathbb{R}^2$ the theory of harmonic functions is closely related to complex analysis in one complex variable, since the real and imaginary part of a holomorphic function are harmonic. It is therefore easy to see that harmonic functions satisfy the maximum principle, Liouville's theorem and the mean value theorem. Subharmonic functions are upper semicontinuous function (see Definition 2.1.1) u such that $\Delta u \geq 0$. They play a big role in potential theory and examples of them are given by $\log |f|$ and $|f|^\alpha$, where f is holomorphic. The main reason for using subharmonic functions is that they are a lot more flexible than harmonic functions. For example, the maximum of two subharmonic functions is subharmonic. This flexibility is clearly noticeable when we are searching for solutions of the Dirichlet problem, that is a harmonic function on a domain Ω with given boundary values. But the Perron method [16, Chapter 2.6] gives a solution by using subharmonic functions,

$$\sup\{v(x); v \text{ subharmonic on } \Omega \text{ and } \limsup_{y \to t} v(y) \leq \varphi(t) \text{ for } t \in \partial\Omega\},$$

where the boundary values are given by the function φ and the boundary satisfies certain regularity conditions.

For more information about potential theory in real variables we refer the reader to the books of Hayman and Kennedy [16], Hörmander [18], and Kellogg [20].

This theory does not work equally well in several complex variables due to the fact that the property of being subharmonic is then not invariant under biholomorphic mappings. The reason is that the class of subharmonic functions is too large in several complex variables. This fact motivates the theory of plurisubharmonic functions and pluripotential theory. The following section will give most of the necessary background for our purposes on plurisubharmonic functions and in Section 2.2 we will develop similar results for quasiplurisubharmonic functions.

2.1 Plurisubharmonic functions

2.1.1 Semicontinuity

Although there is a difference between subharmonic functions and plurisubharmonic functions, they satisfy the same continuity property, that is upper semicontinuity.

Definition 2.1.1 A functions $\varphi : X \to [-\infty, +\infty[$ defined on a topological space X is called *upper semicontinuous* if

$$\limsup_{y\to x} \varphi(y) = \varphi(x), \qquad \text{for every } x \in X. \tag{2.1}$$

Here lim sup is defined as

$$\limsup_{y\to x} \varphi(y) = \lim_{\varepsilon\to 0}(\sup\{\varphi(y); y \in B(x,\varepsilon)\}),$$

when X is a metric space, but for a general topological space

$$\limsup_{y\to x} \varphi(y) = \inf\{\sup\{\varphi(y); y \in U\}; U \text{ a non-empty open neighbourhood of } x\}.$$

It is easy to see using the definition, that a function φ is upper semicontinuous if and only if $\varphi^{-1}([-\infty, a[)$ is open for every $a \in \mathbb{R}$.

Roughly speaking, an upper semicontinuous function can "jump up" at some points but it can't "jump down". For example, the characteristic function of a closed set is upper semicontinuous.

A function φ such that $-\varphi$ is upper semicontinuous is called *lower semicontinuous*. Again, it is easy to verify that a function is continuous if and only if it is both lower semicontinuous and upper semicontinuous.

Proposition 2.1.2 *If φ is an upper semicontinuous function on a compact set K, then there exists $x \in K$ such that $\varphi(x) = \sup_K \varphi < +\infty$.*

Proof: The sets $\varphi^{-1}([-\infty, a[)$, for $a \in \mathbb{R}$, give an open covering of K which has a finite subcovering since K is compact and hence is $\sup_K \varphi < +\infty$. Assume $\varphi(x) < \sup_K \varphi$ for every $x \in K$, then there is a sequence x_j of points in K such that $\varphi(x_j) \nearrow \sup_K \varphi$ and $\varphi(x_1) < \varphi(x_2) < \ldots < \sup_K \varphi$. The sets $\varphi^{-1}([-\infty, \varphi(x_j)[)$ then give a covering of K which does not have a finite subcovering, which is a contradiction. $\square$

Definition 2.1.3 Let $Y \subset X$ be a nonempty set and $\varphi : Y \to [-\infty, +\infty[$ a function which is locally bounded around each point in the closure of Y, $\overline{Y}$. Then we define the *upper semicontinuous regularization* φ^* of φ by

$$\varphi^*(x) = \limsup_{Y \ni y \to x} \varphi(y).$$

The function φ^* is upper semicontinuous on $\overline{Y}$, $\varphi \leq \varphi^*$ on Y, and it is the smallest upper semicontinuous function which is larger than φ, i.e. if $\tilde{\varphi}$ is an upper semicontinuous function such that $\varphi \leq \tilde{\varphi} \leq \varphi^*$, then $\tilde{\varphi} = \varphi^*$.

An important fact about upper semicontinuous function is that they can be approximated by continuous function from above and the limit of a decreasing sequence of upper semicontinuous function is upper semicontinuous. We therefore state the following propositions, for the proofs of them see [21, Lemma 2.3.2 and Proposition 2.3.3].

Proposition 2.1.4 *Let X be a manifold and $\varphi_\alpha, \alpha \in A$ a family of upper semicontinuous functions on X. Then $\varphi = \inf_{\alpha \in A} \varphi_\alpha$ is upper semicontinuous* *and furthermore there is a countable subset $A' \subset A$ such that $\varphi = \inf_{\alpha \in A'} \varphi_\alpha$.*

Proposition 2.1.5 *If $\varphi : X \to [-\infty, +\infty[$ is an upper semicontinuous function on a compact metric space X then there exists a sequence φ_j of continuous functions on X such that for every $x \in X$,*

$$\lim_{j \to \infty} \varphi_j(x) \searrow \varphi(x).$$

2.1.2 Plurisubharmonic functions

This section will contain the most important properties of plurisubharmonic function we need for our studies of disc functionals. These results will be mostly stated without proofs since they can be considered classical. For a more detailed survey of pluripotential theory see Klimek [21]. For a more general study of complex analysis of several variables see Krantz [24] and Hörmander [17] . For complex analysis on manifolds see Fritzsche and Grauert [12], Demailly [7] and Huybrechts [19].

Recall that a real valued function h on U, where U is an open subset of $\mathbb{C}$, is *harmonic* if $\Delta u = 0$, or equivalently $\partial\overline{\partial} u = 0$, where ∂ and $\overline{\partial}$ are the differential operators

$$\partial = \frac{\partial}{\partial z} dz, \qquad \overline{\partial} = \frac{\partial}{\partial \overline{z}} d\overline{z}.$$

Let $D_r(a)$ denote the open disc in $\mathbb{C}$ with center a and radius r. We let denote μ the surface measure on the boundary of $D_r(a)$ and the Lebesgue measure will be denoted by λ.

Proposition 2.1.6 *The following is equivalent for a continuous function h on an open set $U \subset \mathbb{C}$.*

(i) h is harmonic.

(ii) If $\overline{D_r(a)} \subset U$ then

$$h(a) = \frac{1}{\mu(\partial D_r(a))} \int_{\partial D_r(a)} h(x)\, d\mu(x).$$

(iii) If $\overline{D_r(a)} \subset U$ *then*

$$h(a) = \frac{1}{\lambda(D_r(a))} \int_{D_r(a)} h(x)\, d\lambda(x).$$

Definition 2.1.7 An upper semicontinuous function u on an open set $U \subset \mathbb{C}$ is *subharmonic* if it satisfies the following condition: For every relatively compact open subset $G \subset U$ and for every continuous function h on $\overline{G}$ which is harmonic on G

$$u|_{\partial G} \le h|_{\partial G} \qquad \text{implies} \qquad u \le h \text{ on } G.$$

From Proposition 2.1.6 we get three equivalent characterizations of subharmonicity.

Proposition 2.1.8 *The following is equivalent for an upper semicontinuous function* u *on an open set* $U \subset \mathbb{C}$*.*

(i) u *is subharmonic.*

(ii) If $\overline{D_r(a)} \subset U$ *then*

$$u(a) \le \frac{1}{\mu(\partial D_r(a))} \int_{\partial D_r(a)} u(x)\, d\mu(x).$$

(iii) If $\overline{D_r(a)} \subset U$ *then*

$$u(a) \le \frac{1}{\lambda(D_r(a))} \int_{D_r(a)} u(x)\, d\lambda(x).$$

The following formula, which is known as the Riesz representation formula [18, eq. (3.1.8)'], will play a important role both in Chapter 3 and Chapter 4. If u is a subharmonic function on the unit disc $\mathbb{D}$ and continuous on its closure $\overline{\mathbb{D}} = \mathbb{D} \cup \mathbb{T}$ then

$$u(a) = \frac{1}{2\pi} \int_{\mathbb{D}} \log \left| \frac{a - x}{1 - a\overline{x}} \right| \Delta u(x) d\lambda + \int_{\mathbb{T}} \frac{1 - |a|^2}{|x - a|^2} u(x)\, d\sigma(x). \tag{2.2}$$

In particular, when $a = 0$,

$$u(0) = \frac{1}{2\pi} \int_{\mathbb{D}} \log|x| \, \Delta u(x) d\lambda + \int_{\mathbb{T}} u(x) \, d\sigma(x). \tag{2.3}$$

Definition 2.1.9 An upper semicontinuous function u on an open set $X \subset \mathbb{C}^n$ is *plurisubharmonic* if it is subharmonic along every complex line, that is for every $a, b \in \mathbb{C}^n$, the function

$$z \mapsto u(a + bz)$$

is subharmonic on $\{z \in \mathbb{C}; a + bz \in X\}$. We let $\mathcal{PSH}(X)$ denote the family of plurisubharmonic functions on X which are not identically $-\infty$ on any connected component of X.

A function u such that $-u$ is plurisubharmonic is called *plurisuperharmonic*.

Proposition 2.1.10 *Assume X and Y are open sets in $\mathbb{C}^n$ and $\mathbb{C}^m$ respectively, and $f : X \to Y$ is a holomorphic mapping. If $u \in \mathcal{PSH}(Y)$ then $u \circ f \in \mathcal{PSH}(X)$.*

This implies that plurisubharmonicity is invariant under biholomorphic mappings and that we can define plurisubharmonic functions on complex manifolds as follows.

Definition 2.1.11 Let X be a complex manifold. An upper semicontinuous function u on X is plurisubharmonic if the function $u \circ \Phi^{-1}$ is plurisubharmonic on $\Phi(U)$ for every local coordinates $\Phi : U \to \mathbb{C}^n$, $U \subset X$.

Proposition 2.1.12 *The following are equivalent for an upper semicontinuous function u on a complex manifold X.*

(i) *u is in $\mathcal{PSH}(X)$.*

(ii) *$u \circ f$ is subharmonic on $\mathbb{D}$ for every $f \in \mathcal{A}_X$.*

(iii)

$$u(f(0)) \leq \int_{\mathbb{T}} u \circ f \, d\sigma$$

for every $f \in \mathcal{A}_X$.

Here, as mentioned before, $\mathcal{A}_X$ is the set of all closed analytic discs in X and σ is the arc length measure on the unit circle $\mathbb{T}$ normalized to 1.

Proof: Property *(ii)* follows from *(i)* by Proposition 2.1.10. If u satisfies *(ii)* then in particular $z \mapsto u(\Phi^{-1}(a+bz))$ is subharmonic for every $a, b \in \mathbb{C}^n$ and every local coordinates Φ, that is u is plurisubharmonic. We have therefore established that *(i)* and *(ii)* are equivalent. Finally, it is clear from Proposition 2.1.8 that *(ii)* and *(iii)* are equivalent. □

Since we will be working with discs and disc functionals, then condition *(iii)* in the proposition above will be most useful to us. It is generally referred to as the *subaverage property* of plurisubharmonic functions.

Plurisubharmonicity can also be defined using differential operators, similar to the definition of harmonic functions using the Laplacian.

We let d and d^c denote the real differential operators

$$d = \partial + \overline{\partial} \qquad \text{and} \qquad d^c = i(\overline{\partial} - \partial),$$

where

$$\partial = \sum_{j=1}^{n} \frac{\partial}{\partial z_j} dz_j \qquad \text{and} \qquad \overline{\partial} = \sum_{j=1}^{n} \frac{\partial}{\partial \overline{z}_j} d\overline{z}_j. \tag{2.4}$$

Hence, in $\mathbb{C}$ we have $dd^c u = \Delta u dV$ where dV is the standard area form.

If a plurisubharmonic function u is not identically $-\infty$, that is $u \in \mathcal{PSH}(X)$, then u is in $L^1_{\text{loc}}(X)$ and does therefore define a distribution. The $(1,1)$-current $dd^c u$ is then also well defined on X.

We can then characterize plurisubharmonicity using the differential operator $dd^c = 2i\partial\overline{\partial}$. This in particular shows that plurisubharmonicity is a local property.

Proposition 2.1.13 *If $u \in \mathcal{PSH}(X)$ then $dd^c u \geq 0$ in a weak sense.*

Conversely, if u is a locally integrable function on X such that $dd^c u \geq 0$ in a weak sense, then there is a plurisubharmonic function $\tilde{u}$ on X which is equal to u almost everywhere.

For the proof we refer the reader to [21, Theorem 2.9.11].

Note that $dd^c u \geq 0$ is equivalent to the Levi form of u being positive, that is

$$\sum_{j,k=1}^{n} \frac{\partial u}{\partial z_j \partial \overline{z}_k} w_j \overline{w}_k \geq 0,$$

for every $w \in \mathbb{C}^n$. For more information about positive currents and positive forms see [21, Chapter 3.2] and [7, Chapter III.1].

2.2 Quasiplurisubharmonic functions

Plurisubharmonic functions satisfy the maximum principle. This implies that any plurisubharmonic function on a compact complex manifold, for example the complex projective space $\mathbb{P}^n$, is constant. This fact motivates the definition and studying of quasiplurisubharmonic functions. Recall that a plurisubharmonic function u satisfies $dd^c u \geq 0$ in a weak sense. Quasiplurisubharmonic functions on the other hand are such that $dd^c u \geq -\omega$, where ω is a closed $(1,1)$-current. That is $\omega + dd^c u$ is a positive $(1,1)$-current, and usually the family of such currents on a given manifold is large, in particular if the manifold is Kähler and ω is a Kähler form. But the main application of quasiplurisubharmonic functions is for studying metrics on Kähler manifolds [11, 14, 23]. They have also been used to define and study the relative extremal functions [13], global extremal functions [4, 13] and the Green functions [6] on compact manifolds. Furthermore, quasiplurisubharmonic functions have been used to study the projective hull in $\mathbb{P}^n$ [15], which is analogous to the polynomial hull in $\mathbb{C}^n$. It is therefore reasonable to wonder if the disc formula for quasiplurisubharmonic functions presented in Chapter 4 can be used to characterize the projective hull, similar to the characterization of the polynomial hull given by Poletsky's formula (see Section 4.4).

For a detailed survey of quasiplurisubharmonic functions on compact Kähler manifolds see [13].

This section will contain the necessary definition and properties of quasiplurisubharmonic functions we will need in Chapter 4. First a few words about notation. We assume X is a complex manifold of dimension n, $\mathcal{A}_X$ will then be the family of all *closed analytic discs* in X, that is, all holomorphic mappings from a neighbourhood of the closed unit disc $\overline{\mathbb{D}}$ into X. The boundary of the unit disc $\mathbb{D}$ will be denoted by $\mathbb{T}$ and σ will be the arc length measure on $\mathbb{T}$ normalized to 1. Furthermore, $D_r = \{z \in \mathbb{C}; |z| < r\}$ will be the disc centered at zero with radius r.

We start by noting that if ω is a closed, positive $(1,1)$-current on a manifold X, that is a continuous linear functional acting on $(n-1, n-1)$-forms, then locally we have a potential for ω. This means that for every point x there is a neighbourhood U of x and a plurisubharmonic function $\psi : U \to \mathbb{R} \cup \{-\infty\}$ such that $dd^c\psi = \omega$. This allows us to work with things locally in a similar fashion as in the classical case, $\omega = 0$. We will furthermore see that when there is a global potential, that is, when ψ can be defined on all of X, then most of the questions about ω-plurisubharmonic functions turn into questions involving plurisubharmonic functions.

Proposition 2.2.1 *Let X be a complex manifold with the second de Rham cohomology $H^2(X) = 0$, and the Dolbeault cohomology $H^{(0,1)}(X) = 0$. Then every closed positive $(1,1)$-current ω has a global plurisubharmonic potential $\psi : X \to \mathbb{R} \cup \{-\infty\}$, such that $dd^c\psi = \omega$.*

Proof: Since ω is a positive current it is real, and from the fact $H^2(X) = 0$ it follows that there is a real current η such that $d\eta = \omega$. Now write $\eta = \eta^{1,0} + \eta^{0,1}$, where $\eta^{1,0} \in \Lambda'_{1,0}(X, \mathbb{C})$ and $\eta^{0,1} \in \Lambda'_{0,1}(X, \mathbb{C})$. Note that $\eta^{0,1} = \overline{\eta^{1,0}}$ since η is real. We see, by counting degrees, that $\overline{\partial}\eta^{0,1} = \omega^{0,2} = 0$. Then since $H^{(0,1)}(X) = 0$, there is a distribution μ on X such that $\overline{\partial}\mu = \eta^{0,1}$. Hence

$$\eta = \overline{\overline{\partial}\mu} + \overline{\partial}\mu = \partial\overline{\mu} + \overline{\partial}\mu.$$

If we set $\psi = (\mu - \overline{\mu})/2i$, then

$$\omega = d\eta = d(\partial\overline{\mu} + \overline{\partial}\mu) = (\partial + \overline{\partial})(\partial\overline{\mu} + \overline{\partial}\mu) = \partial\overline{\partial}(\mu - \overline{\mu}) = dd^c\psi.$$

Finally, by modifying ψ on a negligible set we may assume it is plurisubharmonic function since ω is positive. $\square$

If we apply this locally to a coordinate system biholomorphic to a polydisc and use the Poincaré lemma we get the following.

Corollary 2.2.2 *For a closed, positive* $(1,1)$*-current* ω *there is locally a plurisubharmonic potential* ψ *such that* $dd^c\psi = \omega$*.*

Note that the difference of two potentials for ω is a pluriharmonic function, thus C^∞. This implies that if ψ and ψ' are two local potentials of ω defined on sets U and U', respectively, then for $x \in U \cap U'$, $\psi(x) = -\infty$ if and only if $\psi'(x) = -\infty$. We therefore make the following definition.

Definition 2.2.3 The *singular set* $\operatorname{sing}(\omega)$ of ω is defined as the union of all $\psi^{-1}(\{-\infty\})$ for all local potentials ψ of ω.

In the following we assume $\omega = \omega_1 - \omega_2$, where ω_1 and ω_2 are closed, positive $(1,1)$-currents. We have plurisubharmonic local potentials ψ_1 and ψ_2 for ω_1 and ω_2, respectively, and we write the potential for ω as

$$\psi(x) = \begin{cases} \psi_1(x) - \psi_2(x) & \text{if } x \notin \operatorname{sing}(\omega_1) \cap \operatorname{sing}(\omega_2) \\ \limsup\limits_{y \to x} \psi_1(y) - \psi_2(y) & \text{if } x \in \operatorname{sing}(\omega_1) \cap \operatorname{sing}(\omega_2) \end{cases}$$

and the singular set of ω is defined as $\operatorname{sing}(\omega) = \operatorname{sing}(\omega_1) \cup \operatorname{sing}(\omega_2)$.

The reason for the restriction to $\omega = \omega_1 - \omega_2$, which is the difference of two positive, closed $(1,1)$-currents, is the following. Our methods rely on the existence of local potentials which are well defined plurisubharmonic functions, not only distributions, for we need to apply Riesz representation theorem to this potential composed with an analytic disc. With $\omega = \omega_1 - \omega_2$ we can work with the local potentials of ω_1 and ω_2 separately, and they are are given by plurisubharmonic functions.

Definition 2.2.4 A function $u : X \to [-\infty, +\infty]$ is called ω*-upper semicontinuous* if for every $a \in \operatorname{sing}(\omega)$,

$$\limsup_{X \setminus \operatorname{sing}(\omega) \ni z \to a} u(z) = u(a)$$

and for each local potential ψ of ω, defined on an open subset U of X, $u+\psi$ is upper semicontinuous on $U \setminus \operatorname{sing}(\omega)$ and locally bounded above around each point of $\operatorname{sing}(\omega)$.

This is equivalent to saying that $\limsup_{\operatorname{sing}(\omega)\not\ni z\to a} u(z) = u(a)$ for every a in $\operatorname{sing}(\omega)$ and that $u+\psi$ extends as

$$\limsup_{\operatorname{sing}(\omega)\not\ni z\to a} (u+\psi)(z), \qquad \text{for } a \in \operatorname{sing}(\omega)$$

to an upper semicontinuous function on U with values in $\mathbb{R} \cup \{-\infty\}$. This extension will be denoted by $(u+\psi)^\dagger$. Note that $(u+\psi)^\dagger$ is not the upper semicontinuous regularization $(u+\psi)^*$ of the function $u+\psi$, but just a way to define the sum on $\operatorname{sing}(\omega)$ where possibly one of the terms is equal to $+\infty$ and the other might be $-\infty$.

Note that the question whether $(u+\psi)^\dagger$ is upper semicontinuous does not depend at all on the values of u at $\operatorname{sing}(\omega)$. The reason for the conditions on u at $\operatorname{sing}(\omega)$ is to ensure that u is Borel measurable and to uniquely determine the function from its values outside of $\operatorname{sing}(\omega)$.

It is easy to see that u is Borel measurable from the fact that $u = (u+\psi) - \psi$ is the difference of two Borel measurable functions on $X \setminus \operatorname{sing}(\omega)$ and that u restricted to the Borel set $\operatorname{sing}(\omega)$ is the increasing limit of upper semicontinuous functions. Hence it is Borel measurable.

Definition 2.2.5 An ω-upper semicontinuous function $u : X \to [-\infty, +\infty]$ is called *ω-plurisubharmonic* if $(u+\psi)^\dagger$ is plurisubharmonic on U for every local potential ψ of ω defined on an open subset U of X. We let $\mathcal{PSH}(X,\omega)$ denote the set of all ω-plurisubharmonic functions on X which are not identically $-\infty$ on any connected component of X.

Similarly we could say that u is in $\mathcal{PSH}(X,\omega)$ if it is ω-upper semicontinuous and $dd^c u \geq -\omega$ in a weak sense. Conversely, if u is a locally integrable function on X such that $dd^c u \geq -\omega$ then there is a function $\tilde{u} \in \mathcal{PSH}(X,\omega)$ such that $\tilde{u} = u$ almost everywhere.

The most important example of ω-plurisubharmonic functions is when $X = \mathbb{P}^n$ and ω is the Fubini-Study Kähler form. It turns out [13, Example 2.2] that these functions are in 1-to-1 correspondence with the Lelong class

$$\mathcal{L} = \{u \in \mathcal{PSH}(\mathbb{C}^n); u \leq \frac{1}{2}\log(1+|z|)^2 + c\},$$

which is used in classical potential theory to define the global extremal function [21, Chapter 5] and to characterize the polynomial hull of a set. It should be noted that the global extremal function has a disc formula [27,28,31], but this formula is of different nature from those studied here. In particular there is no easy way to derive the formulas for the global extremal function from our formulas, as we do for the relative extremal function in Section 4.4.

It turns out that size of the $\mathcal{PSH}(X,\omega)$ is independent of the representative of the cohomology class of ω, and when we look at another representative from the cohomology class then all we do is translate the set of ω-plurisubharmonic functions.

Proposition 2.2.6 *Assume both ω and ω' are the difference of two positive, closed $(1,1)$-currents. If the current $\omega - \omega'$ has a global potential $\chi = \chi_1 - \chi_2 : X \to [-\infty, +\infty]$, where χ_1 and χ_2 are plurisubharmonic functions, then for every $u' \in \mathcal{PSH}(X,\omega')$ the function u defined by $u(x) = u'(x) - \chi(x)$ for $x \notin \operatorname{sing}(\omega') \cup \operatorname{sing}(\omega)$ extends to a unique function in $\mathcal{PSH}(X,\omega)$ and the map $\mathcal{PSH}(X,\omega') \to \mathcal{PSH}(X,\omega)$, $u' \mapsto u$ is bijective.*

Proof: Let $\psi' = \psi_1' - \psi_2'$ be a local potential of ω'. The functions $\psi_1 = \psi_1' + \chi_1$ and $\psi_2 = \psi_2' + \chi_2$ are well defined as the sums of plurisubharmonic functions. Then $\psi = \psi_1 - \psi_2$, extended over $\operatorname{sing}(\omega)$ as before, is a local potential of ω since $\omega = \omega' + dd^c\chi$.

Take $u' \in \mathcal{PSH}(X,\omega')$ and define a function u on X by

$$u(x) = \begin{cases} (u' + \psi')^\dagger(x) - \psi(x) & \text{for } x \in X \setminus \operatorname{sing}(\omega) \\ \limsup\limits_{\operatorname{sing}(\omega) \not\ni y \to x} (u' + \psi')^\dagger(y) - \psi(y) & \text{for } x \in \operatorname{sing}(\omega) \end{cases}$$

This definition is independent of ψ' since any other local potential of ω' differs from ψ' by a continuous pluriharmonic function which cancels out in the definition of u, due to the definition of ψ.

Then $u+\psi = (u'+\psi')^\dagger$ on $X \setminus \text{sing}(\omega)$ where the sum is well defined, since neither u nor ψ are $+\infty$ there. The right hand side is upper semicontinuous so $u+\psi$ is upper semicontinuous on $X \setminus \text{sing}(\omega)$. But $(u'+\psi')^\dagger$ is upper semicontinuous on X so the extension $(u+\psi)^\dagger$ also satisfies $(u+\psi)^\dagger = (u'+\psi')^\dagger$ and is therefore plurisubharmonic since $u' \in \mathcal{PSH}(X,\omega')$. This shows that $u \in \mathcal{PSH}(X,\omega)$.

This map from $\mathcal{PSH}(X,\omega')$ to $\mathcal{PSH}(X,\omega)$ is injective because $u = u' - \chi$ almost everywhere and the extension over $\text{sing}(\omega) \cup \text{sing}(\omega')$ is unique.

By changing the roles of ω and ω' we get an injection in the opposite direction which maps $v \in \mathcal{PSH}(X,\omega)$ to a function $v' \in \mathcal{PSH}(X,\omega)$ defined as $v' = v + \chi$ outside of $\text{sing}(\omega) \cup \text{sing}(\omega')$. These maps are clearly the inverses of each other because if we apply the composition of them to the function $u' \in \mathcal{PSH}(X,\omega')$ we get an ω-upper semicontinuous function which satisfies $(u'-\chi)+\chi = u'$ outside of $\text{sing}(\omega) \cup \text{sing}(\omega')$. Since this function is equal to u' almost everywhere they are the same, which shows that the composition is the identity map. $\square$

Proposition 2.2.7 *If $\varphi\colon X \to [-\infty,+\infty]$ is an ω-upper semicontinuous function we define $\mathcal{F}_{\omega,\varphi} = \{u \in \mathcal{PSH}(X,\omega); u \le \varphi\}$. If $\mathcal{F}_{\omega,\varphi} \ne \emptyset$ then $\sup \mathcal{F}_{\omega,\varphi} \in \mathcal{PSH}(X,\omega)$, and consequently $\sup \mathcal{F}_{\omega,\varphi} \in \mathcal{F}_{\omega,\varphi}$.*

Proof: Assume ψ is a local potential of ω defined on $U \subset X$. For $u \in \mathcal{F}_{\omega,\varphi}$, the function $(u+\psi)^\dagger$ is a plurisubharmonic function on U which satisfies $(u+\psi)^\dagger \le (\varphi+\psi)^\dagger$. The supremum of the family $\{(u+\psi)^\dagger; u \in \mathcal{F}_{\omega,\psi}\} \subset \mathcal{PSH}(U)$ therefore defines a plurisubharmonic function on U,

$$F_\psi(x) = (\sup\{(u+\psi)^\dagger(x); u \in \mathcal{F}_{\omega,\varphi}\})^*,$$

with $F_\psi \le (\varphi+\psi)^\dagger$. We want to emphasize the difference between $\dagger$ and $*$. The extension of the function $u+\psi$ over $\text{sing}(\omega)$, where the sum is possibly not defined, is denoted by $(u+\psi)^\dagger$ but $*$ is used to denote the upper semicontinuous regularization of a function.

Since the difference of two local potentials is a continuous function, the function $(\sup\{(u+\psi)^\dagger; u \in \mathcal{F}_{\omega,\varphi}\})^* - \psi$ is independent of ψ. This means that

$$S = F_\psi - \psi, \quad \text{on } U \setminus \operatorname{sing}(\omega),$$

extended over $\operatorname{sing}(\omega)$ using lim sup, is a well-defined function on X.

Clearly S is ω-plurisubharmonic since $(S+\psi)^\dagger = F_\psi$ which is plurisubharmonic, and S satisfies

$$\sup \mathcal{F}_{\omega,\varphi} + \psi \leq F_\psi = S + \psi \leq \varphi + \psi, \quad \text{on } U \setminus \operatorname{sing}(\omega).$$

This implies

$$\sup \mathcal{F}_{\omega,\varphi} \leq S \leq \varphi, \tag{2.5}$$

on $U \setminus \operatorname{sing}(\omega)$. The latter inequality holds also on $\operatorname{sing}(\omega)$ because of the definition of S at $\operatorname{sing}(\omega)$ and the ω-upper semicontinuity of φ.

Furthermore, if $u \in \mathcal{F}_{\omega,\varphi}$ and $a \in \operatorname{sing}(\omega)$, then

$$u(a) = \limsup_{x\to a} u(x) \leq \limsup_{x\to a}[\sup \mathcal{F}_{\omega,\varphi}(x)] \leq \limsup_{x\to a} S(x) = S(a).$$

Taking supremum over u then shows that the first inequality in (2.5) above holds also on $\operatorname{sing}(\omega)$, that is $\sup \mathcal{F}_{\omega,\varphi} \leq S$. But $S \in \mathcal{F}_{\omega,\varphi}$ by the latter inequality and therefore $S \leq \sup \mathcal{F}_{\omega,\varphi}$. This shows that $\sup \mathcal{F}_{\omega,\varphi} = S \in \mathcal{PSH}(X,\omega)$. □

Proposition 2.2.8 *If* $u, v \in \mathcal{PSH}(X,\omega)$ *then* $\max\{u,v\} \in \mathcal{PSH}(X,\omega)$.

Proof: For any local potential ψ we know that

$$\max\{u,v\} + \psi = \max\{u+\psi, v+\psi\}$$

is upper semicontinuous outside of $\operatorname{sing}(\omega)$ and locally bounded above around each point of $\operatorname{sing}(\omega)$. Therefore, the extension $(\max\{u,v\}+\psi)^\dagger$ is equal to $\max\{(u+\psi)^\dagger, (v+\psi)^\dagger\}$ which is plurisubharmonic, hence $\max\{u,v\}$ is ω-plurisubharmonic. □

Our approach in Chapter 4 depends on the fact that we can we define the pullback of currents by holomorphic maps. This we can do in two very different cases, first if the map is a submersion and secondly if it is an analytic disc not lying in $\text{sing}(\omega)$.

If $\Phi : Y \to X$ is a submersion and ω is a current on X then we can define the inverse image $\Phi^*\omega$ of ω by its action on forms,

$$\langle \Phi^*\omega, \tau \rangle = \langle \omega, \Phi_*\tau \rangle \tag{2.6}$$

where $\Phi_*\tau$ is the direct image of the form τ. For more details see Demailly [7, 2.C.2 Ch. I.]. We use this pullback in Section 4.3 to move the problem from the manifold X to a manifold where $\Phi^*\omega$ has a global potential, see Lemma 4.3.1.

If f is an analytic disc then it is important for us to be able to define the pullback of ω by a f to include ω in the disc functional.

Definition 2.2.9 For $f \in \mathcal{A}_X$ such that $f(0) \notin \text{sing}(\omega)$ we define the *pullback of* ω *by* f, denoted $f^*\omega$, with

$$dd^c(\psi \circ f),$$

where ψ is any local potential of ω. Since the difference of two local potentials is pluriharmonic, this definition is independent of the choice of ψ, and it gives a definition of $f^*\omega$ on all of $\mathbb{D}$.

Note that $\psi \circ f$ is not identically $\pm\infty$ since $f(0) \notin \text{sing}(\omega)$.

If $\omega = \omega_1 - \omega_2$, then we could as well define the positive currents $f^*\omega_1$ and $f^*\omega_2$, using ψ_1 and ψ_2 respectively, and then define $f^*\omega = f^*\omega_1 - f^*\omega_2$. This gives the same result since $\psi \circ f = \psi_1 \circ f - \psi_2 \circ f$ almost everywhere.

It is also possible to look at $f^*\omega$ as a real measure on $\mathbb{D}$, and we let $R_{f^*\omega}$ be its Riesz potential,

$$R_{f^*\omega}(z) = \int_{\mathbb{D}} G_{\mathbb{D}}(z, \cdot)\, d(f^*\omega), \tag{2.7}$$

where $G_{\mathbb{D}}$ is the Green function for the unit disc, $G_{\mathbb{D}}(z, w) = \frac{1}{2\pi} \log \frac{|z-w|}{|1-z\overline{w}|}$.

Since f is a closed analytic disc not lying in $\operatorname{sing}(\omega)$ it follows that $f^*\omega$ is a Radon measure in a neighbourhood of the unit disc, therefore with finite mass on $\mathbb{D}$ and not identically $\pm\infty$.

It is important to note that if we have a local potential ψ defined in a neighbourhood of $\overline{f(\mathbb{D})}$, then the Riesz representation formula (2.3) gives

$$\psi(f(0)) = R_{f^*\omega}(0) + \int_{\mathbb{T}} \psi \circ f \, d\sigma. \tag{2.8}$$

Similar to plurisubharmonic functions, see Proposition 2.1.12, quasiplurisubharmonicity can be characterized by analytic discs.

Proposition 2.2.10 *The following are equivalent for a function u on X.*

(i) *u is in $\mathcal{PSH}(X,\omega)$.*

(ii) *u is ω-upper semicontinuous and f^*u is $f^*\omega$-subharmonic on $\mathbb{D}$ for all $f \in \mathcal{A}_X$ such that $f(\mathbb{D}) \not\subset \operatorname{sing}(\omega)$.*

(iii) *$u \circ f + R_{f^*\omega}$ is subharmonic on $\mathbb{D}$ for every $f \in \mathcal{A}_X$ such that $f(\mathbb{D}) \not\subset \operatorname{sing}(\omega)$.*

Proof: Assume $u \in \mathcal{PSH}(X,\omega)$, take $f \in \mathcal{A}_X$, $h(\mathbb{D}) \not\subseteq \operatorname{sing}(\omega)$, and $a \in \mathbb{D}$. Let ψ be a local potential for ω defined in a neighbourhood U of $f(a)$. Note that $(u+\psi)^\dagger \circ f = (u \circ f + \psi \circ f)^\dagger$, that is, the extension of $(u+\psi)\circ f$ over $\operatorname{sing}(f^*\omega)$ is the same as the extension of $u+\psi$ over $\operatorname{sing}(\omega)$ pulled back by f, for both functions are subharmonic and equal almost everywhere, thus the same. Since $(u+\psi)^\dagger \in \mathcal{PSH}(U)$ and $(u+\psi)^\dagger \circ f = (u \circ f + \psi \circ f)^\dagger$ is subharmonic in a neighbourhood of a we see that $u \circ f$ is $f^*\omega$-subharmonic.

Assume now that *(ii)* holds and let $\psi \in \mathcal{PSH}(U)$ be a local potential for ω. Then $(u+\psi)^\dagger$ is upper semicontinuous, and *(ii)* implies that $(u+\psi)^\dagger \circ f$ is subharmonic on $\mathbb{D}$ for every $f \in \mathcal{A}_U$. Hence $(u+\psi)^\dagger \in \mathcal{PSH}(U)$ and we have *(i)*.

It is clear that *(ii)* and *(iii)* are equivalent since $R_{f^*\omega}$ is a global potential for $f^*\omega$ on $\mathbb{D}$. □

From the Definition 2.2.5 we see that ω-plurisubharmonicity, like plurisubharmonicity, is a local property. Therefore it is sufficient in condition *(ii)* and *(iii)* to look at discs $f \in \mathcal{A}_U$ where U is a neighbourhood of a given point.

2.3 Holomorphic discs, disc functionals and envelopes

If H is a disc functional defined for discs $f \in \mathcal{A}_X$, with $f(\mathbb{D}) \not\subset \operatorname{sing}(\omega)$, then we define the *envelope of* H, denoted EH, on $X \setminus \operatorname{sing}(\omega)$ by

$$EH(x) = \inf\{H(f); f \in \mathcal{A}_X, f(0) = x\}.$$

We then extend EH to a function on X by

$$EH(x) = \limsup_{\operatorname{sing}(\omega) \not\ni y \to x} EH(y), \qquad \text{for } x \in \operatorname{sing}(\omega), \tag{2.9}$$

in accordance with Definition 2.2.4 of ω-upper semicontinuous functions.

If $\Phi : Y \to X$ is a holomorphic function and H a disc functional on $\mathcal{A}_X$, then we can define the pullback $\Phi^* H$ of H by $\Phi^* H(f) = H(\Phi \circ f)$, for $f \in \mathcal{A}_Y$. Every disc $f \in \mathcal{A}_Y$ gives a push-forward $\Phi \circ f \in A_X$ and it is easy to see that

$$\Phi^* EH \leq E\Phi^* H, \tag{2.10}$$

where $\Phi^* EH = EH \circ \Phi$ is the pullback of EH. We have an equality in (2.10) if every disc $f \in \mathcal{A}_X$ has a lifting $\tilde{f} \in A_Y$, $f = \Phi \circ \tilde{f}$.

The most important example of a disc functional in the classical theory when $\omega = 0$, is the *Poisson disc functional* H_φ which is defined by $f \mapsto \int_{\mathbb{T}} \varphi \circ f \, d\sigma$, where φ is a locally integrable function on X. When we study the Poisson disc functional in Chapter 3 we will need the following.

Lemma 2.3.1 *Let φ be an upper semicontinuous function on a complex manifold X and $F \in (D_r \times Y, X)$, where $r > 1$ and Y is a complex manifold, then $y \mapsto H_\varphi(F(\cdot, y))$ is upper semicontinuous. Furthermore, if φ is plurisubharmonic then this function is also plurisubharmonic.*

Proof: Fix a point $x_0 \in Y$ and a compact neighbourhood V of x_0. The function

$\varphi \circ F$ is upper semicontinuous and therefore bounded above on $\mathbb{T} \times V$ so by Fatou's lemma

$$\begin{aligned} \limsup_{x \to x_0} H_\varphi(F(\cdot, x)) &\leq \int_{\mathbb{T}} \limsup_{x \to x_0} \varphi(F(t, x))\, d\sigma(t) \\ = \int_{\mathbb{T}} \varphi(F(t, x_0))\, d\sigma(t) &= H_\varphi(F(\cdot, x_0)), \end{aligned}$$

which shows that the function is upper semicontinuous.

Assume φ is plurisubharmonic and let $h \in A_Y$. Then

$$\begin{aligned} \int_{\mathbb{T}} H_\varphi(F(\cdot, h(s)))\, d\sigma(s) &= \int_{\mathbb{T}} \int_{\mathbb{T}} \varphi(F(t, h(s)))\, d\sigma(t)\, d\sigma(s) \\ &= \int_{\mathbb{T}} \int_{\mathbb{T}} \varphi(F(t, h(s)))\, d\sigma(s)\, d\sigma(t) \\ &\geq \int_{\mathbb{T}} \varphi(F(t, h(0)))\, d\sigma(t) \\ &= H_\varphi(F(\cdot, h(0)), \end{aligned}$$

because for fixed t, the function $s \mapsto \varphi(F(t, h(s)))$ is subharmonic. □

3

Disc formulas for plurisubharmonic functions

Let φ be a function on a complex manifold X with values in $[-\infty, +\infty]$. The *Poisson disc functional* for φ, denoted H_φ, is defined as

$$H_\varphi(f) = \int_{\mathbb{T}} \varphi \circ f \, d\sigma,$$

for $f \in \mathcal{A}_X$, where $\mathcal{A}_X$ is the set of all closed analytic discs in X. The *envelope of* H_φ is then the function $EH_\varphi : X \to [-\infty, +\infty]$, given by

$$EH_\varphi(x) = \inf\{H_\varphi(f); f \in \mathcal{A}_X, f(0) = x\}.$$

Now, if u is a plurisubharmonic function on X that satisfies $u \leq \varphi$, then by the subaverage property of plurisubharmonic functions (property *(iii)* in Proposition 2.1.12) we see that for $f \in \mathcal{A}_X$ with $f(0) = x$, we have

$$u(x) \leq \int_{\mathbb{T}} u \circ f \, d\sigma \leq \int_{\mathbb{T}} \varphi \circ f \, d\sigma = H_\varphi(f).$$

If we take the supremum on the left hand side over all $u \in \mathcal{PSH}(X)$ such that $u \leq \varphi$, and infimum on the right hand side over all $f \in \mathcal{A}_X$ with $f(0) = x$, we get the fundamental inequality

$$\sup\{u(x); u \in \mathcal{PSH}(X), u \le \varphi\} \le \inf\{H_\varphi(f); f \in \mathcal{A}_X, f(0) = x\}. \qquad (3.1)$$

The goal is to show that this is actually an equality,

$$\sup\{u(x); u \in \mathcal{PSH}(X), u \le \varphi\} = \inf\{H_\varphi(f); f \in \mathcal{A}_X, f(0) = x\}. \qquad (3.2)$$

That is done by showing that the function on the right hand side, EH_φ, is in the family on the left hand side. Then EH_φ is obviously not greater than the left hand side and we have an equality. The hard part is to prove the plurisubharmonicity of EH_φ, or equivalently that it satisfies the subaverage property. It is easier to see that $EH_\varphi \le \varphi$, because if $f_x \in \mathcal{A}_X$ is the constant disc which maps everything to $x \in X$ then

$$EH_\varphi(x) \le H_\varphi(f_x) = \int_{\mathbb{T}} \varphi(x)\, d\sigma(t) = \varphi(x). \qquad (3.3)$$

We will also refer to equation (3.2) by using its shorter form

$$\sup \mathcal{F}_\varphi = EH_\varphi,$$

where $\mathcal{F}_\varphi = \{u \in \mathcal{PSH}(X); u \le \varphi\}$ denotes the family of plurisubharmonic functions we are looking at.

We start by looking at the case when φ is upper semicontinuous (Section 3.1). It turns out that it is enough to prove equality (3.2) for continuous functions, because if φ_j is a decreasing sequence of continuous functions converging to φ then we have the limits $\lim_{j\to\infty} EH_{\varphi_j} = EH_\varphi$ and $\lim_{j\to\infty} \sup \mathcal{F}_{\varphi_j} = \sup \mathcal{F}_\varphi$. Furthermore, if equation (3.2) holds for the φ_j's then $\sup \mathcal{F}_{\varphi_j} = EH_{\varphi_j}$ which ensures $\sup \mathcal{F}_\varphi = EH_\varphi$. Proving the result for continuous φ's uses the approach introduced by Bu and Schachermayer in [5]. Their motivation comes from probability theory, more specifically analytic martingales and Hardy martingales on Banach spaces. By using their method we get a very coherent proof of (3.2), see Theorem 3.1.6.

In Section 3.2 we extend this result to functions φ which are of the form

$\varphi = \varphi_1 - \varphi_2$, where φ_1 is upper semicontinuous and φ_2 is plurisubharmonic. This is done by using the result from Section 3.1 and approximating φ by using convolution. This is an extension of a result proved by Edigarian in [10].

For simplicity we assume in Section 3.1 and Section 3.2 that X is an open subset of $\mathbb{C}^n$. But these results can be extended to any complex manifold by using the method developed by Lárusson and Sigurdsson [25,26] and Rosay [35]. This is done in Section 3.3. This method proves the subaverage property of EH_φ on any complex manifold by showing how a "large enough" part of X can be embedded into $\mathbb{C}^n$ where previous results can be applied.

Finally, in Section 3.4 we see how the Poisson disc functional and the Riesz disc functional can be combined into a single disc formula.

3.1 Upper semicontinuous φ's on subsets of $\mathbb{C}^n$

In the following we assume X is an open subset of $\mathbb{C}^n$ and φ is a function on X such that $\mathcal{F}_\varphi \neq \emptyset$.

We will first prove equation (3.2) in the case when φ is continuous. The case when φ is upper semicontinuous then follows from this by taking a decreasing sequence of continuous functions tending to φ.

We start by showing that $\sup \mathcal{F}_\varphi$ is plurisubharmonic. Although this result follows from the plurisubharmonicity of EH_φ, it is worth a proof of its own because it is not directly connected to analytic discs and disc functionals.

Lemma 3.1.1 *If φ is an upper semicontinuous function such that $\mathcal{F}_\varphi \neq \emptyset$, then $\sup \mathcal{F}_\varphi$ is plurisubharmonic.*

Proof: Since $\sup \mathcal{F}_\varphi \leq \varphi$ and φ is upper semicontinuous, then the upper semicontinuous regularization

$$\limsup_{y \to x} \sup \mathcal{F}_\varphi(y) = (\sup \mathcal{F}_\varphi)^*(x),$$

which is a plurisubharmonic function by [21, Theorem 2.9.14], satisfies the inequality $(\sup \mathcal{F}_\varphi)^* \leq \varphi$. This implies $(\sup \mathcal{F}_\varphi)^* \in \mathcal{F}_\varphi$. Then $(\sup \mathcal{F}_\varphi)^* \leq \sup \mathcal{F}_\varphi$ and we have an equality $(\sup \mathcal{F}_\varphi)^* = \sup \mathcal{F}_\varphi$. □

The following lemma plays the main role in proving (3.2) in the case we when φ is continuous.

Lemma 3.1.2 *For a closed arc $A \subset \mathbb{T}$, there exists a sequence of functions $\{p_m\}_m$, analytic in a neighbourhood of $\overline{\mathbb{D}}$ and such that*

- $p_m(0) = 0$.
- $p_m(\mathbb{D}) \subset \mathbb{D}$.
- $\lim_{m\to\infty} p_m(x) = 0$ *for every* $x \in \overline{\mathbb{D}} \setminus A$.
- $\sigma_m \xrightarrow{weakly} \sigma(A)\sigma + (1-\sigma(A))\delta_0$,
  where we define $\sigma_m = (p_m)_*\sigma$ *as the pushforward of* σ *by* p_m.
  This is equivalent to

  $$\int_{\mathbb{T}} f \circ p_m \, d\sigma \to \sigma(A) \int_{\mathbb{T}} f \, d\sigma + (1-\sigma(A))f(0),$$

  for any continuous function f *on* $\overline{\mathbb{D}}$.

Proof: For every m let $A_m \subset \mathbb{T}$ be an open neighbourhood of A in $\mathbb{T}$ such that $A_{m+1} \subset A_m$ and $\cap_m A_m = A$, and let $C_m = A_m \setminus A$. Then define the continuous functions $h_m : \mathbb{T} \to [-m, 0]$ which take the value 0 on A and $-m$ on $\mathbb{T} \setminus A_m$, and are interpolated linearly between these values on C_m. Using the Poisson kernel we can extend h_m to a function which is continuous on $\overline{\mathbb{D}}$ and harmonic on $\mathbb{D}$. The harmonicity of h_m implies that its maximum value, 0, is only taken on A. Since $A_{m+1} \subset A_m$ and by the definition of h_m on $\mathbb{T}$ we see that

$$h_1 \geq \frac{1}{2}h_2 \geq \frac{1}{3}h_3 \geq \dots.$$

This implies $0 > mh_1 \geq h_m$ on $\overline{\mathbb{D}} \setminus A$, in particular $\lim_{m\to\infty} h_m = -\infty$ on $\overline{\mathbb{D}} \setminus A$.

For convenience let $\alpha = \sigma(A)$ and $\alpha_m = \sigma(A_m)$. Then by the mean value property of harmonic functions we see that

$$-m(1-\alpha) \leq h_m(0) = \int_{\mathbb{T}} h_m \, d\sigma \leq -m(1-\alpha_m).$$

Now let $\hat{h}_m$ be the harmonic conjugate of h_m that takes the value 0 at 0, and define the functions $g_m = \exp(h_m + i\hat{h}_m)$. The function g_m is holomorphic on $\mathbb{D}$ and with boundary values almost everywhere on $\mathbb{T}$. It is therefore clear that

- $e^{-(1-\alpha)m} \leq g_m(0) = e^{-(1-\alpha_m)m}$,
- $e^{-m} < |g_m| < 1$ on $\mathbb{D}$,
- $|g_m| = 1$ on A,
- $|g_m| = e^{-m}$ on $\mathbb{T} \setminus A_m$,
- $\lim_{m\to\infty} g_m = 0$ on $\overline{\mathbb{D}} \setminus A$.

To show that $g_m(\sigma) \to \alpha\sigma + (1-\alpha)\delta_0$ let f be a continuously differentiable function on $\overline{\mathbb{D}}$. This ensures that the Fourier series of f converges uniformly. We may assume f is differentiable since differentiable functions are dense in the set of continuous functions on $\overline{\mathbb{D}}$.

Then

$$\int_{\mathbb{T}} f \circ g_m \, d\sigma \to \sigma(A) \int_{\mathbb{T}} f \, d\sigma + (1 - \sigma(A)) f(0)$$

for every continuous function f on $\overline{\mathbb{D}}$. Now note that

$$\int_{\mathbb{T}} f \circ g_m \, d\sigma = \int_A f \circ g_m \, d\sigma + \int_{C_m} f \circ g_m \, d\sigma + \int_{\mathbb{T}\setminus A_m} f \circ g_m \, d\sigma. \qquad (3.4)$$

The second term tends to 0 and the third term tends to $(1-\alpha)f(0)$ as $m \to \infty$. We therefore wish to show that the first term tends to $\alpha \int_{\mathbb{T}} f \, d\sigma$.

Write f on $\mathbb{T}$ as a Fourier series

$$f(x) = \sum_{n=-\infty}^{\infty} a_n x^n, \qquad \text{where } a_n = \frac{1}{2\pi} \int_0^{2\pi} f(e^{it}) e^{-itn} \, dt \qquad (3.5)$$

Note that since $g_m(A) \subset \mathbb{T}$ this Fourier series is well-defined for the composition $f \circ g_m$ on A.

Since the real and imaginary part of g_m are harmonic, $|g_m| \leq 1$ on C_m, and $|g_m| = e^{-m}$ on $\mathbb{T} \setminus A_m$, we see that for $n \geq 1$

$$\begin{aligned}
|\int_A g_m^n \, d\sigma| &= |g_m^n(0) - \int_{\mathbb{T}\setminus A} g_m^n \, d\sigma| \\
&\leq \exp(-(1-\alpha_m)mn) + \sigma(\mathbb{T} \setminus A_m)\|g_m^n\|_{\mathbb{T}\setminus A_m} + \sigma(C_m)\|g_m^n\|_{C_m} \\
&\leq \exp(-(1-\alpha_m)mn) + (1-\alpha_m)e^{-mn} + \sigma(C_m) \xrightarrow[m\to\infty]{} 0,
\end{aligned}$$

where we used the fact that $\mathbb{T} \setminus A = (\mathbb{T} \setminus A_m) \cup C_m$ and $\sigma(C_m) \to 0$.

Since $h_m = 0$ on A, then $g_m^{-n} = \overline{g_m^n}$ on A and we see that for negative powers we have the same result as above, that is for $n \geq 1$,

$$|\int_A g_m^{-n} \, d\sigma| = |\overline{\int_A g_m^n \, d\sigma}| \xrightarrow[m\to\infty]{} 0.$$

We then conclude using the Fourier series (3.5) that

$$\begin{aligned}
\int_A f \circ g_m \, d\sigma = \sum_{n=-\infty}^{\infty} a_n \int_A g_m^n \, d\sigma &= a_0 \int_A d\sigma + \sum_{n=-\infty, n\neq 0}^{\infty} a_n \int_A g_m^n \, d\sigma \\
&\xrightarrow[m\to\infty]{} a_0\alpha = \alpha \frac{1}{2\pi} \int_0^{2\pi} f(e^{it}) \, dt = \sigma(A) \int_{\mathbb{T}} f \, d\sigma,
\end{aligned}$$

which shows along with (3.4) that $g_m(\sigma)$ tends weakly to $\alpha\sigma + (1-\alpha)\delta_0$.

Since h_m is continuous on $\overline{\mathbb{D}}$ and $|g_m| = e^{h_m}$ there is an $r_m < 1$ such that if we define $q_m(x) = g_m(r_m x)$ then

$$|g_m| - \frac{1}{m} \leq |q_m| \leq |g_m| + \frac{1}{m}, \qquad \text{on } \mathbb{T}.$$

Then it follows from Lebesgue theorem that $\int_A q_m^n \, d\sigma \to \alpha\sigma$ as $m \to \infty$, since $f \circ g_m - f \circ q_m \to 0$ almost everywhere on $\mathbb{T}$.

The functions q_m are holomorphic on D_{1/r_m} and they satisfy all the properties desired from p_m except they do not map 0 to 0 since $e^{-\alpha mk} \leq q_m(0) = g_m(0) \leq e^{-\alpha_m mk}$.

To fix this we use automorphisms of $\mathbb{D}$ of the form $z \mapsto \frac{z - q_m(0)}{1 - \overline{q_m(0)}z}$. These maps tend uniformly on $\overline{\mathbb{D}}$ to the identity map because $q_m(0) \to 0$ and

$$\Big|\frac{z - q_m(0)}{1 - q_m(0)z} - z\Big| < \frac{2q_m(0)}{1 - q_m(0)}$$

We therefore define the closed analytic discs p_m as

$$p_m(z) = \frac{q_m(z) - q_m(0)}{1 - \overline{q_m(0)}q_m(z)}.$$

Again by the uniform convergence of the automorphisms we see that the p_m's satisfy $p_m(\sigma) \xrightarrow{\text{weakly}} \alpha\sigma + (1-\alpha)\delta_0$. □

Theorem 3.1.3 *If X is a domain in $\mathbb{C}^n$ and φ is a continuous function on X such that $\mathcal{F}_\varphi \neq \emptyset$, then*

$$EH_\varphi = \sup \mathcal{F}_\varphi.$$

Proof: We already know that $EH_\varphi \leq \varphi$ by (3.3) and that $\sup \mathcal{F}_\varphi \leq EH_\varphi$ by (3.1). It is therefore enough to show that EH_φ is plurisubharmonic, because then $EH_\varphi \in \mathcal{F}_\varphi$ and we have an equality. We will therefore show that EH_φ is upper semicontinuous and that it satisfies the subaverage property of plurisubharmonic functions.

Beginning with the upper semicontinuity, fix $x_0 \in X$ and let $\beta > EH_\varphi(x_0)$. Let $f \in A_X$ be such that $f(0) = x_0$ and $H_\varphi(f) < \beta$. By the continuity of φ there is a neighbourhood U of $\overline{0}$ in $\mathbb{C}^n$ such that

$$H_\varphi(f(\cdot) + x) = \int_{\mathbb{T}} \varphi(f(t) + x)\, d\sigma(t) < \beta, \quad \text{for } x \in U.$$

This implies $EH_\varphi < \beta$ on $x_0 + U$, by the definition of the envelope. This shows that EH_φ is upper semicontinuous. We now turn our attention to the subaverage property of EH_φ, that is in order to prove plurisubharmonicity of EH we need to show that

$$EH_\varphi(x_0) \leq \int_{\mathbb{T}} EH_\varphi(x_0 + y_0 t)\, d\sigma(t), \tag{3.6}$$

for every $y_0 \in \mathbb{C}^n$ such that $x_0 + y_0\overline{\mathbb{D}} \subset X$. To prove (3.6) for fixed y_0, it suffices to show that for every $\varepsilon > 0$ and for every continuous function u such that $EH_\varphi \leq u$, there exists a disc $g \in \mathcal{A}_X$, $g(0) = x_0$ such that

$$H_\varphi(g) \leq \int_{\mathbb{T}} u(x_0 + y_0 t)\, d\sigma(t) + \varepsilon. \tag{3.7}$$

To clarify this better, since EH_φ is upper semicontinuous there is a sequence $\{u_j\}$ of continuous functions such that $u_j \searrow EH_\varphi$, which implies

$$\int_{\mathbb{T}} u_j(x_0 + y_0 t)\, d\sigma(t) \searrow \int_{\mathbb{T}} EH_\varphi(x_0 + y_0 t)\, d\sigma(t).$$

Then there is a function u_{j_0}, and if (3.7) is valid there is a disc g (depending on u_{j_0}), such that

$$EH_\varphi(x_0) \leq H_\varphi(g) \leq \int_{\mathbb{T}} u_{j_0}(x_0 + y_0 t)\, d\sigma(t) + \varepsilon \leq \int_{\mathbb{T}} EH_\varphi(x_0 + y_0 t)\, d\sigma(t) + 2\varepsilon.$$

Since ε was arbitrary we have shown that EH_φ satisfies the subaverage property. Our goal is therefore to construct the disc g in (3.7).

For every $t_0 \in \mathbb{T}$ there is a disc $f \in \mathcal{A}_X$ with $f(0) = 0$ such that

$$H_\varphi(f(\cdot) + x_0 + y_0 t_0) < u(x_0 + y_0 t_0) + \varepsilon.$$

By the continuity of φ and u we can assume there is a closed arc $A \subset \mathbb{T}$ containing t_0 as an inner point such that

$$|\varphi(x_0 + y_0 t + f(s)) - \varphi(x_0 + y_0 t_0 + f(s))| < \varepsilon, \quad \text{for } t \in A, s \in \overline{\mathbb{D}},$$

and

$$|u(x_0 + y_0 t) - u(x_0 + y_0 t_0)| < \varepsilon, \quad \text{for } t \in A.$$

By the compactness of $\mathbb{T}$ we can find finitely many arcs $A_1, \ldots, A_k$, points $x_1, \ldots, x_k$ with $x_j \in A_j$, and closed analytic discs $f_1, \ldots, f_k$, such that the A_j's only intersect at the endpoints, $\cup_j A_j = \mathbb{T}$ and $f_j(0) = 0$, and such that

$$H_\varphi(f_j(\cdot) + x_0 + y_0 t_j) < u(x_0 + y_0 t_j) + \varepsilon, \tag{3.8}$$

$$|\varphi(x_0 + y_0 t + f_j(s)) - \varphi(x_0 + y_0 t_j + f_j(s))| < \varepsilon, \quad \text{for } t \in A_j, s \in \overline{\mathbb{D}}, \tag{3.9}$$

$$|u(x_0 + y_0 t) - u(x_0 + y_0 t_j)| < \varepsilon, \qquad \text{for } t \in A_j. \tag{3.10}$$

Furthermore, let $\delta > 0$ be such that for every $j = 1, \ldots, k$ we have

$$x_0 + y_0 t + f_j(s) + x \in X$$

and

$$|\varphi(x_0 + y_0 t + f_j(s)) - \varphi(x_0 + y_0 t + f_j(s) + x)| < \varepsilon, \tag{3.11}$$

for every $x \in \mathbb{C}^n$, $|x| < \delta$, $t \in A_j$ and $s \in \overline{\mathbb{D}}$.

Now shrink the closed arcs A_j such that they become disjoint and such that

$$\sup_{(t,t_1,\ldots,t_k)\in\mathbb{T}\times D^k_{1+\delta}} |\varphi(x_0 + y_0 t + \sum_j f_j(t_j))| \cdot \sigma(\mathbb{T} \setminus \cup_j A_j) \le \varepsilon \tag{3.12}$$

and

$$\int_{\cup_j A_j} u(x_0 + y_0 t)\, d\sigma(t) \le \int_{\mathbb{T}} u(x_0 + y_0 t)\, d\sigma(t) + \varepsilon \tag{3.13}$$

For each $j = 1, \ldots, k$ let B_j be a closed set in $\overline{\mathbb{D}}$ containing $\overline{0}$ and $\cup_{l=1, l\neq j}^k A_l$. Then by Lemma 3.1.2 there is for every $j = 1, \ldots, k$ a sequence of analytic functions $\{p_{m,j}\}_m$ such that

$$p_{m,j}(0) = 0 \tag{3.14}$$

$$|f_j \circ p_{m,j}(t)| < \frac{\delta}{k}, \text{ for } t \in B_j \tag{3.15}$$

$$p_{m,j}(\mathbb{D}) \subset \mathbb{D} \tag{3.16}$$

$$p_{m,j}(\sigma) \xrightarrow{\text{weakly}} \sigma(A_j)\sigma + (1 - \sigma(A_j))\delta_0. \tag{3.17}$$

The last point implies there is for every j a number m_j such that

$$\int_{A_j} \varphi(x_0 + y_0 t_j + f_j \circ p_{m_j,j}(t))\, d\sigma(t)$$
$$\leq \sigma(A_j) \int_{\mathbb{T}} \varphi(x_0 + y_0 t_j + f_j(t))\, d\sigma(t) + \frac{\varepsilon}{k}. \quad (3.18)$$

To simplify notation we let $p_j = p_{m_j,j}$. Then we define the closed analytic disc $g \in \mathcal{A}_X$ by

$$g(t) = x_0 + y_0 t + \sum_{j=1}^{k} f_j \circ p_j(t).$$

This is a well defined disc in $\mathcal{A}_X$ because of (3.15) and (3.16), and with center x_0 by (3.14).

To conclude the proof

$$\begin{aligned}
H_\varphi(g) &= \int_{\mathbb{T}} \varphi\Big(x_0 + y_0 t + \sum_{j=1}^{k} f_j \circ p_j(t)\Big)\, d\sigma(t) \\
&\leq \int_{\cup_l A_l} \varphi\Big(x_0 + y_0 t + \sum_{j=1}^{k} f_j \circ p_j(t)\Big)\, d\sigma(t) + \varepsilon, && \text{by (3.12)}, \\
&= \sum_{l=1}^{k} \Big(\int_{A_l} \varphi\Big(x_0 + y_0 t + f_l \circ p_l(t) + \sum_{j=1, j\neq l}^{k} f_j \circ p_j(t)\Big)\, d\sigma(t)\Big) + \varepsilon \\
&\leq \sum_{l=1}^{k} \Big(\int_{A_l} \varphi\Big(x_0 + y_0 t + f_l \circ p_l(t)\Big)\, d\sigma(t) + \sigma(A_l)\varepsilon\Big) + \varepsilon, && \text{by (3.11) and (3.15)}, \\
&\leq \sum_{l=1}^{k} \Big(\int_{A_l} \varphi\Big(x_0 + y_0 t_l + f_l \circ p_l(t)\Big)\, d\sigma(t) + \sigma(A_l)\varepsilon\Big) + 2\varepsilon, && \text{by (3.9)}, \\
&\leq \sum_{l=1}^{k} \Big(\sigma(A_l) \int_{\mathbb{T}} \varphi\Big(x_0 + y_0 t_l + f_l(t)\Big)\, d\sigma(t) + \frac{\varepsilon}{k}\Big) + 3\varepsilon, && \text{by (3.18)}, \\
&\leq \sum_{l=1}^{k} \Big(\sigma(A_l) u(x_0 + y_0 t_l) + \sigma(A_l)\varepsilon\Big) + 4\varepsilon, && \text{by (3.8)}, \\
&\leq \sum_{l=1}^{k} \Big(\int_{A_l} u(x_0 + y_0 t)\, d\sigma(t) + \sigma(A_l)\varepsilon\Big) + 5\varepsilon, && \text{by (3.10)}, \\
&\leq \int_{\mathbb{T}} u(x_0 + y_0 t)\, d\sigma(t) + 7\varepsilon, && \text{by (3.13)}.
\end{aligned}$$

Now we turn our attention to the case when the function φ is upper semicontinuous. We will approximate it from above to extend the previous theorem.

Lemma 3.1.4 *Assume φ is an upper semicontinuous function and $\{\varphi_j\}_j$ are continuous functions such that $\varphi_j \searrow \varphi$. Then $EH_{\varphi_j} \searrow EH_\varphi$.*

Proof: It is clear that EH_φ is monotone with respect to φ. That is if $\varphi \leq \tilde{\varphi}$ then $EH_\varphi \leq EH_{\tilde{\varphi}}$, because $H_\varphi(f) \leq H_{\tilde{\varphi}}(f)$ for every $f \in \mathcal{A}_X$. This shows that EH_{φ_j} is a decreasing sequence of function such that $EH_\varphi \leq EH_{\varphi_j}$. Then there is a limit $\lim_{j\to\infty} EH_{\varphi_j} \geq EH_\varphi$.

Fix $x \in X$ and let $\beta > EH_\varphi(x)$. Then there is a disc $f \in \mathcal{A}_X$ such that $f(0) = x$ and

$$H_\varphi(f) < \beta.$$

By the Lebesgue monotone convergence theorem we see that $H_{\varphi_j}(f) \searrow H_\varphi(f)$, therefore there is a j_0 such that $H_{\varphi_j}(f) < \beta$ for $j \geq j_0$. This implies $\lim_{j\to\infty} EH_{\varphi_j}(x) \leq EH_\varphi(x)$, that is

$$\lim_{j\to\infty} EH_{\varphi_j}(x) = EH_\varphi(x).$$

□

Lemma 3.1.5 *Assume φ is an upper semicontinuous function such that $\mathcal{F}_\varphi \neq \emptyset$ and assume $\{\varphi_j\}_{j\in\mathbb{N}}$, are continuous functions such that $\varphi_j \searrow \varphi$. Then* $\sup \mathcal{F}_{\varphi_j} \searrow \sup \mathcal{F}_\varphi$.

Proof: The functions $\sup \mathcal{F}_{\varphi_j}$ form a decreasing sequence of plurisubharmonic functions which do not converge to $-\infty$ since $\mathcal{F}_\varphi \neq \emptyset$ and $\sup \mathcal{F}_\varphi \leq \sup \mathcal{F}_{\varphi_j}$. There is therefore a plurisubharmonic limit S such that

$$\sup \mathcal{F}_\varphi \leq S = \lim_{j\to\infty} \sup \mathcal{F}_{\varphi_j}.$$

It is clear that $S \leq \varphi$ since $S \leq \varphi_j$ for every j, and then $S \in \mathcal{F}_\varphi$ which implies the opposite inequality $S \leq \sup \mathcal{F}_\varphi$. We then have an equality $S = \sup \mathcal{F}_\varphi$. □

Now we can use Lemma 3.1.4, Lemma 3.1.5 and Theorem 3.1.3 to prove equation (3.2) in the case when φ is upper semicontinuous.

Theorem 3.1.6 *If X is a domain in $\mathbb{C}^n$ and φ is an upper semicontinuous function on X such that $\mathcal{F}_\varphi \neq \emptyset$, then*

$$EH_\varphi = \sup \mathcal{F}_\varphi.$$

If $\mathcal{F}_\varphi = \emptyset$ then $EH_\varphi = -\infty$

Proof: Let $\{\varphi_j\}_j$ be continuous functions such that $\varphi_j \searrow \varphi$. Then

$$EH_{\varphi_j} = \sup \mathcal{F}_{\varphi_j}$$

by Theorem 3.1.3. The left hand side tends to EH_φ by Lemma 3.1.4 which is then a plurisubharmonic function. If $\sup \mathcal{F}_\varphi \neq \emptyset$ then the right hand side tends to $\sup \mathcal{F}_\varphi$ by Lemma 3.1.5. These limits must be the same, hence $EH_\varphi = \sup \mathcal{F}_\varphi$. If $\sup \mathcal{F}_\varphi$ is empty then $EH_\varphi = -\infty$ since $-\infty$ is the only plurisubharmonic function dominated by φ. $\square$

3.2 More general φ's on subsets of $\mathbb{C}^n$

In the following we will look at the case when $\varphi = \varphi_1 - \varphi_2$ is the difference of an upper semicontinuous function φ_1 and a plurisubharmonic function φ_2. The main tool we will use to proof equality (3.2) in this case is convolution and we therefore still assume X is an open subset of $\mathbb{C}^n$. Later this result will be generalized to any complex manifold using the Reduction Theorem by Lárusson and Sigurdsson (see Theorem 3.3.1).

As mentioned in Chapter 1 Edigarian [10] proved this for plurisuperharmonic functions $\varphi = -\varphi_2$. Our method resembles his approach.

The first problem we run into when $\varphi = \varphi_1 - \varphi_2$ is that the value of φ is not well-defined when $\varphi_1(x) = -\infty$ and $\varphi_2(x) = -\infty$. Since we intend the envelope EH_φ to be upper semicontinuous then it is reasonable to define

$\varphi : X \to [-\infty, \infty]$ in the following way

$$\varphi(x) = \begin{cases} \varphi_1(x) - \varphi_2(x) & \text{if } \varphi_2(x) \neq -\infty \\ \limsup\limits_{\varphi_2^{-1}(-\infty) \not\ni y \to x} \varphi_1(y) - \varphi_2(y) & \text{if } \varphi_2(x) = -\infty. \end{cases} \tag{3.19}$$

This definition of the function φ should be viewed alongside Lemma 3.2.3 which states roughly that it suffices to look at discs not lying entirely in $\varphi^{-1}(\{-\infty\})$.

Note that φ is an L^1_{loc} function and that the Poisson functional satisfies $H_\varphi = H_{\varphi_1} - H_{\varphi_2}$, when $H_{\varphi_1}(f) \neq -\infty$ or $H_{\varphi_2}(f) \neq -\infty$.

We will now prove that the envelope EH_φ is plurisubharmonic by showing that

$$\lim_{\delta \to 0} EH_{\varphi_\delta} = EH_\varphi, \tag{3.20}$$

where φ_δ is a family of smooth functions defined by convolution which approximate φ. Note that the functions EH_{φ_δ} are plurisubharmonic by Theorem 3.1.6.

Let $\rho : \mathbb{C}^n \to \mathbb{R}$ be a non-negative C^∞ radial function with support in the unit ball $\mathbb{B}$ in $\mathbb{C}^n$ and such that $\int_{\mathbb{B}} \rho \, d\lambda = 1$, where λ is the Lebesgue measure in $\mathbb{C}^n$. For an open set $X \subset \mathbb{C}^n$ we let $X_\delta = \{x \in X; d(x, X^c) > \delta\}$ and if χ is in $L^1_{\text{loc}}(X)$ we define the convolution $\chi_\delta(x) = \int_{\mathbb{B}} \chi(x - \delta y)\rho(y) \, d\lambda(y)$ which is a C^∞ function on X_δ. It is well known that if $\chi \in \mathcal{PSH}(X)$ then $\chi_\delta \geq \chi$ and $\chi_\delta \searrow \chi$ as $\delta \searrow 0$.

The following lemma is the first part in proving the limit (3.20). It mimics the work of Edigarian [10] and uses his ingenious change of variables $y \to ty$ to ensure that the disc g we seek is centered at $f(0)$.

Lemma 3.2.1 *Assume $X \subset \mathbb{C}^n$ is open and $\varphi = \varphi_1 - \varphi_2$ a function on X defined as in (3.19). If $f \in \mathcal{A}_{X_\delta}$, then there exists $g \in \mathcal{A}_X$ such that $f(0) = g(0)$ and $H_\varphi(g) \leq H_{\varphi_\delta}(f)$, and consequently $EH_\varphi|_{X_\delta} \leq EH_{\varphi_\delta}$.*

Proof: Since φ_1 is upper semicontinuous and φ_2 is plurisubharmonic the function $(t, y) \mapsto \varphi(f(t) - \delta y)$ is integrable on $\mathbb{T} \times \mathbb{B}$. By using the change of variables $y \to ty$ where $t \in \mathbb{T}$ and that ρ is radial we see that

$$\begin{aligned} H_{\varphi_\delta}(f) &= \int_{\mathbb{T}}\int_{\mathbb{B}} \varphi(f(t) - \delta y)\rho(y)\, d\lambda(y)\, d\sigma(t) \\ &= \int_{\mathbb{T}}\int_{\mathbb{B}} \varphi(f(t) - \delta t y)\rho(y)\, d\lambda(y)\, d\sigma(t) \\ &= \int_{\mathbb{B}}\Big(\int_{\mathbb{T}} \varphi(f(t) - \delta t y)\, d\sigma(t)\Big)\rho(y)\, d\lambda(y). \end{aligned}$$

From measure theory we know that for every measurable function we can find a point where the function is less than or equal to its integral with respect to a probability measure. Applying this to the function $y \mapsto \int_{\mathbb{T}} \varphi(f(t) - \delta t y)\, d\sigma(t)$ and the measure $\rho\, d\lambda$ we can find $y_0 \in \mathbb{B}$ such that

$$H_{\varphi_\delta}(f) \geq \int_{\mathbb{T}} \varphi(f(t) - \delta t y_0)\, d\sigma(t) = H_\varphi(g),$$

if $g \in A_X$ is defined by $g(t) = f(t) - \delta t y_0$. It is clear that $g(0) = f(0)$. By taking the infimum over f, we see that $EH_{\varphi_\delta} \geq EH_\varphi|_{X_\delta}$. □

Note that $EH_\varphi|_{X_\delta}$ is the restriction of the function EH_φ to X_δ, but not the envelope of the functional H_φ restricted to $\mathcal{A}_{X_\delta}$. There is a subtle difference between these two, and in general they are different because $\mathcal{A}_{X_\delta} \subsetneq \mathcal{A}_X$. Note also that the function EH_{φ_δ} is only defined on X_δ because the convolution φ_δ is defined on X_δ.

Lemma 3.2.2 *If $\varphi = \varphi_1 - \varphi_2$ as above, then for every $f \in \mathcal{A}_X$ there is a limit $\lim_{\delta\to 0} H_{\varphi_\delta}(f) \leq H_\varphi(f)$ and it follows that for every $x \in X$,*

$$\lim_{\delta \to 0} EH_{\varphi_\delta}(x) = EH_\varphi(x).$$

Proof: Let $f \in \mathcal{A}_X$, $\beta > H_\varphi(f)$, and δ_0 be such that $f(\overline{\mathbb{D}}) \in X_{\delta_0}$, and assume $\varphi_2 \circ f \neq -\infty$. Since φ_2 is plurisubharmonic we know that $\varphi_{2,\delta} \geq \varphi_2$ on X_δ for all $\delta < \delta_0$, so

$$H_{\varphi_\delta}(f) = H_{\varphi_{1,\delta}}(f) - H_{\varphi_{2,\delta}}(f) \leq \int_{\mathbb{T}} \sup_{B(f(t),\delta)} \varphi_1\, d\sigma(t) - H_{\varphi_2}(f).$$

The upper semicontinuity of φ_1 implies that the integrand on the right hand side is bounded from above on $\mathbb{T}$ and also that it decreases to $\varphi_1(f(t))$ when $\delta \to 0$. It follows from Lebesgue's monotone convergence theorem that the integral tends to $\int_{\mathbb{T}} \varphi_1 \circ f \, d\sigma = H_{\varphi_1}(f)$ when $\delta \to 0$, that is the right side tends to $H_\varphi(f) < \beta$. We can therefore find $\delta_1 \le \delta_0$ such that

$$\int_{\mathbb{T}} \sup_{B(f(t),\delta)} \varphi_1 \circ f \, d\sigma - H_{\varphi_2}(f) < \beta, \qquad \text{for every } \delta < \delta_1.$$

However, if $\varphi_2 \circ f = -\infty$, then by monotone convergence

$$\begin{aligned} H_{\varphi_\delta}(f) &= \int_{\mathbb{T}} \int_{\mathbb{B}} \varphi(f(t) - \delta y) \rho(y) \, d\lambda(y) \, d\sigma(t) \\ &\le \int_{\mathbb{T}} \sup_{B(f(t),\delta)} \varphi \, d\sigma(t) = \int_{\mathbb{T}} \sup_{B(f(t),\delta) \setminus \varphi_2^{-1}(-\infty)} (\varphi_1 - \varphi_2) \, d\sigma(t) \\ &\xrightarrow[\delta \to 0]{} \int_{\mathbb{T}} \limsup_{y \to f(t)} \big(\varphi_1(y) - \varphi_2(y)\big) \, d\sigma(t) = H_\varphi(f). \end{aligned}$$

To prove that $\lim_{\delta \to 0} EH_{\varphi_\delta}(x) = EH_\varphi(x)$, let $\varepsilon > 0$ and assume $f \in \mathcal{A}_X$, $f(0) = x$, is such that $H_\varphi(f) < EH_\varphi(x) + \varepsilon$. Then there is a δ such that

$$EH_{\varphi_\delta}(x) \le H_{\varphi_\delta}(f) < EH_\varphi(x) + \varepsilon.$$

This along with the fact that $EH_\varphi(x) \le EH_{\varphi_\delta}(x)$ by Lemma 3.2.2 shows that $\lim_{\delta \to 0} EH_{\varphi_\delta} = EH_\varphi$. □

Now that we have established (3.20) we see that we can in fact completely avoid the troublesome set $\varphi_2^{-1}(-\infty)$, that is we do not have to consider discs which lie in this set. Note though that the discs might intersect it, but that intersection will always be of measure zero with respect to the measure σ on $\mathbb{T}$.

Lemma 3.2.3 *If $\varphi = \varphi_1 - \varphi_2$ as before, $f \in \mathcal{A}_X$, $f(\mathbb{D}) \subset \varphi_2^{-1}(-\infty)$, and $\varepsilon > 0$, then there is a disc $g \in \mathcal{A}_X$ such that $g(\mathbb{D}) \not\subset \varphi_2^{-1}(-\infty)$ and $H_\varphi(g) < H_\varphi(f) + \varepsilon$.*

Proof: By Lemma 3.2.2 we can find $\delta > 0$ such that $H_{\varphi_\delta}(f) \le H_\varphi(f) + \varepsilon$. Let

$\tilde{B} = \big\{y \in \mathbb{B}; \{\varphi(f(t) - \delta t y); t \in \mathbb{D}\} \not\subset \varphi_2^{-1}(-\infty)\big\}$, then $\mathbb{B} \setminus \tilde{B}$ is a zero set and as in the proof of Lemma 3.2.2 there is $y_0 \in \tilde{B}$ such that

$$\int_{\mathbb{T}} \varphi(f(t) - \delta t y_0)\, d\sigma(t) \leq \int_{\mathbb{T}} \int_{\tilde{B}} \varphi(f(t) - \delta t y)\rho(y)\, d\lambda(y)\, d\sigma(t) = H_{\varphi_\delta}(f).$$

We define $g \in \mathcal{A}_X$ by $g(t) = f(t) - \delta t y_0$. Then $H_\varphi(g) \leq H_\varphi(f) + \varepsilon$. □

Theorem 3.2.4 *Assume $\varphi = \varphi_1 - \varphi_2$ is the difference of an upper semicontinuous function φ_1 and a plurisubharmonic function φ_2 on a domain X in $\mathbb{C}^n$. If $\mathcal{F}_\varphi \neq \emptyset$ then*

$$EH_\varphi = \sup \mathcal{F}_\varphi.$$

Proof: We start by showing that the envelope EH_φ is upper semicontinuous. Since φ_δ is continuous then EH_{φ_δ} is plurisubharmonic by Theorem 3.1.6, in particular it is upper semicontinuous and does not take the value $+\infty$.

Now, assume $x \in X$ and let $\delta > 0$ be so small that $x \in X_\delta$. By the fact that $EH_{\varphi_\delta} < +\infty$ and $EH_\varphi|_{X_\delta} \leq EH_{\varphi_\delta}$ we see that EH_φ is finite. For $\beta > EH_\varphi(x)$, there is by Lemma 3.2.2 a $\delta > 0$ such that $EH_{\varphi_\delta}(x) < \beta$. Since EH_{φ_δ} is upper semicontinuous there is a neighbourhood $V \subset X_\delta$ of x where $EH_{\varphi_\delta} < \beta$. By Lemma 3.2.1, $EH_\varphi < \beta$ on V, which shows that EH_φ is upper semicontinuous.

Now we show that EH_φ satisfies the subaverage property. Fix a point $x \in X$, an analytic disc $h \in \mathcal{A}_X$, $h(0) = x$ and find δ_0 such that $h(\overline{\mathbb{D}}) \subset X_{\delta_0}$. Note that the function EH_{φ_δ} is plurisubharmonic by Theorem 3.1.3 since φ_δ is continuous. Then Lemma 3.2.1 and the plurisubharmonicity of EH_{φ_δ} gives that for every $\delta < \delta_0$,

$$EH_\varphi(x) \leq EH_{\varphi_\delta}(x) \leq \int_{\mathbb{T}} EH_{\varphi_\delta} \circ h\, d\sigma.$$

When $\delta \to 0$, Lebesgue's theorem along with Lemma 3.2.2 implies that $EH_\varphi(x) \leq \int_{\mathbb{T}} EH_\varphi \circ h\, d\sigma$.

Since $EH_\varphi(x) \leq H_\varphi(x) = \varphi(x)$, where $H_\varphi(x)$ is the functional H_φ evaluated at the constant disc $t \mapsto x$, we see that $EH_\varphi \leq \sup \mathcal{F}_\varphi$. Also, if $u \in \mathcal{F}_\varphi$

and $f \in \mathcal{A}_X$, then

$$u(f(0)) \leq \int_{\mathbb{T}} u \circ f \, d\sigma \leq \int_{\mathbb{T}} \varphi \circ f \, d\sigma = H_\varphi(f).$$

Taking supremum over $u \in \mathcal{F}_\varphi$ and infimum over $f \in \mathcal{A}_X$ we get the opposite inequality, $\sup \mathcal{F}_\varphi \leq EH_\varphi$, and therefore an equality. □

3.3 Generalization to manifolds

We will now extend the result from the previous section to a complex manifold X. This is done by using a theorem of Lárusson and Sigurdsson which is stated below. This theorem does not work specifically with the Poisson disc functional H_φ, because it can be applied to any disc functional which satisfies some conditions, most notably that the corresponding disc function for discs in a domain of holomorphy in $\mathbb{C}^n$ should have a plurisubharmonic envelope.

Theorem 3.3.1 *(Lárusson and Sigurdsson, [26, Theorem 1.2]) A disc functional H on a complex manifold X has a plurisubharmonic envelope if it satisfies the following three conditions.*

(i) The envelope $E\Phi^*H$ is plurisubharmonic for every holomorphic submersion Φ from a domain of holomorphy in affine space into X, where the pull-back Φ^*H is defined as $\Phi^*H(f) = H(\Phi \circ f)$ for a closed disc f in the domain of Φ.

(ii) There is an open cover of X by subsets U with a pluripolar subset $Z \subset U$ such that for every $h \in \mathcal{A}_U$ with $h(\overline{\mathbb{D}}) \not\subset Z$, the function $w \mapsto H(h(w))$ is dominated by an integrable function on $\mathbb{T}$.

(iii) If $h \in \mathcal{A}_X$, $w \in \mathbb{T}$, and $\varepsilon > 0$, then w has a neighbourhood U in $\mathbb{C}$ such that for every sufficiently small closed arc J in $\mathbb{T}$ containing w there is a holomorphic map $F : D_r \times U \to X$, $r > 1$, such that $F(0, \cdot) = h|_U$ and

$$\frac{1}{\sigma(J)} \underline{\int_J} H(F(\cdot, t)) \, d\sigma(t) \leq EH(h(w)) + \varepsilon, \tag{3.21}$$

where the integral on the left hand side is the lower integral, i.e. the supremum of the integrals of all integrable Borel functions dominated by the integrand.

To clarify these condition, the goal is to show that EH satisfies the subaverage property

$$EH(h(0)) \leq \int_{\mathbb{T}} EH \circ h \, d\sigma, \quad \text{for } h \in \mathcal{A}_X.$$

If we look at the integrand, then for every point $t \in \mathbb{T}$ there is a disc $f \in \mathcal{A}_X$, $f(0) = h(t)$ such that $H(f)$ is arbitrary close to $EH(h(t))$. Condition *(iii)* tells us that for a small arc on $\mathbb{T}$ we can have a holomorphic family of discs $F(s,t)$ such that each disc is close to the envelope $EH(h(t))$. This can be viewed as a weak upper semicontinuity of the disc functional H.

We can then cover $\mathbb{T}$ with these arcs, but to be able to embed a neighbourhood of the graph of the F's and h into $\mathbb{C}^N$ then these arcs need to be disjoint. After shrinking the arcs to make them disjoint, condition *(ii)* ensures that the integral over the complement of the arcs is bounded.

When this neighbourhood has been embedded into $\mathbb{C}^N$ then condition *(i)* ensures that there is a disc $\tilde{g}$ in $\mathbb{C}^N$ such that

$$\Phi^* H(\tilde{g}) \leq \int_{\mathbb{T}} E\Phi^* H \circ \tilde{h} \, d\sigma + \varepsilon,$$

where $\tilde{h}$ is a lifting of h to $\mathbb{C}^N$. The disc $g = \Phi \circ \tilde{g} \in \mathcal{A}_X$ then shows that EH satisfies the subaverage property.

Theorem 3.3.2 *Assume X is a connected complex manifold and $\varphi = \varphi_1 - \varphi_2$ is the difference of an upper semicontinuous function φ_1 and a plurisubharmonic function φ_2. If $\mathcal{F}_\varphi =\neq \emptyset$ then*

$$\sup \mathcal{F}_\varphi = EH_\varphi.$$

If $\mathcal{F}_\varphi = \emptyset$ then $EH_\varphi = -\infty$.

Proof: We have to show that H_φ satisfies the three conditions in Theorem 3.3.1. Condition *(i)* follows from Theorem 3.2.4 and condition *(ii)* if we take $U = X$ and $Z = \varphi^{-1}(\{+\infty\})$. Then $H_\varphi(h(w)) = \varphi(h(w))$ which is integrable since $h(0) \notin Z$.

To verify condition *(iii)*, let $h \in \mathcal{A}_X$, $w \in \mathbb{T}$ and $\beta > EH_\varphi(h(w))$. Then there is a disc $f \in \mathcal{A}_X$, $f(0) = h(w)$ such that $H_\varphi(f) < \beta$. Now look at the graph $\{(t, f(t))\}$ of f in $\mathbb{C} \times X$ and let π denote the projection from $\mathbb{C} \times X$ to X. In the proof of [25, Lemma 2.3], Lárusson and Sigurdsson show that by restricting the graph to a disc D_r, $r > 1$, there is a bijection Φ from a neighbourhood of the graph into $\mathbb{D}^{n+1}$ such that $\Phi(t, f(t)) = (t, \overline{0})$. In order to clarify the notation we write $\overline{0}$ for the zero vector in $\mathbb{C}^n$.

If we define $\tilde{\varphi} = \varphi \circ \pi \circ \Phi^{-1}$, then $H_\varphi(f) = H_{\tilde{\varphi}}((\cdot, \overline{0}))$, where $(\cdot, \overline{0})$ represents the analytic disc $t \mapsto (t, 0, \ldots, 0)$. The function $\tilde{\varphi}$ is defined on an open subset of $\mathbb{C}^{n+1}$ which enables us to smooth it using convolution as in the first part of this section.

By Lemma 3.2.2, there is a $\delta \in]0, 1[$ such that $H_{\tilde{\varphi}_\delta}((\cdot, \overline{0})) < \beta$. Since $\tilde{\varphi}_\delta$ is continuous, the function $x \mapsto H_{\tilde{\varphi}_\delta}((\cdot, \overline{0}) + x)$ is continuous. Then there is a neighbourhood $\tilde{U}$ of 0 in $D^n_{1-\delta}$, such that $H_{\tilde{\varphi}_\delta}((\cdot, \overline{0}) + x) < \beta$ for $x \in \tilde{U}$. Let $J \subset \mathbb{T}$ be a closed arc such that $\tilde{h}(J) \subset \tilde{U}$, where $\tilde{h}(t) = \Phi(0, h(t))$. With the same argument as in the proof of Lemma 3.2.1, we can find $y_0 \in \mathbb{B} \subset \mathbb{C}^{n+1}$ such that,

$$\begin{aligned}
\beta &> \frac{1}{\sigma(J)} \int_J H_{\tilde{\varphi}_\delta}\big((\cdot, 0) + \tilde{h}(t)\big)\, d\sigma(t) \\
&= \frac{1}{\sigma(J)} \int_{\mathbb{B}} \Big(\int_J \int_{\mathbb{T}} \tilde{\varphi}\big((s, 0) + h(t) - \delta s y\big)\, d\sigma(s)\, d\sigma(t) \Big) \rho(y)\, d\lambda(y) \\
&\geq \frac{1}{\sigma(J)} \int_J \int_{\mathbb{T}} \tilde{\varphi}\big((s, 0) + \tilde{h}(t) - \delta s y_0\big)\, d\sigma(s)\, d\sigma(t).
\end{aligned}$$

We define the function $F \in (D_r \times U, X)$ by

$$F(s, t) = \pi \circ \Phi^{-1}((s, 0) + \Phi(0, h(t)) - \delta s y_0)$$

and the set $U = h^{-1}(\pi(\Phi^{-1}(\tilde{U})))$.

Then $\tilde{\varphi}((s,0)+\tilde{h}(t)-\delta s y_0)=\varphi(F(s,t)$, and we conclude that

$$\beta > \frac{1}{\sigma(J)}\int_J\int_{\mathbb{T}} \varphi(F(s,t))\,d\sigma(s)\,d\sigma(t) = \frac{1}{\sigma(J)}\int_J H_\varphi(F(\cdot,t))\,d\sigma(t),$$

which shows that the Poisson disc functional H_φ has a plurisubharmonic envelope EH_φ on every complex manifold.

If $\mathcal{F}_\varphi \neq \emptyset$ then this implies $EH_\varphi \leq \sup \mathcal{F}_\varphi$, that is $EH_\varphi = \sup \mathcal{F}_\varphi$ because of inequality (3.1).

However, if $\mathcal{F}_\varphi = \emptyset$ then the function which is identically $-\infty$ is the only plurisubharmonic function which is dominated by φ. We have showed that the envelope EH_φ is plurisubharmonic and that is satisfies $EH_\varphi \leq \varphi$ by (3.3). This implies $EH_\varphi = -\infty$. □

3.4 Merging the Riesz and the Poisson functionals

Let X be an n-dimensional complex manifold, v a plurisubharmonic function on X and $\varphi = \varphi_1 - \varphi_2$, the difference of an upper semicontinuous function φ_1 and a plurisubharmonic function φ_2, as before. We define the *Riesz disc functional for v and φ* by

$$H^R_{v,\varphi}(f) = \frac{1}{2\pi}\int_{\mathbb{D}} \log|\cdot|\,\Delta(v\circ f) + \int_{\mathbb{T}} \varphi\circ f\,d\sigma, \qquad \text{for } f\in\mathcal{A}_X. \tag{3.22}$$

The original Riesz functional studied by Poletsky [33, 34], Lárusson and Sigurdsson [25, 26], and Edigarian [10] is the case when $\varphi = 0$. Their result states that the envelope $EH^R_{v,0}$ is plurisubharmonic and equal to the largest non-positive plurisubharmonic function with a Levi form which is no smaller than the Levi form $\mathcal{L}(v) = \sum_{j,k=1}^n \frac{\partial^2 v}{\partial z_j \partial \bar{z}_k}$ of v, i.e.

$$\sup\{u(x); u\in\mathcal{PSH}(X), \mathcal{L}(u)\geq\mathcal{L}(v), u\leq 0\} \\ = \inf\Big\{\frac{1}{2\pi}\int_{\mathbb{D}} \log|\cdot|\,\Delta(v\circ f); f\in\mathcal{A}_X, f(0)=x\Big\}.$$

We will use here the same approach as these authors, but the more general results for the Poisson disc functional in Section 3.2 will enable us to prove this results for aforementioned $\varphi = \varphi_1 - \varphi_2$.

The Riesz disc functional is closely connected to the Poisson disc functional through the Riesz representation formula (2.3). Fix $x \in X$ and let $f \in \mathcal{A}_X$ be such that $f(0) = x$, then

$$H^R_{v,\varphi}(f) = \frac{1}{2\pi} \int_{\mathbb{D}} \log|\cdot|\, \Delta(v \circ f) + \int_{\mathbb{T}} \varphi \circ f \, d\sigma \tag{3.23}$$

$$= v(f(0)) - \int_{\mathbb{T}} v \circ f \, d\sigma + \int_{\mathbb{T}} \varphi \circ f \, d\sigma = v(x) + H_{\varphi - v}(f). \tag{3.24}$$

From this we see that $EH^R_{v,\varphi} = v + EH_{\varphi - v}$. Since $\varphi - v$ is the difference of an upper semicontinuous function and a plurisubharmonic function then $EH_{\varphi - v}$ is plurisubharmonic, and then $EH^R_{v,\varphi}$ is also plurisubharmonic since it is the sum of two plurisubharmonic functions.

Furthermore, since $EH_{\varphi - v}$ is plurisubharmonic then

$$\mathcal{L}(EH^R_{v,\varphi}) = \mathcal{L}(v) + \mathcal{L}(EH_{\varphi - v}) \geq \mathcal{L}(v), \tag{3.25}$$

and if we look at the constant disc f_x which sends everything to $x \in X$ then we see that

$$EH^R_{v,\varphi}(x) \leq H^R_{v,\varphi}(f_x) = 0 + \varphi(x). \tag{3.26}$$

It is therefore clear that $EH^R_{v,\varphi} \in \{u \in \mathcal{PSH}(X); \mathcal{L}(u) \geq \mathcal{L}(v), u \leq \varphi\}$.

Now, if we assume $f \in \mathcal{A}_X$, $f(0) = x$ and that u is a plurisubharmonic function such that $\mathcal{L}(u) \geq \mathcal{L}(v)$ and $u \leq \varphi$ then by applying the Riesz representation formula (2.3) to the subharmonic function $u \circ f$ we see that

$$\begin{aligned} u(x) &= \int_{\mathbb{D}} \log|\cdot|\, \Delta(u \circ f) + \int_{\mathbb{T}} u \circ f \, d\sigma \\ &\leq \int_{\mathbb{D}} \log|\cdot|\, \Delta(v \circ f) + \int_{\mathbb{T}} \varphi \circ f \, d\sigma \\ &= H^R_{v,\varphi}(f). \end{aligned}$$

By taking the supremum over u on the left hand side and the infimum over f on the right hand side we see that

$$
\begin{aligned}
&\sup\{u(x); u \in \mathcal{PSH}(X), \mathcal{L}(u) \geq \mathcal{L}(v), u \leq \varphi\} \\
&\qquad\qquad \leq \inf\Big\{\frac{1}{2\pi}\int_{\mathbb{D}} \log|\cdot|\,\Delta(v\circ f) + \int_{\mathbb{T}} \varphi\circ f\,d\sigma; f \in \mathcal{A}_X, f(0) = x\Big\}.
\end{aligned}
$$

Since $EH^R_{v,\varphi}$ is in the family on the left hand side this is actually an equality. We have thus proved the following result which combines the disc formulas for the Poisson disc functional and the original Riesz functional into a single formula.

Theorem 3.4.1 *Assume X is a connected complex manifold, v is a plurisubharmonic function on X and $\varphi = \varphi_1 - \varphi_2$ is the difference of an upper semicontinuous function φ_1 and a plurisubharmonic function φ_2. Then $EH^R_{v,\varphi}$ is plurisubharmonic and*

$$
\begin{aligned}
&\sup\{u(x); u \textit{ is plurisubharmonic}, \mathcal{L}(u) \geq \mathcal{L}(v), u \leq \varphi\} \\
&\qquad\qquad = \inf\Big\{\frac{1}{2\pi}\int_{\mathbb{D}} \log|\cdot|\,\Delta(v\circ f) + \int_{\mathbb{T}} \varphi\circ f\,d\sigma; f \in \mathcal{A}_X, f(0) = x\Big\},
\end{aligned}
$$

This result is closely connected to the disc formula for ω-plurisubharmonic functions given in Theorem 4.1.1. More specifically, this is the special case when the current $\omega = -dd^c v$ has a global potential, which is studied in Section 4.2.

4

Disc formulas for quasiplurisubharmonic functions

4.1 Introduction

In this chapter we turn our attention to ω-plurisubharmonic functions. We wish to give a disc formula for the function $\sup \mathcal{F}_{\omega,\varphi}$, where

$$\mathcal{F}_{\omega,\varphi} = \{u \in \mathcal{PSH}(X,\omega)\,;\, u \leq \varphi\}.$$

We assume X is a complex manifold and $\omega = \omega_1 - \omega_2$ is the difference of two closed and positive $(1,1)$-currents. The reason we look at currents on this form is that our methods rely on the currents having a local potential which is a function and not only a distribution. When $\omega = \omega_1 - \omega_2$ then ω_1 and ω_2 both have plurisubharmonic local potentials ψ_1 and ψ_2 by Corollary 2.2.2. Then $\psi = \psi_1 - \psi_2$ is a local potential of ω.

The Poisson disc functional H_φ from before is obviously not appropriate for this task since it fails to take into account the current ω. The remedy is to look at the pullback of ω by an analytic disc. If f is an analytic disc we can define a closed $(1,1)$-current $f^*\omega$ on $\mathbb{D}$, which is locally given as $dd^c(\psi \circ f)$, see Definition 2.2.9. We can also look at $f^*\omega$ as a Radon measure on $\mathbb{D}$.

Furthermore, we let $R_{f^*\omega}$ be the Riesz potential of $f^*\omega$,

$$R_{f^*\omega}(z) = \int_{\mathbb{D}} G_{\mathbb{D}}(z, \cdot)\, d(f^*\omega), \tag{4.1}$$

where $G_{\mathbb{D}}$ is the Green function for the unit disc, $G_{\mathbb{D}}(z,w) = \frac{1}{2\pi} \log \frac{|z-w|}{|1-z\overline{w}|}$.

Assume $\varphi = \varphi_1 - \varphi_2$ is the difference of an ω_1-upper semicontinuous function φ_1 and a plurisubharmonic function φ_2. We define φ at points where both φ_1 and φ_2 take the value $-\infty$ by taking limit superior, identical to definition (3.19). Fix $x \in X$ and let u be an ω-plurisubharmonic function on X such that $u \leq \varphi$ and $f \in \mathcal{A}_X$ a closed analytic disc such that $f(0) = x$ and $f(\mathbb{D}) \not\subset \operatorname{sing}(\omega)$. Then $u \circ f$ is an $f^*\omega$-subharmonic function on $\mathbb{D}$, by Proposition 2.2.10, and since the Riesz potential $R_{f^*\omega}$ is a global potential for $f^*\omega$ on $\mathbb{D}$ we have, by the subaverage property of $u \circ f + R_{f^*\omega}$, that

$$u(f(0)) + R_{f^*\omega}(0) \leq \int_{\mathbb{T}} u \circ f\, d\sigma + \int_{\mathbb{T}} R_{f^*\omega}\, d\sigma.$$

Since $R_{f^*\omega} = 0$ on $\mathbb{T}$ and $u \leq \varphi$, we conclude that

$$u(x) \leq -R_{f^*\omega}(0) + \int_{\mathbb{T}} \varphi \circ f\, d\sigma. \tag{4.2}$$

The right hand side is independent of u so we define the disc functional

$$H_{\omega,\varphi} : \mathcal{A}_X \to [-\infty, +\infty]$$

by

$$H_{\omega,\varphi}(f) = -R_{f^*\omega}(0) + \int_{\mathbb{T}} \varphi \circ f\, d\sigma, \tag{4.3}$$

if $f(\mathbb{D}) \not\subset \operatorname{sing}(\omega)$, and by $H_{\omega,\varphi}(f) = +\infty$ if $f(\mathbb{D}) \subset \operatorname{sing}(\omega)$.

By taking supremum on the left hand side over all $u \in \mathcal{PSH}(X,\omega)$, $u \leq \varphi$, and infimum on the right hand side over all $f \in \mathcal{A}_X$ such that $f(0) = x$ we get the inequality

$$\sup \mathcal{F}_{\omega,\varphi} \leq EH_{\omega,\varphi}. \tag{4.4}$$

The following theorem, which is our main result, shows that this is actually an equality.

Theorem 4.1.1 *Let X be a connected complex manifold, $\omega = \omega_1 - \omega_2$ be the difference of two closed positive $(1,1)$-currents on X, $\varphi = \varphi_1 - \varphi_2$ be the difference of an ω_1-upper semicontinuous function φ_1 in $L^1_{loc}(X)$ and a plurisubharmonic function φ_2, and assume that $\mathcal{F}_{\omega,\varphi}$ is non-empty. Then the function $\sup \mathcal{F}_{\omega,\varphi}$ is ω-plurisubharmonic and for every $x \in X \setminus \operatorname{sing}(\omega)$,*

$$\sup\{u(x); u \in \mathcal{PSH}(X,\omega), u \leq \varphi\}$$
$$= \inf\{-R_{f^*\omega}(0) + \int_{\mathbb{T}} \varphi \circ f \, d\sigma; f \in \mathcal{A}_X, f(0) = x\}.$$

If $\mathcal{F}_{\omega,\varphi}$ is empty, then the right hand side is $-\infty$ for every $x \in X$.

We prove this theorem in Section 4.2 for the case when ω_1 and ω_2 have global potentials. For simplicity we also assume there that $\varphi_2 = 0$. The proof uses the result from Theorem 3.2.4.

In Section 4.3 we prove the general case of the theorem by reducing the problem to the case of global potentials. This is done by using an ω-Reduction Theorem similar to the Reduction Theorem of Lárusson and Sigurdsson, (Theorem 3.3.1). This is done first for $\varphi_2 = 0$, but the general case follows from calculations similar to those for the Riesz disc functional in Section 3.4.

4.2 The case of a global potential

Here we assume $\omega = \omega_1 - \omega_2$ has a global potential $\psi = \psi_1 - \varphi_2$ on X, where ψ_1 and ψ_2 are global potentials of ω_1 and ω_2 respectively. Furthermore, we assume $\varphi = \varphi_1$ is an ω_1-upper semicontinuous function, i.e. $\varphi_2 = 0$.

If $\mathcal{F}_{\omega,\varphi} \neq \emptyset$, then we know that function $\sup \mathcal{F}_{\omega,\varphi}$ is ω-plurisubharmonic by Proposition 2.2.7. To prove Theorem 4.1.1 it is therefore enough to show that $EH_{\omega,\varphi} \in \mathcal{F}_{\omega,\varphi}$, then obviously $EH_{\omega,\varphi} \leq \sup \mathcal{F}_{\omega,\varphi}$ and we have an equality.

First note that if $x \notin \mathrm{sing}(\omega)$ and $f_x \in \mathcal{A}_X$ is the constant disc which maps everything to x, then $f^*\omega = 0$ and $H_{\omega,\varphi} = 0 + \int_{\mathbb{T}} \varphi(x)\, d\sigma = \varphi(x)$, which shows that $EH_{\omega,\varphi} \leq \varphi$.

Our goal is therefore to show that $EH_{\omega,\varphi}$ is ω-plurisubharmonic. Since ω has a global potential we can use this potential to connect the ω-Poisson functional $H_{\omega,\varphi}$ to the classical Poisson functional H_φ.

Lemma 4.2.1 *If $f \in \mathcal{A}_X$ and $\psi = \psi_1 - \psi_2$ is a potential for ω in a neighbourhood of $f(\overline{\mathbb{D}})$ then*

$$H_{\omega,\varphi}(f) + \psi(f(0)) = H_{\varphi+\psi}(f).$$

Proof: By the linearity of $R_{f^*\omega}$ as a function of ω and the Riesz representation (2.3) for $\psi_1 \circ f$ and $\psi_2 \circ f$ we get

$$\begin{aligned}
H_{\omega,\varphi}(f) + \psi(f(0)) &= -R_{f^*\omega}(0) + \int_{\mathbb{T}} \varphi \circ f\, d\sigma + \psi(f(0)) \\
&= -R_{f^*\omega}(0) + \int_{\mathbb{T}} \varphi \circ f\, d\sigma + \psi_1(f(0)) - \psi_2(f(0)) \\
&= -R_{f^*\omega}(0) + \int_{\mathbb{T}} \varphi \circ f\, d\sigma + R_{f^*\omega}(0) + \int_{\mathbb{T}} (\psi_1 - \psi_2) \circ f\, d\sigma \\
&= \int_{\mathbb{T}} (\varphi + \psi_1 - \psi_2) \circ f\, d\sigma = H_{\varphi+\psi}(f).
\end{aligned}$$

□

Proof of Theorem 4.1.1 when ω_1 and ω_2 have global potentials and $\varphi_2 = 0$

Proof: By Lemma 4.2.1 for $x \in X \setminus \mathrm{sing}(\omega)$,

$$EH_{\omega,\varphi}(x) + \psi(x) = \inf\{H_{\omega,\varphi}(f) + \psi(x); f \in \mathcal{A}_X, f(0) = x\} = EH_{\varphi+\psi}(x).$$

Since $\varphi+\psi = (\varphi+\psi_1)-\psi_2$ is the difference of an upper semicontinuous function and a plurisubharmonic function, then Theorem 3.2.4 shows that $EH_{\varphi+\psi}$ is plurisubharmonic, that is $EH_{\omega,\psi}$ is ω-plurisubharmonic □

4.3 Reduction to global potentials

Here we see how a local potential for ω can be constructed on a set, big enough to prove the ω-plurisubharmonicity of EH. The methods used are in principle similar to those in the proof the Reduction Theorem for plurisubharmonic functions (Theorem 3.3.1). The main role is played be Lemma 4.3.1 which shows the existence of these local potentials. Theorem 4.3.3 then shows how the ω-plurisubharmonicity of EH can be reduced to the case when the current has a global potential.

It should be pointed out that Theorem 4.3.3 does not work specifically with the Poisson functional but a general disc functional H. We will however apply the results here to the Poisson functional from Section 4.1, so it is of no harm to think of it in the role of H.

If $\Phi : Y \to X$ is a submersion, the currents $\Phi^*\omega_1$ and $\Phi^*\omega_2$ are well-defined on Y by (2.6). The core in showing the ω-plurisubharmonicity of EH is the following lemma. It produces a local potential of the currents $\Phi^*\omega_1$ and $\Phi^*\omega_2$ in a neighbourhood of the graphs of the discs from condition *(iii)* in Theorem 4.3.3 below. Its functionality and proof are similar to those of the Meyer-Vietoris sequence [37, Theorem 3, Chapter 11]).

Lemma 4.3.1 *Let X be a complex manifold and $\tilde{\omega}$ a positive closed $(1,1)$-current on $\mathbb{C}^2 \times X$. Assume $h \in (D_r, X)$, $r > 1$ and for $j = 1, \ldots, m$ assume $J_j \subset \mathbb{T}$ are disjoint arcs and $U_j \subset D_r$ are pairwise disjoint open discs containing J_j. Furthermore, assume there are functions $F_j \in (D_s \times U_j, X)$, $s > 1$, for $j = 1, \ldots, m$, such that $F_j(0, w) = h(w)$, $w \in U_j$.*

If $K_0 = \{(w, 0, h(w); w \in \overline{\mathbb{D}}\}$ and $K_j = \{(w, z, F_j(z, w); z \in \overline{\mathbb{D}}, w \in J_j\}$ then there is an open neighbourhood of $K = \cup_{j=0}^{m} K_j$ where $\tilde{\omega}$ has a global potential ψ.

Proof: For convenience we let $U_0 = D_r$ and $F_0(z, w) = h(z)$. As before $\overline{0}$ will denote the zero vector in $\mathbb{C}^n$. The graphs of the F_j's are biholomorphic to polydiscs, hence Stein. By slightly shrinking the U_j's and s we can, just as in the proof of Theorem 1.2 in [26], use Siu's Theorem [36] and the proof of [25, Lemma 2.3], define biholomorphisms Φ_j from the polydisc $U_j \times D_s^{n+1}$

onto a neighbourhood of the K_j such that

$$\Phi_j(w, z, \overline{0}) = (w, z, F_j(z, w)), \qquad w \in U_j, z \in D_s, \tag{4.5}$$

for $j = 1, \ldots, m$ and

$$\Phi_0(w, 0, \overline{0}) = (w, 0, h(w)), \qquad w \in U_0. \tag{4.6}$$

Furthermore, we may assume the functions are continuous on the closure of $U_j \times D_s^{n+1}$.

For $j = 1, \ldots, m$ let U_j' and U_j'' be open discs concentric to U_j such that

$$J_j \subset\subset U_j'' \subset\subset U_j' \subset\subset U_j,$$

and B_j a neighbourhood of $\Phi_j(\overline{U_j'} \times \{(0, \overline{0})\})$ defined by

$$B_j = \Phi_j(U_j \times D_{\delta_j}^{n+1})$$

for $\delta_j > 0$ small enough so that

$$B_j \subset \Phi_0(U_0 \times D_s^{n+1}),$$

and

$$B_j \cap K_k = \emptyset, \qquad \text{when } k \neq j \text{ and } k \geq 1.$$

This is possible since $\Phi_j(U_j \times \{(0, \overline{0})\}) \subset \Phi_0(U_0 \times D_s^{n+1})$ and $\Phi_j(U_j \times \{(0, \overline{0})\}) \cap K_k = \emptyset$ if $k \neq j$ and $k \geq 1$.

The compact sets $\Phi_0(\overline{U_0} \setminus U_j' \times \{(0, \overline{0})\})$ and $\Phi_j(\overline{U_j''} \times \overline{D}_s \times \{\overline{0}\})$ are disjoint by (4.6) and (4.5), and likewise $\Phi_0(\overline{U_j'} \times \{(0, \overline{0})\}) \subset\subset B_j$. So there is an $\varepsilon_j > 0$ such that

$$\Phi_0(U_0 \setminus U_j' \times D_{\varepsilon_j}^{n+1}) \cap \Phi_j(U_j'' \times D_s \times D_{\varepsilon_j}^{n}) = \emptyset$$

and

$$\Phi_0(U_j' \times D_{\varepsilon_j}^{n+1}) \subset B_j.$$

Let $\varepsilon_0 = \min\{\varepsilon_1, \ldots, \varepsilon_m\}$ and define the sets $V_0 = \Phi_0(U_0 \times D_{\varepsilon_0}^{n+1})$ and

$V_j = \Phi_j(U_j'' \times D_s \times D^n_{\varepsilon_j})$.

Furthermore, since the graphs of the F_j's, $\Phi_j(U_j \times D_s \times \{\overline{0}\})$, are disjoint for $j \geq 1$ we may assume $V_j \cap V_k = \emptyset$, and similarly that $B_j \cap B_k = \emptyset$ when $j \neq k$ and $j, k \geq 1$.

What this technical construction has achieved is ensuring that the intersection $V_0 \cap V_j$ is contained in B_j, while still letting all the sets V_j and B_j be biholomorphic to polydiscs. Then both $V = \bigcup_{j=1}^m V_j$ and $B = \bigcup_{j=1}^m B_j$ are disjoint unions of polydiscs.

By Corollary 2.2.2 there are local potentials ψ_j of $\tilde{\omega}$ on each of the sets $\Phi_j(U_j \times D_s^{n+1})$, $j = 1, \ldots, m$. Define $\eta' = d^c\psi_0$ on $V_0 \cup B$ and η'' on $V \cup B$ by $\eta'' = d^c\psi_j$ on $V_j \cup B_j$, this is well defined because the $V_j \cup B_j$'s are pairwise disjoint and $V_j \cup B_j \subset \Phi_j(U_j \times D_s^{n+1})$. Since $d\eta' - d\eta'' = \tilde{\omega} - \tilde{\omega} = 0$ on B there is a distribution μ on B satisfying $d\mu = \eta' - \eta''$.

Let χ_1, χ_2 be a partition of unity subordinate to the covering $\{V_0, V\}$ of $V_0 \cup V$. Then

$$\eta = \begin{cases} \eta' - d(\chi_1\mu) & \text{on } V_0 \\ \eta'' + d(\chi_2\mu) & \text{on } V \end{cases}$$

is well defined on $V_0 \cup V$ with $d\eta = \tilde{\omega}$.

If we repeat the topological construction above for $V_0, \ldots, V_m$ instead of $\Phi_j(U_j \times D_s^{n+1})$ we can define sets $V_0', \ldots, V_m'$ and $B_1', \ldots, B_m'$ biholomorphic to polydiscs such that $V_j' \subset V_j$, $B_j' \subset B_j$ and

$$V_0' \cap V_j' \subset B_j' \subset V_0 \cap V_j,$$

and both the B_j''s and the V_j''s are pairwise disjoint. Now let $V' = \bigcup_{j=1}^m V_j'$.

Let ψ' be a real distribution defined on V_0 satisfying $d^c\psi' = \eta' - d\chi_1\mu$ and let ψ'' be a real distribution defined on V satisfying $d^c\psi'' = \eta'' - d\chi_2\mu$. Then $d^c(\psi' - \psi'') = \eta' - \eta'' - d(\chi_1\mu + \chi_2\mu) = 0$. Therefore, on each of the connected sets B_j' we have $\psi' - \psi'' = c_j$, for some constant c_j. Consequently the distribution ψ is well defined on $V_0' \cup V'$ by

$$\psi = \begin{cases} \psi' & \text{on } V_0' \\ \psi'' + c_j & \text{on } V_j' \end{cases}$$

since $V_0' \cap V' \subset B'$ and the V_j''s are disjoint. It is clear that $dd^c\psi = d\eta = \tilde{\omega}$ and since ω is positive we may assume ψ is a plurisubharmonic function. □

We now turn our attention back to the ω-plurisubharmonicity of the envelope EH. We start by showing that it is ω-upper semicontinuous, but this is done separately because it needs weaker assumptions than those needed in Theorem 4.3.3 where we show that EH is ω-plurisubharmonic.

Lemma 4.3.2 *Let X be an n-dimensional complex manifold, H a disc functional on $\mathcal{A}_X$, and $\omega = \omega_1 - \omega_2$ the difference of two positive, closed $(1,1)$-currents on X. The envelope EH is ω-upper semicontinuous if $E\Phi^*H$ is $\Phi^*\omega$-upper semicontinuous for every submersion Φ from a set biholomorphic to an $(n+1)$-dimensional polydisc into X.*

Proof: To show that $EH + \psi$ does not take the value $+\infty$ at $x \in X \setminus \operatorname{sing}(\omega)$, let U be a coordinate polydisc in X centered at x and ψ a local potential of ω on $U \subset X$. Then by (2.10),

$$EH(x)+\psi(x) = EH(\Phi(0,x))+\psi(\Phi(0,x))) \leq E\Phi^*H((0,x))+\psi(\Phi(0,x)) < +\infty,$$

where $\Phi : \mathbb{D} \times U \to U$ is the projection.

Let $\beta > EH(x)+\psi(x)$ and $g \in A_X$ such that $g(0) = x$ and $H(g)+\psi(x) < \beta$. By a now familiar argument in from the proof of [25, Lemma 2.3], there is a biholomorphism Ψ from a neighbourhood of the graph $\{(w, g(w)); w \in \mathbb{D}\}$ into D_s^{n+1}, $s > 1$ such that $\Psi(w, g(w)) = (w, \overline{0})$. If Φ is the projection $\mathbb{C} \times X \to X$ then $\Phi^*\psi = \psi \circ \Phi$ is a local potential of $\Phi^*\omega$ on $\mathbb{C} \times U$. Now, if $\tilde{g} \in \mathcal{A}_{\mathbb{C}\times X}$ is the lifting $w \mapsto (w, g(w))$ of g then

$$E\Phi^*H((0,x)) + \psi(\Phi((0,x)) \leq \Phi^*H(\tilde{g}) + \psi(\Phi((0,x))) = H(g) + \psi(x) < \beta.$$

By assumption there is a neighbourhood $W_0 \times W \subset \mathbb{C} \times U$ of $(0,x)$ such that for $(z_0, z) \in W_0 \times W$,

$$E\Phi^*H((z_0,z)) + \psi(\Phi((z_0,z))) < \beta.$$

Then by (2.10) $EH(z) = \Phi^*EH((z_0,z)) \leq E\Phi^*H((z_0,z))$, which implies

$EH(z) + \psi(z) \leq \beta$ for $z \in W$. This shows that $EH + \psi$ is upper semicontinuous outside of $\operatorname{sing}(\omega)$ and by (2.9), the definition of EH at $\operatorname{sing}(\omega)$, we have shown that EH is ω-upper semicontinuous. □

The following theorem shows that an envelope EH is ω-plurisubharmonic if it satisfies the following conditions. These conditions are very similar to those posed upon the envelope in Theorem 3.3.1 when $\omega = 0$.

Theorem 4.3.3 *(ω-Reduction Theorem): Let X be a complex manifold, H a disc functional on $\mathcal{A}_X$ and $\omega = \omega_1 - \omega_2$ the difference of two positive, closed $(1,1)$-currents on X. The envelope EH is ω-plurisubharmonic if it satisfies the following.*

(i) *$E\Phi^*H$ is $\Phi^*\omega$-plurisubharmonic for every holomorphic submersion Φ from a complex manifold where $\Phi^*\omega$ has a global potential.*

(ii) *There is an open cover of X by subsets U, with ω-pluripolar subsets $Z \subset U$ and local potentials ψ on U, $\psi^{-1}(\{-\infty\}) \subset Z$, such that for every $h \in \mathcal{A}_U$ with $h(\overline{\mathbb{D}}) \not\subset Z$, the function $t \mapsto \big(H(h(t)) + \psi(h(t))\big)^\dagger$ is dominated by an integrable function on $\mathbb{T}$.*

(iii) *If $h \in \mathcal{A}_X$, $h(0) \notin \operatorname{sing}(\omega)$, $t_0 \in \mathbb{T} \setminus h^{-1}(\operatorname{sing}(\omega))$ and $\varepsilon > 0$, then t_0 has a neighbourhood U in $\mathbb{C}$ and there is a local potential ψ in a neighbourhood of $h(U)$ such that for all sufficiently small arcs J in $\mathbb{T}$ containing t_0 there is a holomorphic map $F : D_r \times U \to X$, $r > 1$, such that $F(0, \cdot) = h|_U$ and*

$$\frac{1}{\sigma(J)} \int_J \big(H(F(\cdot, t)) + \psi(F(0,t))\big)\, d\sigma(t) \leq (EH + \psi)(h(t_0)) + \varepsilon.$$

Proof: By Proposition 2.2.10 we need to show that $EH \circ h$ is $h^*\omega$-subharmonic for every $h \in \mathcal{A}_X$, $h(\mathbb{D}) \not\subset \operatorname{sing}(\omega)$ and that EH is ω-upper semicontinuous.

The ω-upper semicontinuity of EH follows from Lemma 4.3.2 so we turn our attention to the subaverage property. We assume $\psi = \psi_1 - \psi_2$ is a local potential of ω defined on an open set U. As with plurisubharmonicity, ω-plurisubharmonicity is a local property so it is enough to prove the subaverage property for $h \in \mathcal{A}_U$, $h(0) \notin \mathrm{sing}(\omega)$. Our goal is therefore to show that

$$EH(h(0)) + \psi(h(0)) \leq \int_{\mathbb{T}} (EH \circ h + \psi \circ h)^{\dagger} \, d\sigma. \tag{4.7}$$

This is automatically satisfied if $EH(h(0)) = -\infty$, and since EH is ω-upper semicontinuous it can only take the value $+\infty$ on $\mathrm{sing}(\omega)$. We may therefore assume $EH(h(0))$ is finite. It is sufficient to show that for every $\varepsilon > 0$ and every continuous function $v : U \to \mathbb{R}$ with $v \geq (EH + \psi)^{\dagger}$, there exists $g \in \mathcal{A}_X$ such that $g(0) = h(0)$ and

$$H(g) + \psi(h(0)) \leq \int_{\mathbb{T}} v \circ h \, d\sigma + \varepsilon. \tag{4.8}$$

Then by definition of the envelope, $EH(h(0)) + \psi(h(0)) \leq \int_{\mathbb{T}} v \circ h \, d\sigma + \varepsilon$ for every v and ε, and (4.7) follows.

Let $r > 1$ such that h is holomorphic on D_r. In the proof of Theorem 1.2 in [26], Lárusson and Sigurdsson show that a function satisfying the subaverage property for all holomorphic discs in X not lying in a pluripolar set Z is plurisubharmonic not only on $X \setminus Z$ but on X. We may therefore assume that $h(\overline{\mathbb{D}}) \not\subset Z$.

Since $h(0) \notin \mathrm{sing}(\omega)$, we have $\psi_1 \circ h(0) \neq -\infty$ and $\psi_2 \circ h(0) \neq -\infty$. Then, by the subaverage property of the subharmonic functions $\psi_1 \circ h$ and $\psi_2 \circ h$, the set $h^{-1}(\mathrm{sing}(\omega))$ is of measure zero with respect to the arc length measure σ on $\mathbb{T}$. The set $h(\mathbb{T}) \setminus \mathrm{sing}(\omega)$ is therefore dense in $h(\mathbb{T})$ and by a compactness argument along with property *(iii)* we can find a finite number of closed arcs $J_1, \dots, J_m$ in $\mathbb{T}$, each contained in an open disc U_j centered on $\mathbb{T} \setminus \mathrm{sing}(\omega)$ and holomorphic maps $F_j : D_s \times U_j \to X$, $s \in]1, r[$ such that $F_j(0, \cdot) = h|_{U_j}$ and, using the continuity of v, such that

$$\underline{\int_{J_j}} \Big(H(F_j(\cdot, t)) + \psi(F(0,t)) \Big) \, d\sigma(t) \leq \int_{J_j} v \circ h \, d\sigma + \frac{\varepsilon}{4} \sigma(J_j). \tag{4.9}$$

We can shrink the discs U_j such that they are relatively compact in D_r and have mutually disjoint closure. Furthermore, by the continuity of v we may assume

$$\int_{\mathbb{T}\setminus\cup_j J_j} |v \circ h| \, d\sigma < \frac{\varepsilon}{4} \tag{4.10}$$

and by condition *(ii)* we may assume

$$\overline{\int_{\mathbb{T}\setminus\cup_j J_j}} H(h(w)) + \psi(h(w))\, d\sigma(w) < \frac{\varepsilon}{4}. \tag{4.11}$$

Our submersion Φ will be the projection $\mathbb{C}^2 \times X \to X$. The manifold in $\mathbb{C}^2 \times X$ where $\Phi^*\omega$ has a global potential will be a neighbourhood of the union of the graphs of h,

$$K_0 = \{(w, 0, h(w)); w \in \overline{\mathbb{D}}\},$$

and the graphs of the F_j's,

$$K_j = \{(w, z, F_j(z, w)); w \in J_j, z \in \overline{\mathbb{D}}\}.$$

By applying Lemma 4.3.1 to both ω_1 and ω_2 there is neighbourhood V of $K = \cup_{j=0}^m K_j$ with potentials Ψ_1 of $\Phi^*\omega_1$ and Ψ_2 of $\Phi^*\omega_2$. Then $\Psi = \Psi_1 - \Psi_2$ is a potential of $\Phi^*\omega$. The $\Phi^*\omega$-plurisubharmonicity of $E\Phi^*H$ given by condition *(i)* ensures that

$$E\Phi^*H(\tilde{h}(0)) + \Phi^*\psi(\tilde{h}(0)) \leq \int_{\mathbb{T}} (E\Phi^*H \circ \tilde{h} + \Phi^*\psi \circ \tilde{h})^\dagger \, d\sigma, \tag{4.12}$$

where $\tilde{h}$ is the lifting $w \mapsto (w, 0, h(w))$ of h to $V \subset \mathbb{C}^2 \times X$.

We know that $\Phi^*EH(\tilde{h}(0)) \leq E\Phi^*H(\tilde{h}(0))$ by inequality (2.10) and since $\Phi^*EH(\tilde{h}(0)) = EH(h(0)) \neq -\infty$ there is a disc $\tilde{g} \in \mathcal{A}_V$ such that $\tilde{g}(0) = \tilde{h}(0)$ and

$$\Phi^*H(\tilde{g}) \leq E\Phi^*H(\tilde{h}(0)) + \frac{\varepsilon}{4}. \tag{4.13}$$

Let $g = \Phi \circ \tilde{g}$ be the projection of $\tilde{g}$ to X, then $g(0) = h(0)$ and $H(g) = \Phi^*H(\tilde{g})$. By the definition of $\tilde{h}$ then the local potential $\Phi^*\psi$ of $\Phi^*\omega$ satisfies $\Phi^*\psi(\tilde{h}) = \psi(h)$. This along with inequalities (4.12) and (4.13) above implies that

$$H(g) + \psi(h(0)) \leq \int_{\mathbb{T}} (E\Phi^*H \circ \tilde{h} + \psi \circ h) \, d\sigma + \frac{\varepsilon}{4}. \tag{4.14}$$

For every $j = 1, \dots, m$ and $w \in J_j$ we have

$$E\Phi^* H(\tilde{h}(w)) \leq \Phi^* H((w, \cdot, F_j(\cdot, w))) = H(F_j(\cdot, w)),$$

because $z \mapsto (w, z, F_j(z, w))$ is a disc in K with center $\tilde{h}(w)$.

This means, by (4.9),

$$\int_{J_j} (E\Phi^* H(\tilde{h}) + \psi \circ h)\, d\sigma \leq \int_{J_j} v \circ h\, d\sigma + \frac{\varepsilon}{4}\sigma(J_j). \tag{4.15}$$

However if $w \in \mathbb{T} \setminus \cup_j J_j$ then

$$E\Phi^* H(\tilde{h}(w)) \leq \Phi^* H(\tilde{h}(w)) = H(h(w)),$$

where $\tilde{h}(w)$ and $h(w)$ on the right hand side are the constant discs at $\tilde{h}(w)$ and $h(w)$. This means, by (4.11), that

$$\int_{\mathbb{T} \setminus \cup_j J_j} (E\Phi^* H(\tilde{h}) + \psi \circ h)\, d\sigma \leq \frac{\varepsilon}{4}. \tag{4.16}$$

Then, first by combining inequality (4.14) with (4.15) and (4.16), and then by (4.10), we see that

$$H(g) + \psi(h(0)) \leq \int_{\cup_j J_j} v \circ h + \frac{\varepsilon}{4}\sigma(\cup_j J_j) + \frac{\varepsilon}{4} + \frac{\varepsilon}{4} \leq \int_{\mathbb{T}} v \circ h + \varepsilon.$$

This shows that the disc g satisfies (4.9) and we are done. □

Proof of Theorem 4.1.1 when $\varphi_2 = 0$

Proof: Finally we can prove Theorem 4.1.1 when $\varphi_2 = 0$ by showing that $H_{\omega,\varphi}$ satisfies the three condition in Theorem 4.3.3.

Condition *(i)* in 4.3.3 follows from the proof in Section 4.2. If $h \in \mathcal{A}_X$ and ψ is a local potential as in Theorem 4.3.3, then condition *(ii)* follows from the fact that $H(h(t)) + \psi(h(t)) = (\varphi(h(t)) + \psi_1(h(t))) - \psi_2(h(t))$ is the difference of an upper semicontinuous function and a plurisubharmonic function. The first term is bounded above on $\mathbb{T}$ and the second one is integrable since $h(\mathbb{D}) \not\subset \operatorname{sing}(\omega)$.

Let $h \in \mathcal{A}_X$, $\varepsilon > 0$ and $t_0 \in \mathbb{T} \setminus h^{-1}(\operatorname{sing}(\omega))$ be as in condition *(iii)* and ψ a local potential for ω in a neighbourhood V' of $x = h(t_0)$. Let $\beta > EH_{\omega,\varphi}(x) + \psi(x) + \varepsilon$. Then there is a disc $f \in \mathcal{A}_X$ such that $f(0) = x$ and $H_{\omega,\varphi}(f) + \psi(x) \leq \beta - \varepsilon/2$. By the proof of [25, Lemma 2.3] there is a neighbourhood V of x in X, $r > 1$ and a holomorphic map $\tilde{F} : D_r \times V \to X$ such that $\tilde{F}(\cdot, x) = f$ on D_r and $\tilde{F}(0, z) = z$ on V. Define $U = h^{-1}(V' \cap V)$ and $F : D_r \times U \to X$ by $F(s,t) = \tilde{F}(s, h(t))$. Then by (2.8),

$$\big(H_{\omega,\varphi}(F(\cdot,t)) + \psi(F(0,t))\big)^\dagger = \int_{\mathbb{T}} (\varphi + \psi)^\dagger \circ F(s,t)\, d\sigma(s). \tag{4.17}$$

Since the integrand is upper semicontinuous on $D_r \times U$, then (4.17) is an upper semicontinuous function of t on U by Lemma 2.3.1. That allows us, by shrinking U, to assume that

$$\big(H_{\omega,\psi}(F(\cdot,t)) + \psi(F(0,t))\big)^\dagger \leq H_{\omega,\varphi}(F(\cdot,t_0)) + \psi(F(0,t_0)) + \frac{\varepsilon}{2}$$

for $t \in U$. Then by the definition of $f = F(\cdot, t_0)$

$$\big(H_{\omega,\varphi}(F(\cdot,t)) + \psi(F(0,t))\big)^\dagger < EH_{\omega,\varphi}(x) + \psi(x) + \varepsilon, \quad \text{for } t \in U.$$

Condition *(iii)* is then satisfied for all arcs J in $\mathbb{T} \cap U$. □

We now finish the proof of our main theorem by showing how the function φ_2 can be integrated into ω and then previous results apply. So, subtracting the function φ_2 from φ_1 can be thought of as just shifting the class $\mathcal{PSH}(X,\omega)$ by $-dd^c\varphi_2$.

End of proof of Theorem 4.1.1

Proof: We define the current $\omega' = \omega - dd^c\varphi_2$ and use the bijection, $u' \mapsto u' - \varphi_2 = u$ between $\mathcal{PSH}(X,\omega')$ and $\mathcal{PSH}(X,\omega)$ from Proposition 2.2.6. Since the positive part of ω and ω' is the same, it is equivalent for φ_1 to be ω_1-upper semicontinuous and ω_1'-upper semicontinuous. Then Theorem 4.1.1 for the case when $\varphi_2 = 0$ can be applied to ω' and φ_1, and for every

$x \notin \operatorname{sing}(\omega') = \operatorname{sing}(\omega) \subset \varphi_2^{-1}(-\infty)$ we infer

$$
\begin{aligned}
& \sup\{u(x); u \in \mathcal{PSH}(X,\omega), u \le \varphi_1 - \varphi_2\} \\
= \; & \sup\{u'(x) - \varphi_2(x); u' \in \mathcal{PSH}(X,\omega'), u' - \varphi_2 \le \varphi_1 - \varphi_2\} \\
= \; & \sup\{u'(x); u' \in \mathcal{PSH}(X,\omega'), u' \le \varphi_1\} - \varphi_2(x) \\
= \; & \inf\{-R_{f^*\omega'}(0) + \int_{\mathbb{T}} \varphi_1 \circ f \, d\sigma; f \in \mathcal{A}_X, f(0) = x\} - \varphi_2(x) \\
= \; & \inf\{-R_{f^*\omega}(0) + R_{f^*dd^c\varphi_2}(0) - \varphi_2(x) + \int_{\mathbb{T}} \varphi_1 \circ f \, d\sigma; f \in \mathcal{A}_X, f(0) = x\} \\
= \; & \inf\{-R_{f^*\omega}(0) + \int_{\mathbb{T}} (\varphi_1 - \varphi_2) \circ f \, d\sigma; f \in \mathcal{A}_X, f(0) = x\}.
\end{aligned}
$$

The last equality follows from the Riesz representation (2.8) applied to the plurisubharmonic function φ_2, which gives $\varphi_2(x) = R_{f^*dd^c\varphi_2}(0) + \int_{\mathbb{T}} \varphi_2 \circ f \, d\sigma$. We also used the fact that $R_{f^*\omega}$ is linear in ω.

To finish the proof we need to show that the equality

$$
\begin{aligned}
& \sup\{u(x); u \in \mathcal{PSH}(X,\omega), u \le \varphi_1 - \varphi_2\} \\
& \qquad = \inf\{-R_{f^*\omega}(0) + \int_{\mathbb{T}} (\varphi_1 - \varphi_2) \circ f \, d\sigma; f \in \mathcal{A}_X, f(0) = x\} \quad (4.18)
\end{aligned}
$$

holds also on $\varphi_2^{-1}(-\infty) \setminus \operatorname{sing}(\omega)$.

The right hand side of (4.18) is ω-upper semicontinuous by Lemma 4.3.2, and it is equal to the function $EH_{\omega',\varphi_1} - \varphi_2$ on $X \setminus \operatorname{sing}(\omega')$. Now assume ψ is a local potential of ω, then $-\varphi_2 + \psi$ is a local potential for ω'. The functions $(EH_{\omega',\varphi_1} - \varphi_2 + \psi)^\dagger$ and $(EH_{\omega,\varphi} + \psi)^\dagger$ are then two upper semicontinuous functions which are equal almost everywhere, thus the same. Furthermore, since EH_{ω',φ_1} is ω'-plurisubharmonic we see that $EH_{\omega,\varphi}$ is ω-plurisubharmonic. This shows that $EH_{\omega,\varphi}$ is in the family $\{u \in \mathcal{PSH}(X,\omega), u \le \varphi\}$, and since $\sup\{u \in \mathcal{PSH}(X,\omega); u \le \varphi\} \le EH_{\omega,\varphi}$ by (4.4) we have an equality not only on $X \setminus \operatorname{sing}(\omega')$ but on $X \setminus \operatorname{sing}(\omega)$, i.e. (4.18) holds on $X \setminus \operatorname{sing}(\omega)$. □

4.4 Relative extremal function

One of the most fruitful applications of the Poisson disc formula (3.2) is for the relative extremal function. The *relative extremal function* u_E for a Borel set $E \subset X$ is defined as

$$u_E(x) = \sup\{u(z); u \in \mathcal{PSH}(X), u \leq 0, u|_E \leq -1\}.$$

In our notation this corresponds to having $\varphi = -\chi_E$, where χ_E is the characteristic function of the set E. If E is open then $-\chi_E$ is upper semicontinuous and Theorem 3.1.6 applies,

$$u_E(x) = \{-\sigma(f^{-1}(E) \cap \mathbb{T}); f \in \mathcal{A}_X, f(0) = x\}.$$

It follows from this, see [25, Theorem 7.4], that for a compact set K and a point p in $\mathbb{C}^n$ the following is equivalent

(i) p is in the polynomial hull of K.

(ii) There is an open ball B containing K and p, such that for every neighbourhood U of K and every $\varepsilon > 0$ there is a disc $f \in \mathcal{A}_B$ with $f(0) = p$ and
$$\sigma(\mathbb{T} \setminus f^{-1}(U)) < \varepsilon.$$

Another application of 3.1.6 is due to Wold [38]. Duval and Sibony [9] proved a characterization of the polynomial hull by using the presence of certain currents. In [38] Wold gives a very interesting proof of their result using the Poisson disc formula.

It will be interesting to see if the disc formulas for the quasiplurisubharmonic functions presented here will give similar applications in the future as those of Poletsky's formula.

To start with we can look at the relative extremal function for quasiplurisubharmonic function, $u_{E,\omega}$. It is defined analogously to u_E, that is

$$u_{E,\omega}(x) = \sup\{u(x); u \in \mathcal{PSH}(X, \omega), u \leq 0, u|_E \leq -1\}.$$

If E is open then $\varphi = -\chi_E$ is upper semicontinuous. Moreover φ is ω_1-upper semicontinuous, because if ψ_1 is a local potential of ω_1 then the sum $-\chi_E + \psi_1$ of two upper semicontinuous function is upper semicontinuous. Then using Theorem 4.1.1 we get the following

$$u_{E,\omega}(x) = \inf\{-R_{f^*\omega}(0) - \sigma(f^{-1}(E) \cap \mathbb{T}); f \in \mathcal{A}_X, f(0) = x\}.$$

Index of Notation

$D_r(a)$	open disc in $\mathbb{C}$ with center a and radius r
$D_r = D_r(0)$	
$\mathbb{D} = D_1$	unit disc
$\mathbb{T} = \partial\mathbb{D}$	unit circle
σ	arc length measure on $\mathbb{T}$ normalized to 1
λ	Lebesgue measure
X	open subset of $\mathbb{C}^n$ or a complex manifold
$\mathcal{A}_X$	set of closed analytic discs in X, that is analytic functions from a neighbourhood of $\overline{\mathbb{D}}$ into X
$\mathcal{PSH}(X)$	set of plurisubharmonic functions on X which are not identically $-\infty$
ω	closed $(1,1)$-current, either positive or the difference of two closed positive currents
$\mathcal{PSH}(X,\omega)$	set of ω-plurisubharmonic functions on X which are not identically $-\infty$
ψ	local potential of ω, $dd^c\psi = \omega$
$\text{sing}(\omega)$	singular set of ω, union of all $\psi^{-1}(-\infty)$
$f^*\omega = dd^c(\psi \circ f)$ (locally)	pullback of f by ω

$G_{\mathbb{D}}(z,w) = \frac{1}{2\pi} \log \lvert \frac{z-w}{1-z\overline{w}} \rvert$	Green function for the unit disc
$R_{f^*\omega}(z) = \int_{\mathbb{D}} G_{\mathbb{D}}(z,\cdot)\, d(f^*\omega)$	Riesz potential of $f^*\omega$
$H_\varphi(f) = \int_{\mathbb{T}} \varphi \circ f \, d\sigma$	Poisson disc functional
$H_{\omega,\varphi} = \int_{\mathbb{T}} \varphi \circ f \, d\sigma - R_{f^*\omega}(0)$	ω-Poisson disc functional
$\mathcal{F}_{\omega,\varphi}$	$u \in \mathcal{PSH}(X,\omega)$ such that $u \leq \varphi$
$\mathcal{F}_\varphi = \mathcal{F}_{0,\varphi}$	$u \in \mathcal{PSH}(X)$ such that $u \leq \varphi$
$\mathbb{P}^n$	complex projective space
$\overline{0}$	zero vector in $\mathbb{C}^n$
χ_E	characteristic function of a set E

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