17th and 18th century European arithmetic in an 18th century Icelandic manuscript

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ABSTRACT

Icelandic arithmetic books from the 18th century, printed and in manuscripts, adhered to the European structure of practical arithmetic textbooks, formed in the late Middle Ages: The number concept, numeration, the four operations in whole numbers and fractions, monetary and measuring units, extraction of roots, ratio, progressions and proportions. A manuscript textbook, *Arithmetica – That is reckoning art*, dated in 1721, deviates from the general model in that it does not treat monetary and measuring units but includes theoretical sections on the number concept and common notions on arithmetic. No specific model for the manuscript has been spotted, while similarities to works by authors as Ramus, Stevin, Suevus, Meichner and Euler were found.

1 Introduction

According to Swetz (1992), basic types of texts on European arithmetics up to the fifteenth century were *theoretical tracts*, transmitting neo-Pythagorean speculations of Nicomachus of Gerasa (ca. 100); *computi*, ancestors of almanacs; *abacus arithmetic* for Roman numerals and *algorisms*, evolving from descendant translations of works of Al-Kwarizmi (ca. 825), such as *Carmen de Algorismo* (c. 1200) by Alexander Villa Dei. *Abacus arithmetic* and *algorisms* that discussed problems related to trade and commerce were called *Practica*, reflecting their practical use. Their influence prevailed in the teaching of arithmetic until the beginning of the 20th century.

Commercial arithmetics were not intended to be *theoretical tracts* devoted to philosophical speculations on the nature of number, rather, they were handbooks on readily usable mathematics. Generally, their content was numeration, monetary and measuring units, the four operations in whole numbers and fractions, i.e. addition, subtraction, multiplication, division, and often extraction of roots, progressions and proportions in the form of *Regula Trium*, the ‘Rule of Three’.

There were two cathedral schools in Iceland until 1800. Their first regulations, issued in 1743, prescribed knowledge in the four arithmetic operations in whole numbers and fractions (Jónsson, 1893). The first substantial printed arithmetic textbooks in Icelandic were published in the 1780s, while textbooks in manuscripts, written in the vernacular, were dispersed from person to person in early modern times. These books generally belonged to the *Practica*.

The existence of the manuscripts shows that Icelanders made efforts to adapt European education and literature to their language as had been customary since the Middle Ages. Before the Lutheran Reformation, in 1550, translations had been made from Latin. Later, texts from the Northern-European protestant countries, written in e.g. German or Danish, seem to have been preferred for translation.

The following manuscripts of arithmetic textbooks from the seventeenth century onwards are preserved in the National and University Library of Iceland – manuscript department:
<table>
<thead>
<tr>
<th>Source</th>
<th>Text</th>
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<tbody>
<tr>
<td>Jónsson, 1892–96. <em>Algorithmus</em>, AM 544, 4to. (1306–08), p. 418.</td>
<td>Even digits are four; 2, 4, 6, 8, and uneven another four; 3, 5, 7, 9. But one is neither as it is not a number but the origin of all numbers.</td>
</tr>
<tr>
<td>Ramus, 1569. <em>Arithmetica libri Duo</em>, p. 1.</td>
<td>Numerus est, secundum quem unum quodque numeratur: ut secundum unitatem unum, secundum binarium duo, secundum ternarium tria; &amp; sic deinceps omnes numeri: Itaque numerus est unitatis aut multitudinis: potesque esse minimum, ut unitas: ...</td>
</tr>
<tr>
<td>Stevin, 1585. <em>L’Arithmetique</em>, edited by Struik, 1958. pp. 494–504.</td>
<td>La partie est de mesime matiere qu’est son enier; Vnité est partie de multitude d’vnitez. Ergo l’vnité est de mesime matiere qu’est la multitude d’vnitez. Mais la matiere de multitude d’vnitez est nombre, Doncques la matiere d’vnité est nombre. Et qui le nie, faict comme celui, qui nie qu’vne piece de pain fait du pain. Nous pourrions aussi dire ainc: Si du nombre donne l’on soubtraict nul nombre, le nombre donne demeure. Soit trois le nombre donne, et de mesme soubrahons vn, qui n’est point nombre comme tu veux. Doncques le nombre donne demeure, c’est à dire qu’il y restera encore trois, ce qui est absurd.</td>
</tr>
<tr>
<td>Cocker’s ARITHMETICK (1715, first published 1677)</td>
<td>The part is of the same matter as its whole, unity is part of a multitude of unities, hence unity is of the same matter as the multitude of unities, But the matter of a multitude of unities is number. Hence the matter of unity is number, Who denies this believes like one who denies that a piece of bread is bread. We can also say: If we subtract no number from a given number, then the given number remains. If three is the given number, and if from this we subtract one, which — as you claim — is no number, then the given number remains, that is thre remains, which is absurd.</td>
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</table>

*Arihtmetica — Margir af þeim Lærdu Mathematicis* Many of the learned mathematicians
Table 1. Elaborations on the number concept in a choice of textbooks.

Clearly, the author of *Arithmetica* is aware of Stevin’s reasoning that one, the unit, is indeed a number.

### 3.2. Common Notions

The *Common Notions* in Euclid’ *Elements*, referring to magnitudes, were found in theoretical books on arithmetic, while they were seldom seen in the *Practicae*. *Arithmetica* contains thirteen *Common Notions*: ‘Incontestable procedures, that is, so obvious rules that each one’s understanding must acknowledge them (p. 7).’

Three of these axioms match Euclid’s *Common Notions* as presented in Heath’s 1956 publication; while other three match Mersenne’s 1644 publication of the *Elements*, Book Seven, see Table 2, below. Both lists of thirteen *Common Notions* refer to measuring, *metitur*, of *metior* : measure, in, Mersenne’s in particular.

The *Common Notions* as listed in Heath’s edition:

1. Things which are equal to the same thing are also equal to one another.
2. If equals be added to equals, the wholes are equal.
3. If equals be subtracted from equals, the remainders are equal.
4. Things which coincide with one another are equal to one another.
5. The whole is greater than the part. (Euclid, 300 BC, vol. 1, p. 155)

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*Arithmetica—That is Reckoning Art.*  
1721/1750, p. 11.
1. Quicumque eiusdem, vel aequulum aequamultiplices fuerint & ipsi inter se sunt aequales.
2. Quorum idem numerus aequamultiplex fuerit, vel quorum aequamultiplices fuerint aequales, & ipsi inter se aequales sint.
3. Quicumque eiusdem numeri, vel aequulum eadem pars, vel eadem partes fuerint, & ipsi inter se sunt aequales.
4. Quorum idem, vel aequales numeri eadem pars, vel eadem partes fuerint, & ipsi inter se sunt aequales.
5. Omnis numeri pars est unitas ab eo denominata, binarii cum numeri unitas pars est ab ipso binario denominata, que dimidia dicitur, ternarii vero unitas est pars, que a ternario denominata tertia dicitur: quaternarii quadra, & tia in alia.
6. Unitas omnem numerum metitur per unitates que in ipso sunt.
7. Omnis numerus seipsum metitur.
8. Si numerus metiatur numerum, & ille, per quem metitur, eundem metitur per eae, que sunt in metente, unitates.
9. Quicumque numerus alium metitur, multiplicans eum, vel multiplicatus ab eo, per quem metitur, illum ipsum producit.
10. Si numerus numerum alium multiplicant, aliquem producend, multiplicant quidem productum metitur per unitates que sunt in multiplicato: multiplicatus vero metitur eundem per unitates que sunt in multiplicante.
11. Quicumque numerus metitur duo, vel plures, metitur quoque eum, qui ex illis componitur.
12. Quicumque numerus metitur aliquem, metiatur quoque eum, quem ille ipse metitur.
13. Quicumque numerus metitur totum & ablatum, eodem reliquum metitur.

Table 2: Comparison of Common Notions, listed in Mercenne’s edition of Euclid’s Elements, Book Seven, to those in Arithmetica – That is Reckoning Art.

Mersenne’s edition also contains ten Axiomata, Communes Notiones decem, in Book One (p. 3), some of which correspond to the Common Notions in Heath’s edition and others to no. 2 and 10 in Arithmetica.

3.3 Numeration

Arithmetic textbooks explain how to write numbers by place value notation, often demonstrated by large numbers. All the five examples on numeration in Arithmetica have corresponding examples in the German textbooks.

<table>
<thead>
<tr>
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<tbody>
<tr>
<td>The number of years from the creation of the world until Christ was born: 3,970 years (p. 4).</td>
<td>3,970 years (p. 4)</td>
<td>3,970 years (p. 3)</td>
<td></td>
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<tr>
<td>The cost of the building of King</td>
<td>13,695,380,050</td>
<td>13,695,380,050</td>
<td>13,695,380,050</td>
</tr>
<tr>
<td>Salomon’s temple: 13,695,380,050 Crowns (p. 5)</td>
<td>Crowns (p. 4)</td>
<td>Crowns (p. 2)</td>
<td>Crowns (p. 22)</td>
</tr>
<tr>
<td>The yearly cost of the government of the Emperor Augustinus: 1,200,000 Crowns (p. 6)</td>
<td>12,000,000 Crowns (p. 5)</td>
<td>12,000,000 Crowns (p. 2-3)</td>
<td>1,200,000 Crowns (p. 22)</td>
</tr>
<tr>
<td>The fortune of Sardanapalus, the King of Assyria: 145,000,000,000 Guilders (p. 6)</td>
<td>154,000,000,000,000 Crowns (p. 6-7)</td>
<td>154,000,000,000,000 Crowns (p. 3-4)</td>
<td>145,000,000,000,000 Guilders (p. 22)</td>
</tr>
<tr>
<td>The number of grains of sand to fill the world: computed by Archimedes as $10^{63}$, the unit with 63 zeros (p. 6)</td>
<td>$8 \times 10^{63}$ (p. 7)</td>
<td>$10^{63}$ (p. 22)</td>
<td></td>
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</table>

Table 3. Examples on numeration in four textbooks.

Two of the above examples match Euler’s text better than the two *Arithmetica Historica*; Sardanapalus’s fortune of 145 billion vs. 154 billion Guilders, and 1.2 million vs. 12 million Crowns of Emperor Augustus’s yearly cost. This supports a hypothesis of a missing ancestor to both Euler’s and the anonymous Icelandic text, as it is highly unlikely that one of them is copied from the other.

### 3.4. Addition and subtraction

The *Arithmetica* explains addition of multi-digit numbers in a well known fashion, still practiced, beginning from the right side, adding up the units, and proceeding to the left. All its examples, but one, listed in Table 4, are contained in the foreign sources by Suevus, Meichsner and Euler, which have more examples.

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<tbody>
<tr>
<td>The age of Methusalem, who according to Holy Scripture was 187 years old when he begat Lamech, whereafter he lived for 782 years, to the age of 969 (p. 10).</td>
<td>969 years (p. 22-23)</td>
<td>969 years (p. 5)</td>
<td>969 years (p. 32)</td>
</tr>
<tr>
<td>Example revealing the ‘current’ year: I want to know how many years there are since the poet Homerus lived. Aulus Gellius writes that he lived 160 years before Rome was built, but the city of Rome was built 752 years before the birth of Christ, and the number of years since Christ was born until now is 1721 years.’ (p. 10)</td>
<td>Number of years since the creation of Earth, 3,970 + 1,590 (current year) (p. 47)</td>
<td>Number of years since the creation of Earth, 3,970 + 1,625 (current year) (p. 4)</td>
<td>The number of years since Homerus lived, 160 plus 752 plus the current year, 1737 (p. 33)</td>
</tr>
<tr>
<td>The number of the Greeks, 880,000, and Trojans, 686,000, deceased in the Trojan War was in total 1,566,000. (p. 11)</td>
<td>1,566,000 men (p. 45-46)</td>
<td>1,566,000 men (p. 32)</td>
<td></td>
</tr>
<tr>
<td>Four men owe me 6,952, 8,346, 6,259 and 5,490 each, a total of 27,047 monetary units.</td>
<td>27,047 Rubles (p. 32)</td>
<td></td>
<td></td>
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</table>

Table 4. Comparison of addition examples in four textbooks.

The examples of the age of Methusalem and of four men owing 27,047 units with the same four amounts, in that case six-dollars, are also found in *Arithmetica Islandica*, dated on its front page in 1716. Several examples count the present year as 1733, possibly a date of its extant copy (pp. 21, 36, 64, 66).
The example of the poet Homerus, revealing the date of all the four books, is also found in Ramus’s (1569) *Arithmeticae libri Duo*, mentioned earlier. ‘Ut si quaeratur quamprimem vixeret Homerus, & respondeatur e Gellio, 160 annis ante conditam Romam, qua condita sit ante natum Christum annis 752. Christum vero natum anno abhinc 1567. addantur hi tres numeri : Summa inductionis indicans Homerum annos abhinc 2479 floruisse, erit hoc modo’ (p. 3). This example testifies that Ramus’s book was written in 1567.

Figure 1: The example in *Arithmetica – That is Reckoning Art* on the number of years since Homerus lived.

Only three subtraction examples are found in our *Arithmetica* and none of them is historical. A demonstration follows on how subtraction and addition can be used for testing each other.

3.5. Multiplication

*Arithmetica – That is Reckoning Art* has several multiplication problems, similar or identical to other sources, such as on the circumference of the earth, see Table 5 below. *Arithmetica* gives an example on the average number of hours in a year, see Figure 2, as does Suevus, and Meichsner too in another book from the same year, *Arithmetica Poetica*. Euler counts the number of hours in a regular year.

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<tbody>
<tr>
<td>Circumference of the earth, 360°·15 = 5,400 miles (p. 23).</td>
<td>360°·15 = 5,400 miles (p. 128).</td>
<td>360°·15 = 5,400 miles (p. 14–15).</td>
<td>360°·105 = 37,800 Werste (p. 69).</td>
</tr>
<tr>
<td>Number of hours in a year, 365·24 + 6 = 8,766 hours (p. 22).</td>
<td>(52·7+1)·360 +6 = 8766 hrs. (p. 127).</td>
<td>52·7·360 = 8,736 hrs. in 52 weeks in <em>Arithmetica Poetica</em> (1625b), (p. 17).</td>
<td>365·24 = 8760 hours (p. 69).</td>
</tr>
<tr>
<td>Size of a military group, 264·100 = 26,400 soldiers (p. 27).</td>
<td></td>
<td></td>
<td>156·97 = 15,132 soldiers (p. 69).</td>
</tr>
<tr>
<td>Fortune in King David’s grave, 3000·600 = 180,000 Crowns (p. 28).</td>
<td>3000·600 = 180,000 Crowns (p. 171–172).</td>
<td></td>
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</tbody>
</table>

Table 5. Comparison of multiplication examples.

The number of soldiers in the military group exceeds the number of adult men in Iceland in the early 18th century, where no army existed.
Figure 2. Computing hours in a year in *Arithmetica – That is Reckoning Art*.

### 3.6. Division

Only two division problems, both concerning calendar computations, are found in other books. Dates of the books are again revealed, as shown in Table 6.

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<tbody>
<tr>
<td>Is the coming year a leap year? Coming year, 1721:4 = 430, remainder 1 (p. 35)</td>
<td>1591:4 = 397, remainder 3 (p. 175)</td>
<td>Check on which of the years 1620–1624 were leap-years (p. 25)</td>
<td></td>
</tr>
<tr>
<td>Golden number of a year (1622+1):19 = 85, remainder 8 (p. 38)</td>
<td>(1591+1):19 = 83, remainder 15 (p. 176)</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Table 6. Comparison of division examples.

The date 1622 in *Arithmetica* could point to copying a book dated that year. Problems of this kind are not found in Euler’s *Rechenkunst*.

Figure 3. Dividing 17,088 by 48 in *Arithmetica – That is Reckoning Art*. The divisor is written below the dividend and disappears in a mess of different products.

### 3.7. Square Root and the Theorem of Pythagoras

The first example on extracting a square root in *Arithmetica – That is Arithmetic Art* concerns a general arranging in a square a group of 54,756 soldiers, exceeding the population of Iceland.

Figure 4. Extracting square root.
In continuation extracting square root is applied to the Pythagorean Theorem, a rule not found in other textbooks inspected. No tower of the kind drawn in the manuscript existed in Iceland, while stories of kings, queens, knights and princesses in castles were a favoured branch of literature.

Root extraction is not contained in the books by Suevus, Meichsner and Euler, while e.g. Gemma Frisius’s (1567) *Arithmeticae practicae methodus facilis* contains extractions of square and cube roots.

![Figure 5. Demonstration of the Pythagorean Theorem:](image)

The height of the tower AB is 30 feet and the width of the dike BC is 28 feet.

\[
30^2 + 28^2 = 1684
\]

\[
1684 - 41^2 = 3
\]

The length of the ladder AC is 41 and \(3/(2\cdot41 + 1)\), that is 41 3/83 feet.

### 3.8. The remaining text

*Arithmetica – That is Reckoning Art* continues with a thorough treatment of fractions and their four operations, and a large section on the *Regula Trium*, applied to proportional problems for centuries (Troppke, 1980, pp. 359–378). The manuscript ceases abruptly late in that section, which reminds the reader that it is a copy and not an original.

The topic of fractions is not contained in the textbooks by Suevus and Meichsner, and *Regula Trium* does not exist in Euler’s book. Examples using the rule, matching those in our *Arithmetica*, have not been found in Suevus’s and Meichsner’s books.

### 4. Summary and conclusions

The quotations and comparisons above witness that the European tradition of arithmetic textbooks was practiced in Iceland. The Icelandic *Arithmetica – That is Reckoning Art* has clear connections to the books *Arithmetica Historica, Die lübliche Rechenkunst* by Suevus (1593) and *Arithmetica Historica. Das ist: Rechenkunst* by Meichsner (1625a), both originating in Lutheran protestant towns.

There are even more examples common to Euler’s *Einleitung zur Rechenkunst* than the other two books. Euler’s book contains, however, neither root extractions nor the *Regula Trium*, and its second part is devoted to ‘benanntien Zahlen’, monetary and measuring units, which *Arithmetica* does not touch upon. Moreover, Euler’s book
was published in 1738, while the Icelandic *Arithmetica* probably origins in 1721, when Euler was 14 years old.

Probably there existed more books of similar origin. More work is needed to trace the origin of *Arithmetica – That is Reckoning Art*.

**Acknowledgement**
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**REFERENCES**

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