ISOMORPHISMS BETWEEN CONSECUTIVE PATTERN CLASSES

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Isomorphisms between consecutive pattern classes

by

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Abstract

We consider isomorphisms between consecutive pattern classes. This was done for classical patterns classes by Albert et al. in 2013. Smith showed in 2006 that there are no non-trivial automorphisms of the full poset of classical pattern containment. We show that the same holds for the full poset of consecutive pattern containment. Smith also showed that the set of non-recognizable permutations is finite in the classical case. We will show that this set is infinite in the consecutive case. Then we provide a consecutive pattern classes whose growth rate approaches 1 with a continuum of automorphisms, which is different from the classical case.

Einsmótanir milli flokka samliggjandi mynstra

Lína Viðarsdóttir

Janúar 2015

Útdráttur

Í þessari ritgerð er fjallað um einsmótanir milli flokka samliggjandi mynstra. Einsmótanir milli flokka klassískra mynstra voru rannsakaðar af Albert, Atkinson og Claesson árið 2013. Árið 2006 sýndi Smith að einu sjálfmótanir klassíska mynstra hlaðmengisins væru einföldu samhverfurnar. Við sýnum fram á að það sama gildi fyrir hlaðmengi samliggjandi mynstra. Smith sýndi einnig fram á að mengi óþekkjanlegra umraðana er endanlegt þegar um ræðir klassísk mynstur. Við sýnum hins vegar fram á að í tilfelli samliggjandi mynstra er mengi óþekkjanlegra umraðana óendanlegt. Einnig sýnum við að til er samliggjandi mynstra flokkur með vaxtarhraða sem nálgast 1 sem hefur óteljanlega margar sjálfmótanir, en þetta er frábrugðið því sem gerist í tilfelli klassískra mynstra.

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Chapter 1

Introduction

A *permutation* of length n is a string of length n whose letters are the integers from the set $\{1, 2, \dots n\}$, each appearing only once. The string $\pi = 53214$ is a permutation of length 5. The set of all permutations of length n is denoted with S_n. Let $\pi = \pi_1\pi_2 \dots \pi_n$ be a permutation. The first $n-1$ letters of π, $\pi_1\pi_2 \dots \pi_{n-1}$, make up the *prefix* of π and the last $n-1$ letters of π, $\pi_2\pi_3 \dots \pi_n$ make up the *suffix* of π.

Let π be a string of length n with distinct letters from $\{1, 2, \dots, m\}$, where $m \geq n$. The *flattening* of π, denoted $\pi^{\downarrow}$, is a permutation where the smallest letter in π has been replaced with the letter 1, the next smallest letter replaced with 2 and so on until the largest letter in π is replaced with the letter n. The flattening of the string 5724 is 3412. Two strings π and π' are *similar* or have the same *relative order* if $\pi^{\downarrow} = \pi'^{\downarrow}$, denoted $\pi \sim \pi'$. The flattening of both 5724 and 7814 is 3412 and thus the strings 5724 and 7814 are similar.

Let $\pi = \pi_1\pi_2 \dots \pi_n$ be a string. A *lifting* of π by the integer a is the string $\pi^{\uparrow a} = \pi'_1\pi'_2 \dots \pi'_n$, where

$$\pi'_i = \begin{cases} \pi_i & \text{if } \pi_i < a \\ \pi_i + 1 & \text{if } \pi_i \geq a. \end{cases}$$

A *lowering* of π by the integer a is the string $\pi^{\downarrow a} = \pi'_1\pi'_2 \dots \pi'_n$, where

$$\pi'_i = \begin{cases} \pi_i & \text{if } \pi_i < a \\ \pi_i - 1 & \text{if } \pi_i \geq a. \end{cases}$$

We use these operations, lifting and lowering, and the notation, the uparrow and downarrow, also for sets of strings. In those cases, the operation is used on all the strings in the set. In general, let f be a map on strings and let A be a set of strings. Then A^f is a set where the map f has been applied to all the strings in the set A.

An *occurrence* of a *classical pattern*, p, in a permutation π is a subsequence of letters in π that have the same relative order as the letters in p. If we look at the permutation 31524 we can see that the letters 354 have the same relative order as the classical pattern 132, so 354 is an occurrence of the classical pattern 132 in the permutation 31524. In fact the permutation has two other occurrences of this classical pattern, 132. These occurrences are 152 and 154. A permutation that has an occurrence of a pattern is said to *contain* the pattern.

Definition 1. Let π be a permutation of length n. The *classical shadow* of π is the set

$$\Delta(\pi) = \{p \in S_{n-1} : p \text{ is a classical pattern in } \pi\}.$$

As an example we have $\Delta(31524) = \{3142, 2143, 3124, 2413, 1423\}$. A permutation that does not contain a pattern is said to *avoid* the pattern. The set of all permutations that avoid a pattern p is denoted by $\mathrm{Av}(p)$.

A set C of permutations is a *classical pattern class* if every for every permutation in C it holds that every classical pattern contained in the permutation is an element in C. The set of all permutations is a partially ordered set under the classical pattern containment order. We will call it the *classical poset*, denoted S.

Definition 2. Let A and B be classical pattern classes. A map $f : A \to B$ is an *isomorphism* if the classical pattern p is contained in $\pi \in A$ if and only if p^f is a classical pattern contained in $\pi^f \in B$. If $A = B$, the map f is called an *automorphism*.

In terms of the classical shadow an equivalent definition is as follows. A bijective map is an isomorphism if and only if $\Delta(\pi)^f = \Delta(\pi^f)$.

Definition 3. A permutation, π is said to be *recognizable* by its shadow if no other permutation has the the same shadow as π. A permutation that is not recognizable by its shadow is said to be *non-recognizable*.

Simion and Schmidt observed in [9] that the classical poset has eight automorphisms induced by the eight symmetries of the square. These symmetries are the identity map, reverse, complement, inverse and compositions of those maps. It was assumed for many years that these were the only automorphisms of the classical poset but it was not proved until 2006 by Smith in [10]. Smith also showed that all permutations of length at least five are recognizable by their classical shadow, implying that the set of non-recognizable permutations is finite. Then in 2013, instead of looking at the full classical poset and automorphisms on the poset Albert, Atkinson and Claesson looked at pattern classes and isomorphisms between those classes. In [1] they studied the following question:

> What order-preserving isomorphisms are there between pattern classes beyond the restrictions of the eight symmetries?

They were able to give a complete answer to this question by finding all maximal pattern classes and maximal isomorphisms between them. Here 'maximal' means that these pattern classes could not be extended to a larger class.

Our aim for this thesis was to do similar research as Smith and Albert et al. did for the classical poset but for the consecutive poset. We will start by introducing consecutive patterns and the consecutive shadow, which is the basis of the thesis. Then in the following chapters we will go through our main results. In Chapter 3 we show that the set of non-recognizable permutations is infinite. In Chapter 4 we show that there are no non-trivial automorphisms of the full consecutive poset. Then in Chapter 5 we find a consecutive pattern class, whose growth rate converges to 1, that has a continuum of automorphisms. Thus we show that in some sense the consecutive shadow behaves the same way as the classical one, and in some sense it behaves differently.

Chapter 2

Preliminaries

An occurrence of a *consecutive pattern* p in a permutation π is a sequence of adjacent letters in π whose letters are in the same relative order as the letters in p.

Example 4. The permutation 31524 has one occurrence of the consecutive pattern 132, the subsequence 152.

As in the case of classical pattern containment the set of all permutations is a partially ordered set under the consecutive pattern containment order, called the *consecutive poset*.

Definition 5. Let π be a permutation of length n. The *consecutive shadow* of π is the set

$$\Delta(\pi) = \{p \in S_{n-1} : p \text{ is a consecutive pattern in } \pi\}.$$

Note that this definition is an analogue to Definition 1, for the classical shadow. From now on we will mostly be working with the consecutive shadow (not the classical one) so for shorthand we will call it the shadow. If there is any doubt we will state which one we are talking about. We also use the same symbol for the consecutive shadow as we used for the classical one.

Given a permutation $\pi = \pi_1\pi_2\ldots\pi_n$ there are at most two consecutive patterns of length $n-1$ contained in π. These patterns are $(\pi_1\pi_2\ldots\pi_{n-1})^{\downarrow\pi_n}$ and $(\pi_2\pi_3\ldots\pi_n)^{\downarrow\pi_1}$. Therefore

$$\begin{aligned}\Delta(\pi) &= \{(\pi_1\pi_2\ldots\pi_{n-1})^{\downarrow\pi_n}, (\pi_2\pi_3\ldots\pi_n)^{\downarrow\pi_1}\} \\ &= \{\pi_1\pi_2\ldots\pi_{n-1}, \pi_2\pi_3\ldots\pi_n\}^{\downarrow}.\end{aligned}$$

Example 6. The permutation $\pi = 14263758$ has the consecutive shadow $\Delta(14263758) = \{1426375, 3152647\}$.

To make it easier to talk about permutations and their consecutive shadow we use the following terminology.

Definition 7. Let π be a permutation whose consecutive shadow is $\Delta(\pi) = \{\alpha, \beta\}$. The permutations α and β are said to be π's *parents* and π their *child*. Two permutations that have a child or children together are said to be *mates* and they form a *couple*. Below we will show that a couple can either have one, two or three children (with the exception of one occurrence of *quads*). So a permutation can either be an *only child*, a *twin* or a *triplet*. A permutation which is a twin or a triplet is said to have *siblings*.

Example 8. Let $\pi = 14263758$. By Example 6 we have

$$\Delta(14263758) = \{1426375, 3152647\},$$

so the permutations 1426375 and 3152647 are π's parents. The permutation $\pi' = 31527486$ has the same shadow as π and this makes π and π' siblings. In the next chapter we are given a method (Proposition 18) we can use to see that π and π' do not have any other sibling, which shows that they are twins.

The monotone permutations, $\text{id}_n = 12\dots n$ and $\text{id}^{\text{r}}{}_n = n(n-1)\dots 1$, behave differently from other permutations in terms of the consecutive shadow as the following example demonstrates.

Example 9. The monotone permutations have the shadows

$$\begin{aligned}
\Delta(\text{id}_n) &= \{(12\dots(n-1))^{\downarrow n}, (23\dots n)^{\downarrow 1}\}\\
&= \{12\dots(n-1)\}\\
&= \{\text{id}_{n-1}\},\\
\Delta(\text{id}^{\text{r}}{}_n) &= \{(n(n-1)\dots 2)^{\downarrow 1}, ((n-1)(n-2)\dots 1^{\downarrow n}\}\\
&= \{(n-1)(n-2)\dots 1\}\\
&= \{\text{id}^{\text{r}}{}_{n-1}\}.
\end{aligned}$$

The monotone permutations are the only permutations where

$$(\pi_1\pi_2\dots\pi_{n-1})^{\downarrow\pi_n} = (\pi_2\pi_3\dots\pi_n)^{\downarrow\pi_1}.$$

So all permutations except the monotone ones have two parents. Since the monotone permutations are the only permutations that have only one parent and id and id^{r} do not have the same parent (with the exception when $n = 2$) they cannot be siblings and therefore the monotone permutations are only children.

As in the classical case we define isomorphisms between consecutive pattern classes. The definition is as follows.

Definition 10. Let A and B be consecutive pattern classes. A map $f : A \to B$ is an *isomorphism* if the consecutive pattern p is contained in $\pi \in A$ if and only if p^f is a consecutive pattern contained in $\pi^f \in B$. If $A = B$, the map f is called an *automorphism*.

In terms of the consecutive shadow this gives that a bijective map is an isomorphism if and only if $\Delta(\pi)^f = \Delta(\pi^f)$.

Chapter 3

Recognizable permutations

In this chapter we show that the set of non-recognizable permutations is infinite in the consecutive case. That is we will prove the following theorem.

Theorem 11. *The set of non-recognizable permutations is infinite.*

This result is different from the classical case. If you recall from the Introduction, Smith showed that all permutations of length at least five are recognizable by their classical shadow. That is, the set of non-recognizable permutations is finite in the classical case.

The non-recognizable permutations are the permutations that share a shadow with another permutation that is, permutations that have siblings. So we would like to know how many permutations have siblings, and moreover we would like to describe these permutations. In order to do so we must first find out which permutations form a couple and how many children they can have. This depends on the definition below.

Definition 12. Let $\alpha = \alpha_1\alpha_2 \dots \alpha_n$ and $\beta = \beta_1\beta_2 \dots \beta_n$ be two permutations.

1. If one of the following similarities,

$$\alpha_1\alpha_2 \dots \alpha_{n-1} \sim \beta_2\beta_3 \dots \beta_n, \tag{3.1}$$

$$\alpha_2\alpha_3 \dots \alpha_n \sim \beta_1\beta_2 \dots \beta_{n-1} \tag{3.2}$$

holds we say that α and β *agree*.

2. If one of the following equalities,

$$\alpha_1\alpha_2 \dots \alpha_{n-1} = \beta_2\beta_3 \dots \beta_n, \tag{3.3}$$

$$\alpha_2\alpha_3 \dots \alpha_n = \beta_1\beta_2 \dots \beta_{n-1} \tag{3.4}$$

holds we say that α and β *agree exactly*.

Remark 13. Definition 12 gives that the permutations α and β agree exactly if and only if they are of the form $a\gamma$ and γa.

We note that if two permutations agree exactly they will also agree, more precisely (3.3) implies (3.1) and (3.4) implies (3.2). Also, for both equalities (3.3) and (3.4) to hold we must have $\alpha = \beta = 1$ or $\{\alpha, \beta\} = \{12, 21\}$. This gives the following proposition.

Proposition 14. *Given two permutations α and β, whose length is larger than two, there are five mutually exclusive possibilities;*

1. *α and β do not agree, that is neither* (3.1) *nor* (3.2) *holds.*
2. *α and β agree exactly one way, that is one of* (3.3) *holds or* (3.4).
3. *α and β agree one way only and not exactly, that is either* (3.1) *or* (3.2) *holds but neither* (3.3) *nor* (3.4) *holds.*
4. *α and β agree both ways but not exactly, that is both* (3.1) *and* (3.2) *hold but neither* (3.3) *nor* (3.4) *holds.*
5. *α and β agree both ways and exactly, that is both* (3.1) *and* (3.2) *hold and either* (3.3) *or* (3.4) *holds.*

Below we will give an example of each of the five cases from the proposition above.

Example 15.

1. The permutations $\alpha = 1235764$ and $\beta = 7652431$ do not agree. If we look at α's prefix and β's suffix we have

$$\alpha_1\alpha_2 \dots \alpha_6 = 123576 \sim 123465$$

and

$$\beta_2\beta_3 \dots \beta_7 = 652431.$$

Then we look at α's suffix and β's prefix. Now

$$\alpha_2\alpha_3 \dots \alpha_7 = 235764 \sim 124653$$

and

$$\beta_1\beta_2 \dots \beta_6 = 765243 \sim 654132.$$

It is easy to see that neither (3.3) nor (3.4) from Definition 12 holds so α and β do not agree.

2. Now look at the permutations $\alpha = 6751423$ and $\beta = 7514236$. We have

$$\alpha_1\alpha_2\ldots\alpha_6 = 675142 \sim 564132$$

and

$$\beta_2\beta_3\ldots\beta_7 = 514236.$$

Since $\alpha_1\alpha_2\ldots\alpha_6 \nsim \beta_2\beta_3\ldots\beta_7$ we know that α and β will not agree both ways. But let's look at the other way. We get

$$\alpha_2\alpha_3\ldots\alpha_7 = 751423$$

and

$$\beta_1\beta_2\ldots\beta_6 = 751423.$$

Now we can see that α and β agree exactly.

3. Let's look at the permutations $\alpha = 2156734$ and $\beta = 1457236$. We first look at α's prefix and β's suffix and get

$$\alpha_1\alpha_2\ldots\alpha_6 = 215673 \sim 214563$$

and

$$\beta_2\beta_3\ldots\beta_7 = 457236 \sim 346125.$$

Now look at α's suffix and β's prefix and we get

$$\alpha_2\alpha_3\ldots\alpha_7 = 156734 \sim 145623$$

and

$$\beta_1\beta_2\ldots\beta_6 = 145723 \sim 145623.$$

Here we can see that $\alpha_2\alpha_3\ldots\alpha_7 \sim \beta_1\beta_2\ldots\beta_6$ and $\alpha_1\alpha_2\ldots\alpha_6 \nsim \beta_2\beta_3\ldots\beta_7$ so α and β agree one way only.

4. The permutations $\alpha = 5736241$ and $\beta = 7462513$ agree both ways but not exactly. This is so because

$$\alpha_1\alpha_2\ldots\alpha_6 = 573624 \sim 462513$$

and

$$\beta_2\beta_3\ldots\beta_7 = 462513,$$

so the prefix of α and the suffix of β agree. We also have

$$\alpha_2\alpha_3\ldots\alpha_7 = 736241 \sim 635241$$

and

$$\beta_1\beta_2\ldots\beta_6 = 746251 \sim 635241,$$

so the suffix of α and the prefix of β also agree.

5. Let's look at the permutations $\alpha = 4152637$ and $\beta = 1526374$. We get

$$\alpha_1\alpha_2\ldots\alpha_6 = 415263$$

and

$$\beta_2\beta_3\ldots\beta_7 = 526374 \sim 415263.$$

We can see that α and β agree one way, but will they agree the other way and will they possibly agree exactly? We have

$$\alpha_2\alpha_3\ldots\alpha_7 = 152637$$

and

$$\beta_1\beta_2\ldots\beta_6 = 152637.$$

So the answer is, yes α and β agree both ways and exactly.

Now that we know what it means for two permutations to agree we can state the following.

Lemma 16. *Two permutations are mates, that is have a child or children together, if and only if they agree.*

Proof. First assume α and β are mates. Then they must have at least one child together. Let $\pi = \pi_1\pi_2\ldots\pi_{n+1}$ be their child. By Definition 5 we know that π's parents are $\alpha \sim \pi_1\pi_2\ldots\pi_n$ and $\beta \sim \pi_2\pi_3\ldots\pi_{n+1}$ and it is easy to see that the prefix of β is similar to the suffix of α, so α and β agree.

Now, assume that α and β agree and in the way that $\alpha_1\alpha_2\ldots\alpha_{n-1} \sim \beta_2\beta_3\ldots\beta_n$. We want to show that α and β are mates. We let γ be the permutation $\gamma = (\alpha_1\alpha_2\ldots\alpha_{n-1})^{\downarrow\alpha_n} = (\beta_2\beta_3\ldots\beta_n)^{\downarrow\beta_1}$. The permutation $(\beta_1\gamma^{\uparrow\beta_1})^{\uparrow\alpha_n}\alpha_n$ will have the parents $\beta_1\gamma^{\uparrow\beta_1} = \beta$ and $\gamma^{\uparrow\alpha_n}\alpha_n = \alpha$. Hence, α and β have the same child and must therefore be mates. $\square$

This lemma tells us that if two permutations do not agree they will not have a child or children together. But if the two permutations agree they will have at least one child. So now the question is, how many children can they have? The answer to this question is given in Proposition 18, below. Before we state the proposition we give one lemma that will be needed in the proof of the proposition.

Lemma 17. *If α and β agree both ways and $\alpha_1 = \beta_1$ and $\alpha_n = \beta_n$ then $\alpha = \beta$ is monotone.*

Proof. This obviously holds for $n = 1$ and $n = 2$, so we assume $n \geq 3$. Let the permutations

$$\alpha = \alpha_1\alpha_2 \dots \alpha_{n-1}\alpha_n,$$
$$\beta = \beta_1\beta_2 \dots \beta_{n-1}\beta_n$$

agree both ways and $\alpha_1 = \beta_1 = a$ and $\alpha_n = \beta_n = b$. Then

$$(a\alpha_2\alpha_3 \dots \alpha_{n-1})^{\downarrow b} = (\beta_2\beta_3 \dots \beta_{n-1}b)^{\downarrow a}, \quad (3.5)$$
$$(\alpha_2\alpha_3 \dots \alpha_{n-1}b)^{\downarrow a} = (a\beta_2\beta_3 \dots \beta_{n-1})^{\downarrow b}. \quad (3.6)$$

We want to prove that $\alpha = \beta$ is monotone. We know that $a \neq b$ so without loss of generality we assume $a > b$ and show that $\alpha = \beta$ is decreasing, by induction. Note that $\alpha = \beta$ if and only if $\alpha_i = \beta_i$ for all $1 \leq i \leq n$, and it is decreasing if $\alpha_i = \alpha_{i-1} - 1$, for all $1 < i \leq n$. Since $\alpha_1 = \beta_1 = a$ this is the same as showing that $\alpha_i = a - (i - 1)$. For $i = 2$ we want to show that $\alpha_2 = \beta_2 = a - 1$. We have from (3.5) that

$$\begin{aligned} {\beta_2}^{\downarrow a} &= a^{\downarrow b} \\ &= a - 1, \quad \text{since } a > b. \end{aligned}$$

If $\beta_2 < a$ we get $\beta_2 = a - 1$ but if $\beta_2 > a$ we have $\beta_2 - 1 = a - 1$ which gives that $\beta_2 = a$, but that contradicts the fact that $\beta_1 = a$ so $\beta_2 = a - 1$. Similarly from (3.6) we get that

$$\begin{aligned} {\alpha_2}^{\downarrow a} &= a^{\downarrow b} \\ &= a - 1. \end{aligned}$$

If $\alpha_2 < a$ we get $\alpha_2 = a - 1$ but if $\alpha_2 < a$ we have that $\alpha_2 = a$, which contradicts that $\alpha_1 = a$, so $\alpha_2 = a - 1$. Now we have shown that $\alpha_2 = \beta_2 = a - 1$.

Assume this holds for $3 \leq k \leq n-2$. By (3.5) we have

$$\begin{aligned} {\beta_{k+1}}^{\downarrow a} &= {\alpha_k}^{\downarrow b} \\ &= (a-(k-1))^{\downarrow b} \\ &= (a-k+1)^{\downarrow b} \\ &= a-k+1-1 \qquad (*) \\ &= a-k. \end{aligned}$$

Following is a small explanation for the $(*)$-step. The induction hypothesis gives that the first k letters of α and β are $a(a-1)(a-2)\dots(a-k+1)$. None of these letters could have been b since the permutation can only have one b and we know that $\alpha_n = \beta_n = b$. So, since $a > b$ we get that $\beta_k = \alpha_k = a-k+1 \geq b$. Which gives that ${\alpha_k}^{\downarrow b} = \alpha_k - 1$.

Now we have that ${\beta_{k+1}}^{\downarrow a} = a-k$ but we want to know what β_{k+1} is. Assume $\beta_{k+1} > a$ then $\beta_{k+1} - 1 = a-k$ or $\beta_{k+1} = a-k+1$, but that cannot hold since by induction hypothesis we have that $\beta_k = a-k+1$, and as said before we can only have one copy of each letter. Hence $\beta_{k+1} < a$ and $\beta_{k+1} = a-k$. Similarly, by 3.6 we can show that $\alpha_k + 1 = a-k$. □

Proposition 18. *Let α and β be a couple, where the length of α and β is at least 4 and $\alpha \neq \beta$.*

1. *If α and β agree one way only but not exactly the couple has an only child.*
2. *If α and β agree exactly one way the couple has twins.*
3. *If α and β agree both ways but not exactly the couple has twins.*
4. *If α and β agree both ways and exactly the couple has triplets.*

We want α and β to be of length 4 or more since there is one example of quads: namely all permutations of length 3 except the monotone ones are siblings and their parents are 12 and 21.

Proof. Let $\alpha = \alpha_1\alpha_2\dots\alpha_n$ and $\beta = \beta_1\beta_2\dots\beta_n$ be a couple, $\alpha \neq \beta$ and $|\alpha| = |\beta| \geq 4$. Since α and β are a couple they must have at least one child. We let $\pi = \pi_1\pi_2\dots\pi_{n+1}$ be such a child.

1. Let α and β agree one way but not exactly with $\alpha_2\alpha_3\dots\alpha_n \sim \beta_1\beta_2\dots\beta_{n-1} \sim \gamma$. We can write

$$\alpha = \alpha_1\gamma^{\uparrow\alpha_1} \text{ and } \beta = \gamma^{\uparrow\beta_n}\beta_n.$$

Here $\alpha_1 \neq \beta_n$, because otherwise α and β would agree exactly. By Definition 5 π's parents are

$$p_1 = (\pi_1\pi_2 \dots \pi_n)^{\downarrow \pi_{n+1}} \text{ and } p_2 = (\pi_2\pi_3 \dots \pi_{n+1})^{\downarrow \pi_1}.$$

It's easy to see that p_1 and p_2 will agree one way. So since $\alpha_2\alpha_3 \dots \alpha_n \sim \beta_1\beta_2 \dots \beta_{n-1}$ we get

$$p_1 = \alpha = \alpha_1\gamma^{\uparrow \alpha_1} \text{ and } p_2 = \beta = \gamma^{\uparrow \beta_n}\beta_n.$$

Now let's look at what we know about π in terms of p_1 and p_2. We can see that π must either be α where β_n has been added to the back of it or β where α_1 has been added to the front of it. From this it looks like we have two possibilities for π, that is π is either

$$\pi' = (\alpha_1\gamma^{\uparrow \alpha_1})^{\uparrow \beta_n}\beta_n \text{ or } \pi'' = \alpha_1(\gamma^{\uparrow \beta_n}\beta_n)^{\uparrow \alpha_1}.$$

But let's take a better look at these two permutations. We know that $\alpha_1 \neq \beta_n$. Assume $\beta_n < \alpha_1$. Then α_1 is greater in value than at least one letter in α, namely $\alpha_1 - 1$. Let $\alpha_1 - 1$ be k places to the right of α_1 so it is the $(k+1)$st letter in α, and the kth letter in γ. This relative order should be the same in π, that is since $\alpha \sim \pi_1\pi_2 \dots \pi_n$ it follows that $\pi_1 > \pi_{k+1}$.

Let's now look at $\pi'' = \alpha_1(\gamma^{\uparrow \beta_n}\beta_n)^{\uparrow \alpha_1}$. Now we want to see what happens between the first and the $(k+1)$st letters in π''. The first thing we do to get π'' is $\gamma^{\uparrow \beta_n}$, since β_n is smaller than α_1 we get that $\gamma_k = \alpha_1 - 1$ becomes α_1 in $\gamma^{\uparrow \beta_n}$. The next thing we do is $(\gamma^{\uparrow \beta_n}\beta_n)^{\uparrow \alpha_1}$, and now the kth letter becomes $\alpha_1 + 1$ in $(\gamma^{\uparrow \beta_n}\beta_n)^{\uparrow \alpha_1}$.

Next we put α_1 in front of $(\gamma^{\uparrow \beta_n}\beta_n)^{\uparrow \alpha_1}$ and we have π''. So α_1 is smaller in value than the letter that is k places to the right of it. So the relative order of the elements has changed. Therefore π'' cannot be a child of α and β if $\beta_n < \alpha_1$. This change in relative order does not happen for π'. With similar arguments we get that when $\alpha_1 < \beta_n$ a change in relative order happens for π' but not for π''. Hence, we have shown that if α and β agree only one way the couple $\{\alpha, \beta\}$ will have only one child of the form

$$\pi' = (\alpha_1\gamma^{\uparrow \alpha_1})^{\uparrow \beta_n}\beta_n = (\alpha_1 + 1)(\gamma^{\uparrow \alpha_1})^{\uparrow \beta_n}\beta_n \text{ if } \beta_n < \alpha_1$$
$$\pi'' = \alpha_1(\gamma^{\uparrow \beta_n}\beta_n)^{\uparrow \alpha_1} = \alpha_1(\gamma^{\uparrow \beta_n})^{\uparrow \alpha_1}(\beta_n + 1) \text{ if } \alpha_1 < \beta_n.$$

We get the same result if α and β agree with $\alpha_1\alpha_2 \dots \alpha_{n-1} \sim \beta_2\beta_3 \dots \beta_n$.

2. Let α and β agree exactly one way. Without loss of generality let $\alpha_2\alpha_3 \dots \alpha_n = \beta_1\beta_2 \dots \beta_{n-1} = \gamma$. Let $\alpha_1 = \beta_n = a$ and we get $\alpha = a\gamma$ and $\beta = \gamma a$. As before

π's parents are

$$p_1 = (\pi_1\pi_2 \dots \pi_n)^{\downarrow \pi_{n-1}} \text{ and } p_2 = (\pi_2\pi_3 \dots \pi_{n+1})^{\downarrow \pi_1}.$$

Since α and β only agree exactly one way we have

$$p_1 = \alpha = a\gamma \text{ and } p_2 = \beta = \gamma a.$$

Because otherwise, the parents would agree both ways. This gives two possibilities for π, that is π can be

$$\pi' = (a\gamma)^{\uparrow a} a = (a+1)\gamma^{\uparrow a} a, \text{ or}$$
$$\pi'' = a(\gamma a)^{\uparrow a} = a\gamma^{\uparrow a}(a+1).$$

Both of these permutations give us the right α and β if we go from the permutation to its parents. Hence, we have shown that if α and β agree exactly one way then they will have two children.

3. Let α and β agree both ways but not exactly. By Definition 12 we get

$$\alpha = \alpha_1{\gamma_1}^{\uparrow \alpha_1} \text{ and } \alpha = {\gamma_2}^{\uparrow \alpha_n}\alpha_n$$
$$\beta = {\gamma_1}^{\uparrow \beta_n}\beta_n \text{ and } \beta = \beta_1{\gamma_2}^{\uparrow \beta_1}.$$

We note that $\alpha_1 \neq \beta_n$ and $\alpha_n \neq \beta_1$. As before we have π's parents,

$$p_1 = (\pi_1\pi_2 \dots \pi_n)^{\downarrow \pi_{n+1}} \text{ and } p_2 = (\pi_2\pi_3 \dots \pi_{n+1})^{\downarrow \pi_1}.$$

Since α and β agree both ways we have,

$$p_1 = \alpha = \alpha_1{\gamma_1}^{\uparrow \alpha_1} = {\gamma_2}^{\uparrow \alpha_n}\alpha_n$$

and

$$p_2 = \beta = {\gamma_1}^{\uparrow \beta_n}\beta_n = \beta_1{\gamma_2}^{\uparrow \beta_1}.$$

We get the following alternatives for π:

$$(p_1)^{\uparrow \beta_n}\beta_n, \quad \alpha_1(p_2)^{\uparrow \alpha_1}, \quad \beta_1(p_1)^{\uparrow \beta_1}, \quad (p_2)^{\uparrow \alpha_n}\alpha_n.$$

With same arguments as in (1) we get that α and β will have the children

$$\begin{aligned}\pi' &= {p_1}^{\uparrow\beta_n}\beta_n = (\alpha_1{\gamma_1}^{\uparrow\alpha_1})^{\uparrow\beta_n}\beta_n \\ &= (\alpha_1+1)({\gamma_1}^{\uparrow\alpha_1})^{\uparrow\beta_n}\beta_n \quad \text{if } \beta_n < \alpha_1, \\ \pi'' &= \alpha_1{p_2}^{\uparrow\alpha_1} = \alpha_1({\gamma_1}^{\uparrow\beta_n}\beta_n)^{\uparrow\alpha_1} \\ &= \alpha_1({\gamma_1}^{\uparrow\beta_n})^{\uparrow\alpha_1}(\beta_n+1) \quad \text{if } \alpha_1 < \beta_n\end{aligned}$$

and

$$\begin{aligned}\pi''' &= \beta_1{p_1}^{\uparrow\beta_1} = \beta_1({\gamma_2}^{\uparrow\alpha_n}\alpha_n)^{\uparrow\beta_1} \\ &= \beta_1({\gamma_2}^{\uparrow\alpha_n})^{\uparrow\beta_1}(\alpha_n+1) \quad \text{if } \beta_1 < \alpha_n, \\ \pi^{(4)} &= {p_2}^{\uparrow\alpha_n}\alpha_n = (\beta_1{\gamma_2}^{\uparrow\beta_1})^{\uparrow\alpha_n}\alpha_n \\ &= (\beta_1+1)({\gamma_2}^{\uparrow\beta_1})^{\uparrow\alpha_n}\alpha_n \quad \text{if } \alpha_n < \beta_1.\end{aligned}$$

So α and β can at most have two children, one depending on α_1 and β_n and another one depending on α_n and β_1. Next we show that the number of children must be exactly two, that is for any $\alpha_1, \alpha_n, \beta_1$ and β_n the two permutations must be different. First we assume $\beta_n < \alpha_1$ and $\beta_1 < \alpha_n$. The two permutations we get are π' and π''' and we want to see if they can be the same permutation. Assume $\pi' = \pi'''$. This implies $\beta_n = \alpha_n + 1$ and $\beta_1 = \alpha_1 + 1$, which gives

$$\beta_1 = \alpha_1 + 1 > \beta_n + 1 = \alpha_n + 2 > \alpha_n.$$

This is a contradiction since we assumed $\beta_1 < \alpha_n$. Hence $\pi' \neq \pi'''$.

Now assume $\beta_n < \alpha_1$ and $\alpha_n < \beta_1$ and that $\pi' = \pi^{(4)}$. This implies, $\alpha_1+1 = \beta_1+1$ and $\alpha_n = \beta_n$ which gives that $\alpha_1 = \beta_n$ and $\alpha_n = \beta_1$ and by Lemma 17 we get $\alpha = \beta$. But this is also a contradiction. With similar arguments we can show that $\pi'' \neq \pi'''$ and $\pi'' \neq \pi^{(4)}$. Therefore, we have shown that if α and β agree both ways but not exactly they will have twins.

4. Let α and β agree both ways and exactly. Assume they agree exactly with

$$\alpha_2\alpha_3\ldots\alpha_n = \beta_1\beta_2\ldots\beta_{n-1} = \gamma_1$$

and let γ_2 be the permutation $\alpha_1\alpha_2\ldots\alpha_{n-1} \sim \beta_2\beta_3\ldots\beta_n \sim \gamma_2$. Assume α and β agree exactly with γ_1 and that they agree but not exactly the other way, that is with

γ_2. Since they agree exactly in this way $\alpha_1 = \beta_n = a$. We have

$$\alpha = a\gamma_1 \text{ and } \alpha = {\gamma_2}^{\uparrow\alpha_n}\alpha_n$$
$$\beta = \gamma_1 a \text{ and } \beta = \beta_1{\gamma_2}^{\uparrow\beta_1}.$$

We notice that $a \neq \alpha_n$ and $a \neq \beta_1$. As before π's parents are

$$p_1 = (\pi_1\pi_2\ldots\pi_n)^{\downarrow\pi_{n+1}} \text{ and } p_2 = (\pi_2\pi_3\ldots\pi_{n+1})^{\downarrow\pi_1},$$

and as in (3) we get,

$$p_1 = \alpha = a\gamma_1 = {\gamma_2}^{\uparrow\alpha_n}\alpha_n$$

and

$$p_2 = \beta = \gamma_1 a = \beta_1{\gamma_2}^{\uparrow\beta_1}.$$

We get the following possibilities for π:

$$(p_1)^{\uparrow a}a, \quad a(p_2)^{\uparrow a}, \quad (p_2)^{\uparrow\alpha_n}\alpha_n, \quad \beta_1(p_1)^{\uparrow\beta_1}$$

With similar arguments as in part 2 above, we get that α and β will have both of the children

$$\pi' = (p_1)^{\uparrow a}a = (a\gamma_1)^{\uparrow a}a = (a+1){\gamma_1}^{\uparrow a}a$$

and

$$\pi'' = a(p_2)^{\uparrow a} = a(\gamma_1 a)^{\uparrow a} = a{\gamma_1}^{\uparrow a}(a+1).$$

Then by using the same arguments as in part (1) we get that α and β will either have

$$\pi''' = (\beta_1{\gamma_2}^{\uparrow\beta_1})^{\uparrow\alpha_n}\alpha_n = (\beta_1+1)({\gamma_2}^{\uparrow\beta_1})^{\uparrow\alpha_n}\alpha_n \text{ if } \alpha_n < \beta_1$$

or

$$\pi^{(4)} = \beta_1({\gamma_2}^{\uparrow\alpha_n}\alpha_n)^{\uparrow\beta_1} = \beta_1({\gamma_2}^{\uparrow\alpha_n})^{\uparrow\beta_1}(\alpha_n+1) \text{ if } \beta_1 < \alpha_n.$$

and not both as their child. So α and β have at most three children. Now we will show that for any a, β_1 and α_n these three children are all different, so the children will be exactly three. First let's look at the case if $\alpha_n < \beta_1$ then we get the three children π', π'' and π'''. We know that $\pi' \neq \pi''$. Assume $\pi' = \pi'''$. Then we get $a = \alpha_1$ but as noted above this cannot happen so $\pi' \neq \pi'''$. Now assume $\pi'' = \pi'''$, then $a = \beta_1 + 1$ and $\alpha_n = a + 1$ this gives

$$\alpha_n = a + 1 = \beta_1 + 2 = \alpha_n$$

which gives that $\beta_1 < \alpha_n$, but that is a contradiction. Therefore $\pi'' \neq \pi'''$. With similar arguments we can show that $\pi' \neq \pi^{(4)}$ and $\pi'' \neq \pi^{(4)}$. Hence, we have shown that if α and β agree both ways and exactly they will have triplets. We get the same result using similar arguments if α and β agree exactly with $\alpha_1\alpha_2\ldots\alpha_{n-1} = \beta_2\beta_3\ldots\beta_n$ and not exactly with $\alpha_2\alpha_3\ldots\alpha_n \sim \beta_1\beta_2\ldots\beta_{n-1}$. □

The following lemma follows directly form Propositions 14 and 18.

Lemma 19. *A non-monotone permutation π is an only child if and only if π's parents agree one way only and not exactly.*

The following example will demonstrate Proposition 18. We will be using the same permutations as in Example 15. The enumeration in the example below is according to the enumeration in Proposition 18 not the enumeration in Example 15.

Example 20.

1. The permutations $\alpha = 2156734$ and $\beta = 1457236$ agree one way but not exactly with $\gamma = 145623$, by Example 15 (3). According to Proposition 18 (1) α and β should have an only child and from the proof we have that the child should be of the form $\pi'' = \alpha_1(\gamma^{\uparrow\beta_n})^{\uparrow\alpha_1}(\beta_n + 1)$, since $\alpha_1 = 1 < 6 = \beta_n$. We get,

$$\begin{aligned}\pi'' &= 2(145623^{\uparrow 6})^{\uparrow 2}(6+1)\\ &= 2(145723)^{\uparrow 2}7\\ &= 21568347.\end{aligned}$$

   Using Definition 5 we can verify that $\pi'' = 21568347$'s parents are indeed $\alpha = 2156734$ and $\beta = 1457236$. If we now look at

$$\begin{aligned}\pi' &= (\alpha_1\gamma^{\uparrow\alpha_1})^{\uparrow\beta_n}\beta_n\\ &= (2145623^{\uparrow 2})^{\uparrow 6}\\ &= (2156724)^{\uparrow 6}6\\ &= 21578346,\end{aligned}$$

   and use Definition 5 we see that $\pi' = 21578346$ parents are 1467235 and 2156734, not α and β.

2. The permutations $\alpha = 6751423$ and $\beta = 7514236$ agree exactly with $\gamma = 751423$, see Example 15 (2). By Proposition 18 (2) and its proof we know that α and β have the following children,

$$\pi' = (a+1)\gamma^{\uparrow a}a \qquad\qquad \pi'' = a\gamma^{\uparrow a}(a+1)$$
$$= (6+1)(751423)^{\uparrow 6}6 \qquad\qquad = 6(751423)^{\uparrow 6}(6+1)$$
$$= 78514236, \qquad\qquad = 68514237.$$

3. The permutations $\alpha = 5736241$ and $\beta = 7462513$ agree both ways but not exactly with $\gamma_1 = 635241$ and $\gamma_2 = 462513$, see Example 15 (4). Here $\beta_n < \alpha_1$ and $\alpha_n < \beta_1$ therefore by Proposition 18 (3)

$$\pi' = (\alpha_1+1)({\gamma_1}^{\uparrow \alpha_1})^{\uparrow \beta_n}\beta_n \qquad\qquad \pi^{(4)} = (\beta_1+1)({\gamma_2}^{\uparrow \beta_1})^{\uparrow \alpha_n}\alpha_n$$
$$= (5+1)(736241)^{\uparrow 3}3 \qquad\qquad = (7+1)(462513)^{\uparrow 1}1$$
$$= 68472513, \qquad\qquad = 85736241$$

are α and β's children. This can be verified by using Definition 5. Let's now look at π'' and π'''. We have

$$\pi'' = \alpha_1({\gamma_1}^{\uparrow \beta_n}\beta_n)^{\uparrow \alpha_1} \qquad\qquad \pi''' = \beta_1({\gamma_2}^{\uparrow \alpha_n}\alpha_n)^{\uparrow \beta_1}$$
$$= 5(7462513)^{\uparrow 5} \qquad\qquad = 7(5736241)^{\uparrow 7}$$
$$= 58472613, \qquad\qquad = 75836241.$$

Now by using Definition 5 we get that π'''s parents are 4736251 and 7462513 and the parents of π''' are 6472513 and 5736241.

4. The permutations $\alpha = 4152637$ and $\beta = 1526374$ agree both ways and exactly with $\gamma_1 = 152637$ and $\gamma_2 = 415263$, by Example 15 (5). Here $\beta_1 < \alpha_n$ therefore by Proposition 18 (4) and its proof we get that

$$\pi' = (a+1){\gamma_1}^{\uparrow a}a = (4+1)(152637)^{\uparrow 4}4 = 51627384$$
$$\pi'' = a{\gamma_1}^{\uparrow a}(a+1) = 4(152637)^{\uparrow 4}(4+1) = 41627385$$
$$\pi^{(4)} = \beta_1({\gamma_2}^{\uparrow \alpha_n})^{\uparrow \beta_1}(\alpha_n+1) = 1(415263)^{\uparrow 1}(7+1) = 15263748$$

are α's and β's children. This can be verified using Definition 5. Let's look at π''' which should not be a child of α and β. We have

$$\pi''' = (\beta_1{\gamma_2}^{\uparrow \beta_1})^{\uparrow \alpha_n}\alpha_n$$
$$= (1526374)^{\uparrow 7}7$$
$$= 15263847.$$

By using Definition 5 we get that the parents of π''' are 4152736 and 1526374.

Now we have answered the question of how many children a couple can have. But as mentioned before that was just one step towards the goal of this chapter, which was to prove Theorem 11. Now we will state a few lemmas which will be used to prove Theorem 11. The first lemma gives gives us the permutations that form couples that have triplets.

Lemma 21. *Let α and β be two permutations of length n that form a couple. The couple has triplets if and only if n is odd and*

$$\{\alpha, \beta\} = \{1\tau nm, m1\tau n\}$$

or the reverse

$$\{\alpha, \beta\}^r = \{n\tau^r 1m, mn\tau^r 1\}$$

where

$$\tau = (m+1)2(m+2)3\cdots(n-1)(m-1)$$

and m is the middle element $m = \frac{n+1}{2}$.

Before we prove this lemma it is good to look at an example. In Example 20 (4) we showed that the permutations $\alpha = 1526374$ and $\beta = 4152637$ have triplets. So according to the lemma above they should be of odd length, which they are, their length is 7, and they should take either of the form mentioned. By looking at the permutations we can see that $\alpha = 1\tau nm$ and $\beta = m1\tau n$. Now we will go through the proof.

Proof. Let α and β be two permutations of length n. For α and β to have triplets they must agree both ways and exactly, by Proposition 18. For α and β to agree exactly one way we let $\alpha = ma_1a_2 \ldots a_k$ and $\beta = a_1a_2 \ldots a_km$, where $k = n-1$. Since α and β also agree the other way it must hold that

$$ma_1a_2a_3 \ldots a_{k-1} \sim a_2a_3 \ldots a_km. \tag{3.7}$$

First we assume k is odd, then α and β are of even length. If $m > a_2$ we get that $a_2 > a_4$ and so on until $a_{k-1} > m$ which gives that $ma_2a_4 \ldots a_{k-1}m$ is a decreasing sequence. Similarly if $m < a_2$ we get that $ma_2a_4 \ldots a_{k-1}m$ is an increasing sequence. But neither case can be true, since m is on both ends. Therefore k must be even.

For an even k we have that $ma_2a_4a_6 \ldots a_k$ is a monotone sequence, either increasing or decreasing, and the same holds for $ma_{k-1}a_{k-3} \ldots a_1$. We also get that if

$$m > a_1 \Rightarrow a_2 > a_3 \Rightarrow a_4 > a_5 \Rightarrow \cdots \Rightarrow a_k > m.$$

This and the fact that $ma_{k-1}a_{k-3}\dots a_1$ is a monotone sequence gives that m must be greater than all the the letters in odd positions and since $ma_2a_4a_6\dots a_k$ is also a monotone sequence m must be smaller than all the letters in even positions. Since k is even the number of odd and even a's is the same, or $\frac{k}{2} = \frac{n-1}{2}$. This gives that $m = \frac{n-1}{2} + 1 = \frac{n+1}{2}$ and $a_1 = 1$ and $a_n = n$. So $\alpha = m1\tau n$ and $\beta = 1\tau nm$. Similarly, if

$$m < a_1 \Rightarrow a_2 < a_3 \Rightarrow a_4 < a_5 \Rightarrow \dots \Rightarrow a_k < m,$$

m must be smaller than the odd a's and larger than all the even a's, which gives that m is again the middle element, $m = \frac{n+1}{2}$, but $a_1 = n$ and $a_k = 1$. Therefore $\alpha = mn\tau'1$ and $\beta = n\tau'1m$.

Now what is left, is to show the construction of τ and τ'. We will show the construction of τ and using the same approach the construction of τ' can be found as well. As stated above (3.7) the following must hold for the permutations $m1\tau n$ and $1\tau nm$,

$$m1\tau \sim \tau nm.$$

This is the same as

$$(m1\tau)^{\downarrow n} = (\tau nm)^{\downarrow 1}$$

which gives

$$m1\tau_1\tau_2\cdots\tau_{n-3} = (\tau_1 - 1)(\tau_2 - 1)\cdots(\tau_{n-3} - 1)(n-1)(m-1). \tag{3.8}$$

Now we can compute the letters of τ. We get

$$\begin{aligned}
&\tau_1 - 1 = m \Rightarrow \tau_1 = m + 1,\\
&\tau_2 - 1 = 1 \Rightarrow \tau_2 = 2,\\
&\tau_3 - 1 = \tau_1 \Rightarrow \tau_3 = m + 1 + 1 = m + 2,\\
&\tau_4 - 1 = \tau_2 \Rightarrow \tau_4 = 2 + 1 = 2,
\end{aligned}$$

and so on. So we get that

$$\tau_k = m + \frac{k+1}{2}$$

when k is odd and

$$\tau_k = \frac{k+2}{2}$$

when k is even. To verify these formulas we look at the two last letters of τ. They are τ_{n-4} and τ_{n-3}. By looking at 3.8 we see that $\tau_{n-4} = n - 1$ which fits the formula above, see below. We showed before that n must be odd so $k = n - 4$ is even. Therefore

$$\begin{aligned}\tau_{n-4} &= m + \frac{k+1}{2} \\ &= m + \frac{n-4+1}{2} \\ &= m + \frac{n-1}{2} + 1 + \frac{n-3}{2} \\ &= \frac{2n-4}{2} + 1 \\ &= n - 1.\end{aligned}$$

Similarly, for τ_{n-3} if looking at 3.8 we see that $\tau_{n-3} = m - 1$. Now we have $k = n - 3$ even so we get

$$\begin{aligned}\tau_{n-3} &= \frac{k+2}{2} \\ &= \frac{n-3+2}{2} \\ &= \frac{n-1}{2} \\ &= \frac{2(m-1)-1+1}{2} \\ &= \frac{2(m-1)}{2} \\ &= m - 1.\end{aligned}$$

Hence, we have shown that

$$\tau = (m+1)2(m+2)3\cdots\left(m + \frac{k+1}{2}\right)\frac{k+2}{2}\cdots(n-1)(m-1). \qquad \square$$

By combining the lemma above and the findings from case (4) in the proof of Proposition 18 we get the exact form for all triplets.

Lemma 22. *If π', π'' and π''' are triplets whose length is $n+1$ either*

$$\begin{aligned}\{\pi', \pi'', \pi'''\} = \{&(m+1)1\tau^{\uparrow m}(n+1)m, \\ &m1\tau^{\uparrow m}(n+1)(m+1), \\ &1(m+1)2\tau^{\uparrow 1}(n+1)\}\end{aligned}$$

or the reverse

$$\begin{aligned}\{\pi', \pi'', \pi'''\} =& \{(m+1)(n+1)\tau^{r\uparrow m}1m, \\ & m(n+1)\tau^{r\uparrow m}1(m+1), \\ & (n+1)\tau^{r\uparrow 1}2(m+1)1\},\end{aligned}$$

where m and τ take the same values as in Lemma 21.

As for the previous lemma it is good to look at an example. Again we use Example 20 (4). There we showed that the permutations $\alpha = 4152637$ and $\beta = 1526374$ have the triplets $51627384, 41627385, 15263748$. We have said that $\alpha = m1\tau n$ and $\beta = 1\tau nm$ and if we look at the triplets we can see that they have the following forms

$$\begin{aligned}51627384 &= (m+1)1\tau^{\uparrow m}(n+1)m \\ 41627385 &= m1\tau^{\uparrow m}(n+1)(m+1) \\ 15263748 &= (n+1)\tau'^{\uparrow 1}2(m+1)1.\end{aligned}$$

Proof. Given a set of triplets we know that their parents must either be $\{1\tau nm, m1\tau n\}$ or the reverse, $\{n\tau^r 1m, mn\tau^r 1\}$. If the former, we get from the proof of Proposition 18 (4), that $\alpha = m1\tau n$ and $\beta = 1\tau nm$. So we have that $a = m, \alpha_n = n$ and $\beta_1 = 1$, we also get that $\gamma_1 = 1\tau n$ and ${\gamma_2}^{\uparrow \alpha_n} = {\gamma_2}^{\uparrow n} = m1\tau$. Therefore, the triplets are

$$\begin{aligned}\pi' &= (a+1){\gamma_1}^{\uparrow a}a \\ &= (m+1)(1\tau n)^{\uparrow m}m \\ &= (m+1)1\tau^{\uparrow m}(n+1)m,\end{aligned}$$

$$\begin{aligned}\pi'' &= a{\gamma_1}^{\uparrow a}(a+1) \\ &= m(1\tau n)^{\uparrow m}(m+1) \\ &= m1\tau^{\uparrow m}(n+1)(m+1),\end{aligned}$$

and

$$\begin{aligned}\pi''' &= \beta_1({\gamma_2}^{\uparrow \alpha_n})^{\uparrow \beta_1}(\alpha_n+1) \\ &= 1(m1\tau)^{\uparrow 1}(n+1) \\ &= 1(m+1)2\tau^{\uparrow 1}(n+1).\end{aligned}$$

□

This lemma gives us that there are two pairs of triplets of every even length, the parents are of odd length. We can also see that for both pairs of triplets we have permutations of the form $a\gamma(a+1)$ and $(a+1)\gamma a$. But do we know anything more about these permutations? The answer is yes, and the result is stated in the lemma below.

Lemma 23. *If a non-monotone permutation π is of either of the form $a\gamma(a+1)$ or $(a+1)\gamma a$ then π's parents must agree exactly.*

Proof. If $\pi = a\gamma(a+1)$ then π's parents are $a\gamma'$ and $\gamma' a$, where $\gamma' = (\gamma)^{\downarrow a+1} = \gamma^{\downarrow a}$, so they will agree exactly. □

This and Proposition 18 give us that a permutation of either of the forms mentioned in the lemma must either be a triplet or a twin. Therefore we get the following.

Lemma 24. *The permutations π and π' are twins whose parents agree exactly if and only if $\{\pi, \pi'\} = \{a\gamma(a+1), (a+1)\gamma a\}$ and π and π' are none of the permutations in Lemma 21*

Proof. The fact that if π and π' are twins whose parents agree exactly, then $\{\pi, \pi'\} = \{a\gamma(a+1), (a+1)\gamma a\}$ was proved in the course of the proof of Proposition 18. The other direction is proved as follows. Let $\pi = a\gamma(a+1)$ and $\pi' = (a+1)\gamma a$, then $\Delta(\pi) = \Delta(\pi') = \{a\gamma, \gamma a\}$. That is π' and π's parents are $a\gamma$ and γa. Hence, their parents agree exactly. By Proposition 18 we have that if two permutations agree exactly then they will either have twins or triplets. Since we said that π and π' are none of the permutations in Lemma 21 they are not triplets so they must be twins. □

Another nice thing for these permutations is that they are not hard to count. If $\pi = a\gamma(a+1)$ and $\pi' = (a+1)\gamma a$ are permutations of length n then we have $(n-2)!$ different possibilities for γ and a can take $n-1$ different values, since we are counting the pairs. Hence, the total number of $\pi = a\gamma(a+1)$ and $\pi' = (a+1)\gamma a$ permutation pairs is $(n-1)(n-2)! = (n-1)!$. Therefore, when n is odd there are $(n-1)!$ pairs of twins that have parents that agree exactly and no triplets. Note, that this is not the total number of twins since we can also have twins whose parents agree both ways but not exactly. Then when n is even we have $(n-1)! - 2$ set of twins whose parents agree exactly and two set of triplets.

Now we have gathered all the information we need to be able to show that we have infinitely many non-recognizable permutations in the consecutive poset, that is we can proof Theorem 11.

Proof of Theorem 11. As stated before, the non-recognizable permutations are those that have siblings, namely the twins and triplets. In Lemma 24 we showed that all pairs of permutations of the form

$$\{\pi, \pi'\} = \{a\gamma(a+1), (a+1)\gamma a\}$$

are twins or a part of a triplet. These permutations exist on every level and more precisely there are $(n-1)!$ such pairs of length n. Hence, the number of non-recognizable permutations is infinite. □

The result above proves one of our main results. The set of non-recognizable permutations in the case of consecutive pattern containment is different from this set in the classical case. In the consecutive case it is infinite while it is finite in the classical case. Another result of ours is that there are no non-trivial automorphisms of the consecutive poset and this will be the topic of the following chapter.

Chapter 4

Non-trivial automorphisms of the consecutive poset

The goal of this chapter is to show that the only automorphisms of the full consecutive poset are the trivial maps or the known symmetries, same as for the classical case. But we note that the the trivial maps are eight in the classical case while they are four in the consecutive case. In the consecutive case the symmetries are the identity map, reverse and complement and compositions of these maps. Our aim is to prove the following theorem.

Theorem 25. *There are no non-trivial automorphisms of the full consecutive poset.*

To be able to provide the proof of Theorem 25 we must know a bit more about how permutations that are siblings look like in relation to each other. In order to do so we introduce *overlap-permutations*. First we give the definition of overlap-permutations, provide examples and go through some of the most relevant properties of the overlap-permutations. Then we will introduce some lemmas that will be used in the proof, which we will provide at the end of this chapter.

4.1 Overlap-permutations

Definition 26. Let $\pi = \pi_1\pi_2 \ldots \pi_n$ be a permutation. If

$$\pi_1\pi_2 \ldots \pi_{n-k} \sim \pi_{k+1}\pi_{k+2} \ldots \pi_n$$

π is said to be a k*-overlap-permutation*.

The overlap-permutations are related to the autocorrelation of patterns, see [8]. There are only two 1-overlap-permutations, the monotone permutations. By the definition above we have that a permutation π is a 2-overlap-permutation if

$$\pi_1\pi_2\dots\pi_{n-2} \sim \pi_3\pi_4\dots\pi_n.$$

In this paper we will only be working with 2-overlap-permutations so from now on we will skip the letter 2 and call them *overlap-permutations*, for shorthand.

To better understand the structure of overlap-permutations we provide two examples.

Example 27. The monotone permutations $\mathrm{id} = 12\dots n$ and $\mathrm{id}^{\mathrm{r}} = n(n-1)\dots 1$ are both overlap-permutations since we have

$$12\dots(n-2) \sim 34\dots n$$

and

$$n(n-1)\dots 3 \sim (n-3)(n-4)\dots 1.$$

Example 28. Given the permutation $\pi = 978563412$ we have

$$\pi_1\pi_2\dots\pi_7 = 9785634 \sim 7563412$$

and

$$\pi_3\pi_4\dots\pi_9 = 8563412 \sim 7563412.$$

This gives that π is an overlap-permutation.

Before we go any further we must mention that similar to the fact that the monotone permutations behave a bit differently than other permutations in relation to their consecutive shadow they do also behave differently in relation to the overlap condition. As stated before, in Example 9, the monotone permutations are the only permutations that have only one parent and in Example 9 we showed that the monotone permutations are only children. Then in Example 27 we showed that the monotone permutations are overlap-permutations. These facts combined are stated in the following lemma, for reference purposes.

Lemma 29. *The monotone permutations are only children and overlap-permutations.*

Later in this chapter we will show that the monotone permutations are the only permutations that have this property, that is they are the only permutations that are only children and overlap-permutations.

The condition that the overlap-permutations must fulfill, that is

$$\pi_1\pi_2\ldots\pi_{n-2} \sim \pi_3\pi_4\ldots\pi_n,$$

is very limiting. Because of this condition the overlap-permutations have many properties. These properties help us to get an idea of how the overlap-permutations must look like. Some of these properties are also used in the proofs in this chapter. Therefore, we will state the most important properties of the overlap-permutations in the following lemma.

Lemma 30. *Let $\pi = \pi_1\pi_2\ldots\pi_n$ be an overlap-permutation.*

1. *Both the odd sequence, $\pi_1\pi_3\ldots$, and the even sequence, $\pi_2\pi_4\ldots$, are monotone.*
2. *The pair $\pi_1\pi_2$ has the same relative order as $\pi_3\pi_4$ which has the same relative order as $\pi_5\pi_6$ and so on. Similarly, $\pi_1\pi_2\pi_3 \sim \pi_3\pi_4\pi_5 \sim \pi_6\pi_7\pi_8$ and so on.*

Proof. We only provide the proof for part (1) as the other part is trivial.

1. Let us first consider the odd sequence. Since π is an overlap-permutation we have

$$\pi_1\pi_2\pi_3\ldots\pi_{n-2} \sim \pi_3\pi_4\pi_5\ldots\pi_n.$$

If we assume $\pi_1 < \pi_3$ we get that $\pi_3 < \pi_5$, which gives that $\pi_5 < \pi_7$ and so on. So the odd sequence is decreasing. If we assume $\pi_1 > \pi_3$ we get that the odd sequence is increasing. Similarly, we can show that the even sequence must be monotone as well. □

From the proof we can see that the whether the odd and the even sequences are decreasing or increasing only depends on the first two letters of the sequences, respectively. Therefore, the two sequences can be both increasing, both decreasing or one increasing and the other decreasing. Below we will provide an example showing different overlap-permutations.

Example 31.

1. The permutation $\pi = 14263758$ is an overlap-permutation where both sequences are increasing. The permutation is showed in Figure 1.

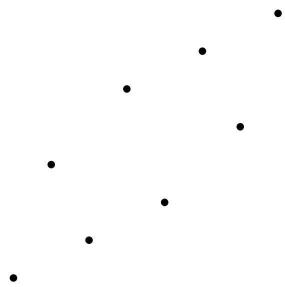


Figure 4.1: Both sequences increasing

2. The permutation $\pi = 58372614$ is an overlap-permutation where both the odd and the even sequence is decreasing. See Figure 2.

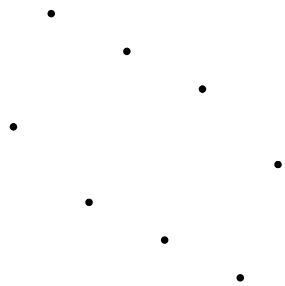


Figure 4.2: Both sequences decreasing

3. The permutation $\pi = 45362718$ is an overlap-permutation where the odd sequence is decreasing and the even sequence is increasing. See Figure 3.

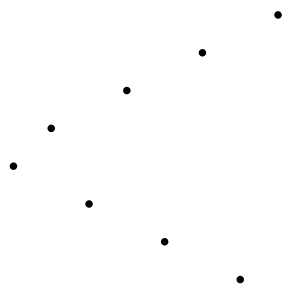


Figure 4.3: The odd sequence is decreasing and the even sequence is increasing

4. The permutation $\pi = 54637281$ is an overlap-permutation where the odd sequence is increasing and the even sequence is decreasing. See Figure 4.

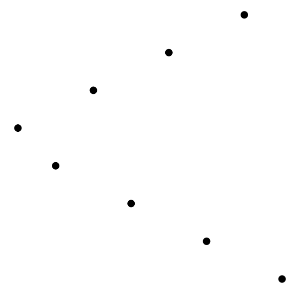


Figure 4.4: The odd sequence is increasing and the even sequence is decreasing

We can see from this example that the overlap-permutations have different structure depending on whether the odd and the even sequences are monotone in different directions or in the same direction. If they are both monotone in the same direction, the structure depends on whether they are both increasing or both decreasing.

Now, we will look into the structure of the overlap-permutations in more details. But first we give the following definition.

Definition 32. A permutation whose letters are arranged in such a way that every other letter forms an increasing sequence and the other letters form a decreasing sequence and either all the increasing letters are greater than all the decreasing letters or vice versa is called an *alternating wedge*. The wedge is either *pointing to the left* or *pointing to the right*. If the increasing terms are greater in value than the decreasing terms the wedge is pointing to the left and has either of the two forms shown in Figure 4.5.

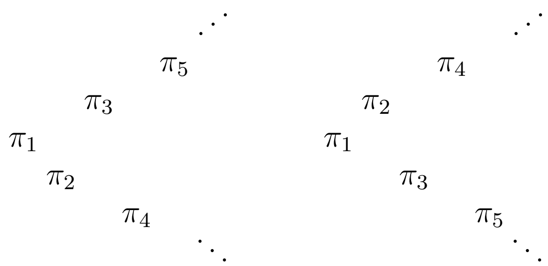


Figure 4.5: Alternating wedges pointing to the left

If the decreasing terms are greater in value than the increasing terms the wedge is pointing to the right and has either of the two forms shown in Figure 4.6.

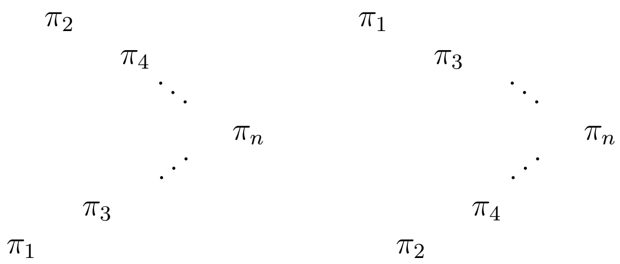


Figure 4.6: Alternating wedges pointing to the right

Now let's look at overlap-permutations that have both the odd and the even sequence monotone in different directions. We get the following lemma.

Lemma 33. *A non-monotone overlap-permutation* $\pi = \pi_1\pi_2\ldots\pi_n$ *is an alternating wedge if the odd and the even sequences are monotone in different directions.*

Proof. We have four cases:

1. The odd sequence is increasing and the even sequence is decreasing and $\pi_1\pi_2$ is an ascent.
2. The odd sequence is increasing and the even sequence is decreasing and $\pi_1\pi_2$ is a descent.
3. The odd sequence is decreasing and the even sequence is increasing and $\pi_1\pi_2$ is an ascent.
4. The odd sequence is decreasing and the even sequence is increasing and $\pi_1\pi_2$ is a descent.

Let us first consider case (1). We assume n is even. First we place π_1. The letter π_2 must be larger in value than π_1 since $\pi_1\pi_2$ is an ascent. Now we think about the next two letters, we know that π_3 must be larger than π_1, since the odd sequence is increasing. Can π_3 be larger than π_2? No, because $\pi_3\pi_4$ is an ascent, so π_3 has to be smaller than π_4 which has to be smaller than π_2. This gives that the ascent $\pi_3\pi_4$ is between $\pi_1\pi_2$ in values. With the same arguments, the next two letters, the ascent $\pi_5\pi_6$, must be between $\pi_3\pi_4$ in values, and so on. Hence, if n is even it is clear that π must be an alternating wedge pointing to the right, where all the letters in even positions must be larger in value than the letters in the odd positions. The form of the permutation is shown in Figure 4.7.

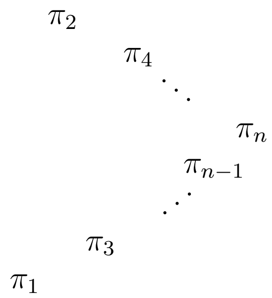


Figure 4.7: Alternating wedge of even length pointing to the right, starting with an ascent

Let's look at what happens if n is odd. The same applies until we have to place the last letter, π_n. That is, the letters $\pi_{n-2}\pi_{n-1}$ are an ascent that is between the ascent $\pi_{n-4}\pi_{n-3}$ in values. Now, where can we place the last letter? It has to be larger than π_{n-2} since the odd sequence is increasing, but can it be larger than π_{n-1}? No, because as stated before the sequence $\pi_1\pi_2\pi_3 \sim \pi_{n-2}\pi_{n-1}\pi_n$. We know that $\pi_1\pi_2\pi_3 \sim 132$, which gives that π_n must be between π_{n-2} and π_{n-1} in values. So, π must be an alternating wedge pointing to the left, where the letters in even positions must be larger in value than the letters in the odd ones. Figure 4.8 shows the form of the permutation.

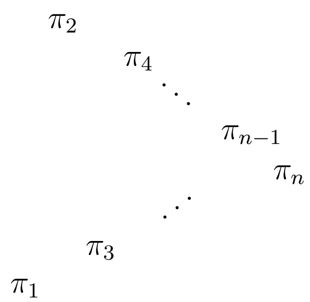


Figure 4.8: Alternating wedge of odd length pointing to the right, starting with an ascent

Now consider case (2). First we place the descent $\pi_1\pi_2$. Next we want to place $\pi_3\pi_4$. The letter π_3 must be larger in value than π_1 so there is only one possible place to place it now. The letter π_4 can also only be placed on one place. That is, it must be smaller in value than π_2, since the even sequence is decreasing. It must also be smaller in value than π_3, since the pair $\pi_3\pi_4$ is a descent, which holds because π_3 is larger than π_2. The same applies for the pair $\pi_5\pi_6$, that is, π_5 will be larger in value than π_5 and π_6 smaller in value than π_4, and so on. This gives that, π will be an alternating wedge pointing to the left, where the letters in odd positions are be larger than the letters in the even positions. The form of π is shown in Figure 4.9.

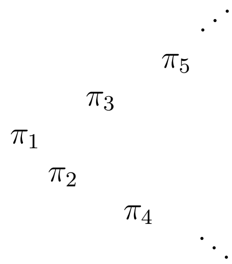


Figure 4.9: Alternating wedge pointing to the left, starting with a descent

The permutations in case (3) are the complement of the permutations in case (2). That is, they are the complement of the permutations in Figure 4.9. Hence they are alternating wedges of the form shown in Figure 4.10.

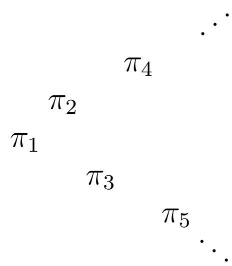


Figure 4.10: Alternating wedge pointing to the left, starting with an ascent

Similarly, the permutations in case (4) are the complement of the permutations in case (1). The form of the even length permutation is shown in Figure 4.11 and the form of the odd one in Figure 4.12.

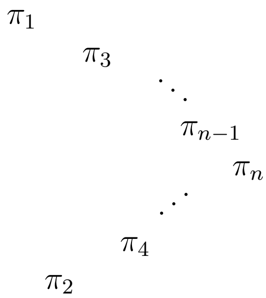


Figure 4.11: Alternating wedge pointing to the right, starting with a descent

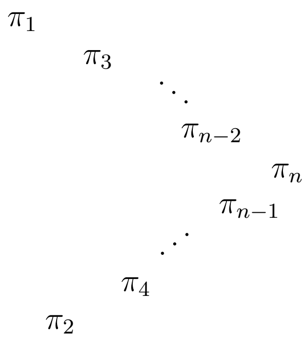


Figure 4.12: Alternating wedge pointing to the right, starting with a descent

□

Now we look at what happens if both the odd and the even sequence are monotone in the same direction. As before we will have four cases:

1. Both sequences are decreasing and $\pi_1\pi_2$ is a descent.
2. Both sequences are decreasing and $\pi_1\pi_2$ is an ascent.
3. Both sequences are increasing and $\pi_1\pi_2$ is a descent.
4. Both sequences are increasing and $\pi_1\pi_2$ is an ascent.

Let's start by looking at case (1). Let $\pi = \pi_1\pi_2 \ldots \pi_n$ be an overlap-permutation that meets the requirements from case (1). If we try to form this permutation as we did in Lemma 33. We start with the descent $\pi_1\pi_2$. When placing the letter π_3 there is only one condition we have to fulfill that is that the value of π_3 is less than the value of π_1, since the odd sequence is decreasing. We note that π_3 cannot be smaller in value than π_2 because if so, π would be monotone by Lemma 30 part (2). When placing the letter π_4 we have two conditions to fulfill, first we have that $\pi_1\pi_2 \sim \pi_3\pi_4$ so since $\pi_1 > \pi_2$ the value of π_4 must be smaller than the value of π_3. Then, since the even sequence is also decreasing, π_4 must be smaller in value than π_2. So far we have only had one possible position to place each letter. But what happens now when we place the letter π_5? The conditions we have to fulfill here are, $\pi_5 < \pi_3$, $\pi_1\pi_2 \sim \pi_3\pi_4$ and $\pi_1\pi_2\pi_3 \sim \pi_3\pi_4\pi_5$. Now we can see that since π_3 is between π_1 and π_2 in value π_5 must be between π_3 and π_4 in value. But based on the conditions we have to fulfill we don't know where in relation to π_2 the letter π_5 can be placed. Therefore we have two possibilities of π_5, either it is between π_2 and π_4 in value or between π_2 and π_3 in value. Setting $\pi_5 < \pi_2$ leads to Figure 4.13, as we now show. Regardless of which possibility for π_5 we choose the letter π_6 must be smaller in value than π_4, for all conditions to be fulfilled, so we have only possible position for π_6.

When placing π_7, $\pi_7 < \pi_5$ and $\pi_2\pi_3\pi_4\pi_5 \sim \pi_4\pi_5\pi_6\pi_7$, are the two conditions that must be fulfilled in order for π to be an overlap permutation. There are some more conditions but they are fulfilled by fulfilling these two mentioned. Now we can see that before we can place π_7 we must know which possibility has been chosen for π_5. Assume we choose the first possibility for π_5, that is, π_5 is between π_2 and π_4 in value. Then we get that $\pi_2\pi_3\pi_4\pi_5 = 3412$, so so for the condition $\pi_2\pi_3\pi_4\pi_5 \sim \pi_4\pi_5\pi_6\pi_7$ to be fulfilled π_7 must be larger in value than π_6 and since both π_2 and π_3 are larger in value than π_4 and π_5, both π_4 and π_5 must be larger in value than π_6 and π_7. Hence, π_7 is between π_4 and π_6 in value. Now when placing π_8 and π_9 we can see that the same holds, that is, in order for the condition $\pi_4\pi_5\pi_6\pi_7 \sim \pi_6\pi_7\pi_8\pi_9$ to be fulfilled, π_8 must be less than π_6 and π_9 must be between π_8 and π_9 in value. And the same holds for the next pair, $\pi_{10}\pi_{11}$ and so on. So by choosing the first possibility for π_5 we fix the rest of the permutation. The form of π is shown in Figure 4.13.

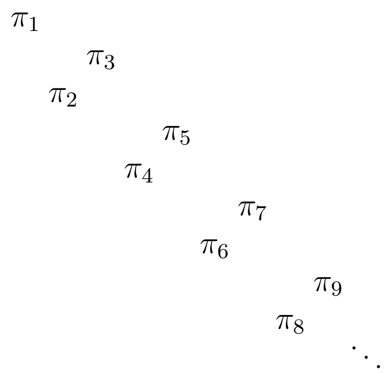


Figure 4.13: Both sequences decreasing, starting with a descent

Had we chosen $\pi_5 > \pi_2$ we would have ended arrived at Figure 4.14 with analogous arguments.

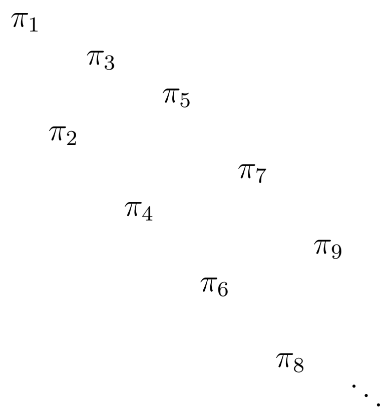


Figure 4.14: Both sequences decreasing, starting with a descent

From this example we get a good idea of how an overlap permutation from (1) looks like. We can think of it as the permutation $a_1b_1a_2b_2\dots$ where some fixed number of a's, let's say d, are larger in value than b_1, and after that only one a can be between each b, that is a_d will be the only a between b_1 and b_2 and a_{d+1} the only a between b_2 and b_3 and so on. This result is recorded in the following lemma.

Lemma 34. *Let π be an overlap-permutation of length n, where the odd and the even sequence are both decreasing and $\pi_1\pi_2$ is a descent. Then there exists $d \in \mathbb{N}$ such that π is the permutation*

$$a_1b_1a_2b_2\dots a_{\frac{n}{2}}b_{\frac{n}{2}},$$

if n is even and the permutation

$$a_1b_1a_2b_2\dots b_{\frac{n-1}{2}}a_{\frac{n+1}{2}}$$

if n is odd, where $b_i > a_k$ if and only if $k > i$ and $k - i \geq d$, where $k, i \in \mathbb{N}$.

We note that $\mathbb{N} = \{0, 1, \dots, \infty\}$. Let's look again at Figures 4.13 and 4.14. We can see that $d = 2$ for the former one and for the latter one we have $d = 3$.

Now let's look at permutation from (2), that is, both sequences are decreasing and $\pi_1\pi_2$ is an ascent. The letters π_1, π_2 and π_3 are fixed, that is, they can only be placed in one way because $\pi_1\pi_2$ is an ascent and π_3 must be smaller than π_1 so $\pi_1\pi_2\pi_3 = 231$. When placing π_4 we know that it must be larger than π_3 since $\pi_2 > \pi_1$ and it must be smaller than π_2. But there are no constraints on whether it should be larger or smaller than π_1, therefore we have two possibilities for π_4, (similar to π_5 above). Regardless of π_4 there is only one possibility for π_5, it must be smaller than π_3. But now when placing π_6 we must know where π_4 was placed. As for the case above, if π_4 is less than π_1 the rest of the permutation is fixed and π will be of form shown in Figure 4.15.

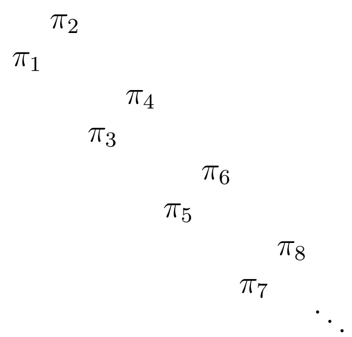


Figure 4.15: Both sequences decreasing, starting with an ascent

However, if π_4 is larger than π_1 we will have two possibilities for π_6, it must be between the values of π_3 and π_4 but it can either be smaller or larger than π_1 and if π_6 is smaller in value than π_1 the rest of the permutation is fixed but if π_6 is larger than π_1 in value there are two possibilities for π_8, either smaller or larger than π_1, and so on. The odd letters are always fixed.

Figure 4.16 shows the form of π if π_4 and π_6 are larger than π_1 but π_8 is smaller than π_1.

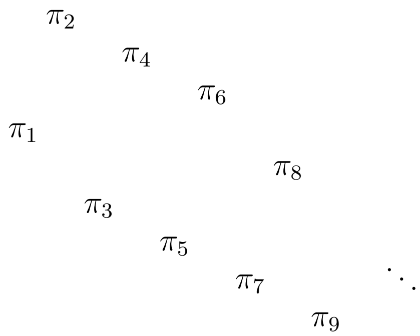


Figure 4.16: Both sequences decreasing, starting with an ascent

Similarly, as for (1), we get an idea of how permutations from (2) look like. As before we let d denote the number of a's that are larger in value than b_1 and the result is recorded in the following lemma.

Lemma 35. *Let π be an overlap-permutation of length n, where the odd and the even sequence are both decreasing and $\pi_1\pi_2$ is a ascent. Then there exists $d \in \mathbb{N}$ such that π is the permutation*

$$b_1a_1b_2a_2 \dots b_{\frac{n}{2}}a_{\frac{n}{2}},$$

if n is even and the permutation

$$b_1a_1b_2a_2 \dots a_{\frac{n-1}{2}}b_{\frac{n+1}{2}}$$

if n is odd, where $b_i > a_k$ if and only if $k > i$ and $k - i \geq d$, where $k, i \in \mathbb{N}$.

By looking at Figure 4.15 again we can see that here $d = 1$ and in Figure 4.16 $d = 3$

Now we move on to case (3). These permutations are the complement of the permutations in case (2) so we get the following lemma.

Lemma 36. *Let π be an overlap-permutation of length n, where the odd and the even sequences are both increasing and $\pi_1\pi_2$ is an ascent. Then there exists $d \in \mathbb{N}$ such that π*

must be the permutation

$$a_1 b_1 a_2 b_2 \dots a_{\frac{n}{2}} b_{\frac{n}{2}},$$

if n is even and the permutation

$$a_1 b_1 a_2 b_2 \dots b_{\frac{n-1}{2}} a_{\frac{n+1}{2}}$$

if n is odd, where $b_k > a_i$ if and only if $k > i$ and $k - i \geq d$, where $k, i \in \mathbb{N}$.

One example of a permutation from case 3 is shown in Figure 4.17.

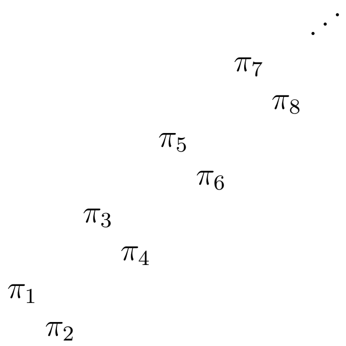


Figure 4.17: Both sequences increasing, starting with a descent

We had two possibilities when placing the letter π_4, it had to be either larger than π_1 and smaller than π_3 or smaller than π_1 and larger than π_2, and we chose the former one. It turns out that if π_4 is larger than π_1 the rest of the permutation will be fixed. If, however, π_4 is smaller than π_1, we will have two possibilities for π_6, either smaller or larger than π_1. If π_6 is larger than π_1 the rest of the permutation is fixed but if it is smaller than π_1, we will have two possibilities for π_8, either smaller or larger than π_1 and so on. The value of d for this permutation is $d = 1$.

Next we look at case (4). The permutations in this case are the complement of the permutations in case (1) which gives the following lemma.

Lemma 37. *Let π be an overlap-permutation of length n, where the odd and the even sequences are both increasing and $\pi_1\pi_2$ is a descent. Then there exists $d \in \mathbb{N}$ such that π must be the permutation*

$$b_1 a_1 b_2 a_2 \dots b_{\frac{n}{2}} a_{\frac{n}{2}},$$

if n is even and the permutation

$$b_1 a_1 b_2 a_2 \dots a_{\frac{n-1}{2}} b_{\frac{n+1}{2}}$$

if n is odd, where $b_k > a_i$ if and only if $k > i$ and $k - i \geq d$, where $k, i \in \mathbb{N}$.

In Figure 4.18 we show one example of a permutation from case (4), where $d = 2$.

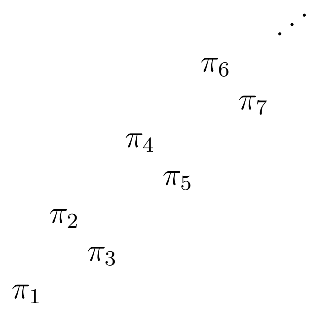


Figure 4.18: Both sequences increasing, starting with an ascent

Now that we have explained the exact structure of the non-monotone overlap-permutations we should have obtained good enough understanding of these permutations to be able to state the relevant lemmas and properties of these permutations. We will start by proving that the monotone permutations are the only permutations that are only children and overlap-permutations. We have shown that monotone permutations are only children and overlap-permutations, see Lemma 29. Below we will show that all other only child permutations are not overlap-permutations.

Lemma 38. *Let π be a non-monotone permutation. If π is an only child then π is not an overlap-permutation.*

Proof. Let $\pi = \pi_1\pi_2 \dots \pi_n$ be a non-monotone only child. We have that $\Delta(\pi) = \{\pi_1\pi_2 \dots \pi_{n-1}, \pi_2\pi_3 \dots \pi_n\}^{\downarrow}$. The parents obviously agree one way. Assume π is an overlap-permutation. Then

$$\pi_1\pi_2 \dots \pi_{n-2} \sim \pi_3\pi_4 \dots \pi_n.$$

Which gives that π's parents agree both ways and by Proposition 18 they must therefore have twins which contradicts the assumption that π is an only child. □

Another property of the overlap-permutations is stated below.

Lemma 39. *A non-monotone permutation π is an overlap-permutation if and only if π's parents agree both ways.*

Proof. We can use similar arguments as we used in Lemma 38, to show that a non-monotone permutation is an overlap-permutation if its parents agree both ways.

The other direction is straightforward, that is if π's parents agree both ways then

$$\pi_1\pi_2\ldots\pi_{n-2} \sim \pi_3\pi_4\ldots\pi_n,$$

so by definition π is an overlap-permutation. □

4.2 Lemmas used in the proof of Theorem 25

In this section we will state a few lemmas that will be used in the proof of Theorem 25. To prove most of these lemmas we will be using the properties of the overlap-permutations.

Lemma 40. *The permutation $\pi = a(a+1)(a+2)\gamma y$ is an only child if and only if $y \neq a-1$.*

Proof. First we want to show that if π is an only child then $y \neq a-1$ but that is the same as showing that if $y = a-1$ then π is not an only child. Let $y = a-1$ then $\pi = a(a+1)(a+2)\gamma(a-1)$ so by Lemma 23 we have that π's parents must agree exactly so π is not an only child.

Now we show that if $y \neq a-1$ then π must be an only child. We know that the monotone permutations are only children so by Lemma 19 we have that π is an only child if it is a monotone or if π's parents agree one way only and not exactly. Since $y \neq a-1$ we know that π is neither of the form $b\tau(b+1)$ nor $(b+1)\tau b$ so π's parents do not agree exactly. Obviously π's parents, $\alpha = (a(a+1)(a+2)\gamma)^{\downarrow y}$ and $\beta = ((a+1)(a+2)\gamma y)^{\downarrow a}$, agree one way. That is, π's parents agree one way and not exactly. And if the parents agree the other way we get

$$x(x+1)(x+2)\gamma_1\gamma_2\ldots\gamma_{n-2}\gamma_{n-1} \sim (x+2)\gamma_1\gamma_2\ldots\gamma_n y.$$

which only holds if α and β are monotone which implies that π is monotone and therefore an only child. Hence, either of the two must hold: π is not monotone and its parents agree one way only and not exactly so π is an only child. Or π is monotone and therefore an only child. □

Lemma 41. *The permutation $\pi = y\beta a(a+1)(a+2)$ is an only child if and only if $y \neq a+3$.*

Proof. This can be proved using similar methods as for Lemma 40. □

Lemma 42. *Let a and n be integers such that $1 \leq a \leq n-2$. Let γ be a string of length n whose letters are the integers from the set $\{1, 2, \ldots, n\} \setminus \{a, a+1\}$, each integer only appearing once. For any such string γ except for $\gamma = (a+2)\tau(a-1)$, either*

$$\gamma a(a+1) \text{ or } a(a+1)\gamma$$

is an only child, but not both.

Proof. Let $\pi = \gamma a(a+1)$ and $\pi' = a(a+1)\gamma$, where $\gamma = \gamma_1\tau\gamma_n$. We want to show that if $\gamma \neq (a+2)\tau(a-1)$ then either π or π' is an only child, but not both. To have $\gamma \neq (a+2)\tau(a-1)$ we have three cases:

1. Let $\gamma_1 = a+2$ but $\gamma_n \neq a-1$. Then

$$\pi = (a+2)\tau\gamma_n x(a+1)$$
$$\pi' = a(a+1)(a+2)\tau\gamma_n.$$

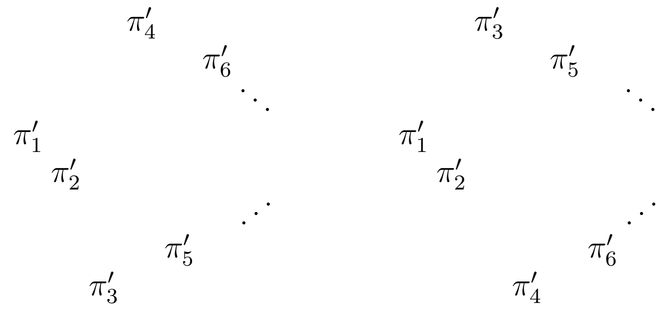
The permutation π' is an only child, by Lemma 40. But the parents of π agree exactly so π is not an only child.

2. Let $\gamma_1 \neq a+2$ but $\gamma_n = a-1$. Then

$$\pi = \gamma_1\tau(a-1)x(a+1)$$
$$\pi' = x(x+1)\gamma_1\tau(x-1).$$

Now, π is an only child by Lemma 41 but the parents of π' agree exactly by Lemma 23, so π' is not an only child.

3. Let $\gamma_1 \neq a+2$ and $\gamma_n \neq a-1$. We want to show that if π has siblings, π' must be an only child and vice versa. For π to have siblings π's parents must either agree exactly or both ways. But since $\gamma_1 \neq a+2$ and $\gamma_1 \neq a$, π's parents will not agree exactly, by Lemma 23. So π's parents must agree both ways. This gives that π must be an overlap-permutation, see Lemma 39. By Lemmas 33, 34, 35, 36, 37 we can see that for π to be an overlap permutation and end with the letters $a(a+1)$, π must either be an alternating wedge pointing to the right, see Figures 4.7 and 4.11 or π is a permutation where the odd and the even sequence are both decreasing, see Figures 4.13 and 4.15, so $\pi = n(n-2)(n-1)\ldots12$ or $\pi = (n-1)n(n-3)(n-2)\ldots12$. Let's now look at π'. If π is an alternating wedge pointing to the right π' will be of either of the two following forms:



If π is of the other form, π' will either be of the form $12n(n-2)(n-1)\ldots$ or $12(n-1)n(n-3)(n-2)\ldots$. Now we can see, by the lemmas stated above that π' is not an overlap permutation so its parents will not agree both ways and by Lemma 23 π' is not of the correct form for its parents to agree exactly. Hence, π' is an only child.

With a similar method we can show that if π' has siblings π must be an only child.

Hence, we have shown that if $\gamma \neq (a+2)\tau(a-1)$ then either of the permutations $\gamma a(a+1)$ or $a(a+1)\gamma$ is an only child.

□

Lemma 43. *Let a and n be integers such that $1 \leq a \leq n-3$. Let γ be a string of length n whose letters are the integers from the set $\{1, 2, \ldots, n\} \setminus \{a, a+1\}$, each integer only appearing once. For any such string γ except for $\gamma = (a-1)\tau(a+2)$, either*

$$\gamma(a+1)a \text{ or } (a+1)a\gamma$$

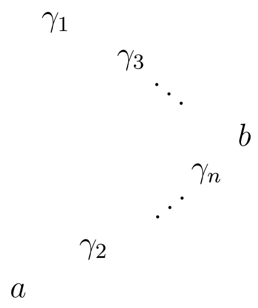
is an only child, but not both.

Proof. This fact can be proved using the same arguments as for the proof of Lemma 42. □

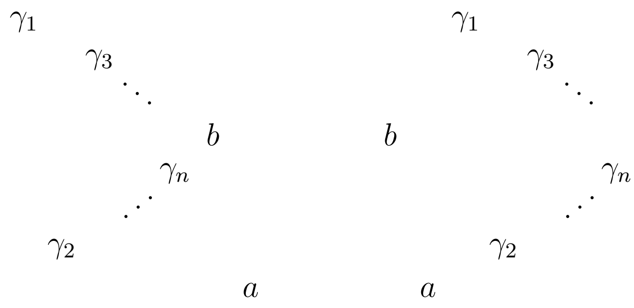
Corollary 44. *By combining the results from Lemmas 42 and 43 we get that for any γ at least one of the permutations $a(a+1)\gamma, (a+1)a\gamma, \gamma a(a+1)$ or $\gamma(a+1)a$ is an only child.*

Lemma 45. *If $a\gamma b$ is an overlap-permutation then neither γba nor $ba\gamma$ are overlap-permutations.*

Proof. Let $\pi = a\gamma b$ be an overlap-permutation. By Lemmas 33, 34, 35, 36, 37 we know that π can be of 8 different forms. Let π be an alternating wedge from Figure 4.7. That is π is of the following form:



So π' and π'' will be of the following forms, respectively:



By looking at Lemmas 33, 34, 35, 36, 37 stated above it is easy to see that neither π' nor π'' can be overlap-permutations. With similar arguments we can show that it doesn't matter which one of the 8 different forms π takes, both π' and π'' will not be overlap-permutations. □

4.3 Proof of Theorem 25

Now we have stated all the lemmas we need for the proof. But before we state the proof we give one definition.

Definition 46. Let α and β be two permutations that form a couple. The set of children the couple has is denoted by $\nabla(\alpha,\beta)$ and $|\nabla(\alpha,\beta)|$ is the number of children the couple has.

Proof of Theorem 25. For $f : S \to S$ to be an automorphism it must hold that $\Delta(\pi)^f = \Delta(\pi^f)$ for all $\pi \in S$. Let π and π' be two permutations. Then π and π' must have the same number of children as π^f and π'^f. This is so because f being an isomorphism gives that the children of π and π' must map to permutations whose parent are π^f and π'^f. That is, the children of π and π' must map to the children of π^f and π'^f, so it must hold that $|\nabla(\pi,\pi')| = |\nabla(\pi^f,\pi'^f)|$.

We will use induction. It is obvious that $f|S_{n\leq 2}$ must be a trivial map. Let's look at the permutations of length three. Since, there where no non-trivial automorphisms for permutations of length two we may assume $12 \mapsto 12$ and $21 \mapsto 21$. The permutation 123 has a unique shadow, that is it is the only permutation that has the shadow $\{12\}$, so it must map to itself. The same holds for the permutation 321, because it has the unique shadow, $\{21\}$. The other four permutations of length 3 have the same shadow, $\{12, 21\}$. Therefore, we would like see if we these four permutations can be interchanged in some way. Below we have listed all possible couples on level 3 along with the number of children the couple has.

$$\begin{array}{lll} |\nabla(123,132)| = 1, & |\nabla(132,213)| = 3, & |\nabla(213,321)| = 2, \\ |\nabla(123,213)| = 1, & |\nabla(132,312)| = 2, & |\nabla(231,312)| = 3, \\ |\nabla(123,231)| = 2, & |\nabla(132,321)| = 2, & |\nabla(231,321)| = 1, \\ |\nabla(123,312)| = 2, & |\nabla(213,231)| = 2, & |\nabla(312,321)| = 1. \end{array}$$

We note that the permutations 132 and 231 do not form a couple and are therefore not in the list above, the same holds for the complement permutations $213, 312$. We can see that the only two couples that have triplets are $\{132, 213\}$ and $\{231, 312\}$. Therefore, we have that either

$$\{132, 213\}^f = \{231, 312\} \text{ and } \{231, 312\}^f = \{132, 213\}$$

or

$$\{132, 213\}^f = \{132, 213\} \text{ and } \{231, 312\}^f = \{231, 312\}.$$

Let's begin by looking at the former case. We have the following possibilities:

1. $132 \mapsto 231$ and $213 \mapsto 312$
2. $132 \mapsto 312$ and $213 \mapsto 231$.

By looking at the list of possible couples we can see that for both of these possibilities the condition, $|\nabla(\pi, \pi')| = |\nabla(\pi^f, \pi'^f)|$, is broken. For example, for (1) we have that $|\nabla(123, 132)| = 1$ but

$$|\nabla(123^f, 132^f)| = |\nabla(123, 231)| = 2.$$

Now for the latter case we have the following possibilities:

1. $132 \leftrightarrow 213$, $231 \mapsto 231$ and $312 \mapsto 312$
2. $132 \mapsto 132$, $213 \mapsto 213$ and $231 \leftrightarrow 312$
3. $132 \leftrightarrow 213$ and $231 \leftrightarrow 312$

4. $132 \leftrightarrow 132$, $213 \leftrightarrow 213$, $231 \leftrightarrow 231$ and $312 \leftrightarrow 312$.

The first two possibilities cannot hold, since for (1), the couple $\{132, 312\}$ exists but not the couple $\{132^f, 312^f\} = \{213, 312\}$, and for (2), the couple $\{132, 312\}$ exists but not the couple $\{132^f, 312^f\} = \{132, 231\}$. If we now look at the third possibility and the list of possible couples we can see that for this possibility everything holds. But this map is reverse-complement and we assumed $12 \leftrightarrow 12$ and $21 \leftrightarrow 21$ for $n = 2$ which is indeed also reverse-complement. Similarly, the last possibility is the identity map and so is $12 \leftrightarrow 12$ and $21 \leftrightarrow 21$. Hence, we have shown that $f|S_{n\leq 3}$ is a trivial map.

Let $f : S \to S$ be a map so that $f|S_{n\geq 3}$ is the identity map. Then we want to show that f must also be the identity map on the next level, that is for all $\pi \in S_{n+1}$ it holds that $\pi^f = \pi$.

Because of the conditions for f stated above and since the induction hypothesis states that all permutations of length n and shorter must be mapped to themselves we have that a permutation of length $n + 1$ can either be mapped to itself or to one of its siblings. So we have four cases.

Let π be a permutations of length $n + 1$.

1. π is an only child: Since π has no siblings π must be map to itself.
2. π's parents agree exactly one way. This gives that π has a twin, π'. By Lemma 24 we have that $\{\pi, \pi'\} = \{x\alpha(x + 1), (x + 1)\alpha x\}$. Without loss of generality we assume $\pi = x\alpha(x+1)$ and $\pi' = (x+1)\alpha x$. Let $\sigma = \alpha(x+1)x$. The permutations π and σ agree exactly so they have at least two children, by Proposition 18. By Lemma 21 we get that these permutations do not fulfill the criteria to have three children, so $|\nabla(\pi, \sigma)| = 2$. Which gives that π and σ cannot agree both ways, so

$$x\alpha_1\alpha_2 \ldots \alpha_{n-1}\alpha_n \nsim \alpha_2\alpha_3 \ldots \alpha_n(x + 1)x.$$

   Let's now look at π' and σ. It is easy to see that they agree at least one way but not exactly with the suffix of π' and prefix of σ. But do they agree the other way? For that to happen we must have

$$(x + 1)\alpha_1\alpha_2 \ldots \alpha_n \sim \alpha_2\alpha_3 \ldots \alpha_n(x + 1)x$$

But above we showed that this cannot happen. So we have

$$|\nabla(x\alpha(x+1), \alpha(x+1)x)| = 2$$
$$|\nabla((x+1)\alpha x, \alpha(x+1)x)| = 1.$$

With similar arguments as here above we can show that there are three other cases. With $\sigma = (x+1)x\alpha$

$$|\nabla(x\alpha(x+1), (x+1)x\alpha)| = 2$$
$$|\nabla((x+1)\alpha x, (x+1)x\alpha)| = 1,$$

with $\sigma = x(x+1)\alpha$

$$|\nabla((x+1)\alpha x, x(x+1)\alpha)| = 2$$
$$|\nabla(x\alpha(x+1), x(x+1)\alpha)| = 1$$

and $\sigma = \alpha x(x+1)$

$$|\nabla((x+1)\alpha x, \alpha x(x+1))| = 2$$
$$|\nabla(x\alpha(x+1), \alpha x(x+1))| = 1.$$

By Corollary 44 we have that at least one of these σ's must be an only child and therefore must map to itself by part 1. Without loss of generality assume $\sigma = \alpha(x+1)x$ is an only child. Now if $\pi^f = \pi'$ the following should hold, $|\nabla(\pi^f, \sigma^f)| = |\nabla(\pi', \sigma)|$. But we showed above that this is not true. Hence, π must be mapped to itself.

3. π's parents agree both ways but not exactly. This gives that π has a twin, π'. Since their parents agree both ways π and π' are overlap-permutations, by Lemma 39. Let $\pi = a\tau b$ and $\pi' = c\gamma d$, now by Lemma 45 we know that both $\sigma = \tau ba$ and $\sigma' = ba\tau$ are not overlap-permutations. Which gives that their parents do not agree both ways so both σ and σ' must be permutations falling under (1) or (2), so they must both map to themselves.

   Let's look at π and σ. It is easy to see that these two permutations agree exactly so we have

$$|\nabla(\pi, \sigma)| \geq 2.$$

   Similarly,

$$|\nabla(\pi, \sigma')| \geq 2.$$

But let's now look at π' and σ. These two permutations cannot agree both ways, because that would imply that σ would be an overlap-permutation. Similarly, π' and σ' cannot agree both ways. But can π' and σ agree exactly and π' and σ' agree exactly? Let π' and σ agree exactly. Then either $\tau b = \gamma d$ or $\tau_2\tau_3 \dots \tau_n ba = c\gamma$. If the former statement holds we get that $\pi = \pi'$ which is not true so it must hold that $\tau_2\tau_3 \dots \tau_n ba = c\gamma = \alpha$. Which gives that $\pi' = \alpha d$ so $\sigma = d\tau_2\tau_2 \dots \tau_n ba$ or in other words $\tau_1 = d$. Now assume π' and σ' agree exactly. Then either $a\tau = c\gamma$ or $ba\tau_1\tau_2 \dots \tau_{n-1} = \gamma d$. As before the former implies that $\pi = \pi'$ which is not true so we have that $ba\tau_1\tau_2 \dots \tau_{n-1} = \gamma d$. But this cannot be true since here $\tau_1 \neq d$ so π' and σ' do not agree exactly. This gives that,

$$|\nabla(\pi', \sigma)| \leq 2 \text{ and } |\nabla(\pi', \sigma')| \leq 1.$$

So since σ' must be mapped to itself and

$$|\nabla(\pi, \sigma')| \neq |\nabla(\pi', \sigma')|$$

π and π' cannot be interchanged, so π must map to itself.

Similarly, if π' and σ' agree exactly then π' and σ cannot agree exactly. Which gives that

$$|\nabla(\pi, \sigma)| \neq |\nabla(\pi', \sigma)|.$$

So we get the same result, that is π must be mapped to itself.

4. π's parents agree both ways and exactly, so π is a triplet. By Lemma 21 we know that π and its two siblings are either

$$\{(m+1)1\tau m, m1\tau n(m+1), 1(m1\tau)^{\uparrow 1}(n+1)\}$$

or the reverse,

$$\{(m+1)n\tau 1m, mn\tau 1(m+1), (n+1)(\tau 1m)^{\uparrow 1}1\}.$$

It suffices to look at the former case. We want to show that these three permutations, the triplets, must all map to themselves. By using the same arguments as in (2) we can show that $(m+1)n\tau 1m, mn\tau 1(m+1)$ cannot be interchanged. Using the same arguments as in (3) we can show that $(n+1)(\tau 1m)^{\uparrow 1}1$ cannot be interchanged with either of the other two. So all three permutations must be mapped to themselves. □

Now we have shown that there are no non-trivial automorphisms of the consecutive poset which was indeed the same result as for the classical case. But what happens if instead of looking at the consecutive poset we look at consecutive pattern classes? This question will be the topic of the next chapter.

Chapter 5

A consecutive pattern class with a continuum of automorphisms

We mentioned in the introduction that in [1] Albert et al. moved the focus from the classical poset to classical pattern classes and studied order-preserving isomorphisms between such classes. What they were able to do was to find all maximal pattern classes and maximal isomorphisms between them. Our aim was to see if we would get similar results for consecutive pattern classes. But soon after we started this study we saw that our results would most likely be different from the classical case, partly because of the fact that in the consecutive case we have infinitely many non-recognizable permutations, see Theorem 11 while in the classical case they are finite. We were not able to find all maximal pattern classes nor all maximal isomorphisms between them. In what follows we show how different the classical case is compared to the consecutive case is. To do so we show that we can find a consecutive pattern class whose growth rate converges to 1 that has a continuum of automorphisms. So we can find a consecutive pattern class, almost as big as the consecutive poset, that has a continuum of automorphisms while in the classical case finitely many maximal isomorphisms were found.

We start by introducing a consecutive pattern class we call the ladder permutation class. Then we introduce the maps we use and show that they are automorphisms. At last we show that we have found a consecutive pattern class with a continuum of automorphisms with a growth rate that converges to 1.

5.1 The ladder permutation class

Definition 47. Let L be the set of all alternating wedges pointing to the left, see Figures 4.9 and 4.10, in addition to the permutations $1, 12$ and 21. The permutations in this set will be called the *ladder permutations*.

The set L, of ladder permutations, can also be defined in terms of consecutive pattern avoidance:

Proposition 48. *Let π be a permutations. Then π is a ladder permutation if and only if $\pi \in \mathrm{Av}(123, 132, 312, 321)$.*

Proof. By looking at the Figures 4.9 and 4.10, it is easy to see that ladder permutations must avoid the consecutive patterns $123, 132, 312$ and 321.

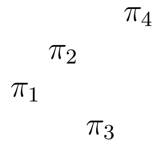
Let π be a permutation. To show that this statement also holds the other way we must show that if π avoids these permutations then it must be a ladder permutation. This is obvious if $\pi = 1, 12, 21$. So we assume π is a permutation of length more than 2 that avoids $123, 132, 312$ and 321. Now, if we try to construct the permutation with this in mind we get the following. The letters π_1 and π_2 can be placed either as a descent or an ascent, assume it is ascent, so we have

$$\begin{array}{cc} & \pi_2 \\ \pi_1 & \end{array}$$

Now when placing π_3 we must be careful. First we see that no matter where we place π_3 we will not be able to create an occurrence of 312 and 321, since π starts with an ascent. Then we see that the value of π_3 cannot be larger than the value of π_2 because, if so π would contain 123. Similarly, if π_3 is between π_1 and π_2 in values π would contain 132. Hence, there is only one place where we can but π_3, that is the value of π_3 must be smaller than the value of π_1. So we have,

$$\begin{array}{ccc} & \pi_2 & \\ \pi_1 & & \\ & & \pi_3 \end{array}$$

Similarly, for π_4 we get that the value of π_4 cannot be smaller that the value of π_3 because then π would contain 321, the value cannot be between π_2 and π_3, because otherwise π would contain 312. Hence, π_4 must be larger in value then π_2. So now π looks as follows.



By continuing this way we can see that π will become a ladder permutation of the form showed in Figure 4.10. Similarly, if π starts with a descent it would become a ladder permutation of the form Figure 4.9. □

From now on the set L will be called the *ladder permutation class*. There are exactly two ladder permutations of each length, except for of course of length 1. The first ladder permutations are shown in Figure 5.1. We will be using the letters α_i and β_i, with $\alpha_i < \beta_i$ in the lexicographic order, to denote the two ladder permutations of length i, or in other words the ladder permutations on level i.

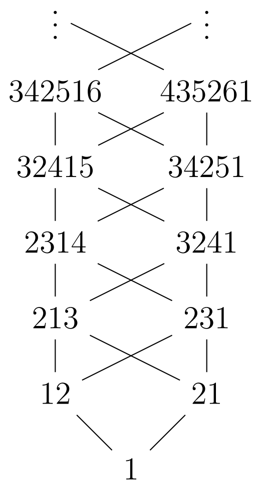


Figure 5.1: The ladder permutations on levels 1 to 6

In Figure 5.1 the α_i permutations are on the left side and the β_i permutations on the right side. The word ladder is used for this pattern class because the two ladder permutations on level i have the same parents, which are the two ladder permutations on level $i-1$. In the figure this is denoted with lines between a permutations and their parents. This fact gives the ladder permutation class a nice property which is stated below.

Lemma 49. *If a permutation π avoids a ladder permutation of length i, then it will avoid all ladder permutations of length $j > i$.*

Proof. What we want to prove is that if π avoids p_i (where $p_i = \alpha_i$ or $p_i = \beta_i$) then π will avoid α_j and β_j for $0 < i < j$. We have said that the parents of a ladder permutation are the two ladder permutations one level down. In other words p_i contains both α_{i-1} and β_{i-1} which both contain α_{i-2} and β_{i-2} and so forth. This gives that if π avoids p_i then it must also avoid α_{i-1} and β_{i-1} because they are both contained within p_i. Then if it avoids α_{i-1} it must also avoid both α_{i-2} and β_{i-2} since they are contained with in π and so on. So this must hold for all α_j and β_j where $0 < j < i$. □

Now we have introduced the ladder permutation class and some of its properties. So now it's time to introduce the map used.

5.2 A pattern class whose growth rate converges to 1

Definition 50. We define the map $g_i : S \to S$ as follows

$$\pi^{g_i} = \begin{cases} \pi & \text{if } \pi \neq \alpha_i, \beta_i \\ \beta_i & \text{if } \pi = \alpha_i, \\ \alpha_i & \text{if } \pi = \beta_i. \end{cases}$$

The map g_i maps all permutations to themselves, except for the ladder permutations, which are interchanged within a level.

It is not so hard to see that this map is an automorphism on the ladder permutation class. But remember we said that we found a pattern class whose growth rate converges to 1, and that does not hold for the ladder permutation class. Hence, we will introduce another pattern class. We will not only introduce one new pattern class, we will indeed introduce a cluster of pattern classes, where each pattern class is denoted with C_i and it is a union of the ladder permutation class and the avoiders of the ladder permutations α_i and β_i. Now we will show that g_i is an automorphism on C_i.

Lemma 51. *Let $C_i = \mathrm{Av}(\alpha_i, \beta_i) \cup L$. The map $g_i|_{C_i}$ is an automorphism, for $i \geq 3$.*

Proof. For $g_i|_{C_i}$ to be an automorphism it must be a bijection and for each $\pi \in C_i$ it must hold that $\Delta(\pi)^{g_i} = \Delta(\pi^{g_i})$.

First we note that the restriction $g_i|_{C_i}$ is obviously a bijection, by Lemma 49. Now let $\pi \in C_i = \mathrm{Av}(\alpha_i, \beta_i) \cup L$ then we have the following three cases:

1. We have, $\pi \in \mathrm{Av}(\alpha_i, \beta_i)$. We want to show that

$$\Delta(\pi)^{g_i} = \Delta(\pi^{g_i}).$$

If π is monotone then it has one parent, that is also monotone. The monotone permutations are not elements of the ladder and must therefore be mapped to themselves, so we get $\Delta(\pi)^{g_i} = \Delta(\pi^{g_i}) = \Delta(\pi^{g_i})$. If π is not monotone it has two parents. Let $\Delta(\pi) = \{\gamma, \tau\}$. We know that $\pi^{g_i} = \pi$ and $\tau, \gamma \in \mathrm{Av}(\alpha_i, \beta_i)$. Hence,

$$\begin{aligned}\Delta(\pi)^{g_i} &= \{\gamma, \tau\}^{g_i}\\ &= \{\gamma, \tau\}\\ &= \Delta(\pi)\\ &= \Delta(\pi^{g_i}).\end{aligned}$$

2. $\pi = \alpha_i$ or $\pi = \beta_i$. We have that $\Delta(\alpha_i) = \Delta(\beta_i) = \{\alpha_{i-1}, \beta_{i-1}\}$ and $\alpha_i^{g_i} = \beta_i$. This gives,

$$\begin{aligned}\Delta(\alpha_i)^{g_i} &= \{\alpha_{i-1}, \beta_{i-1}\}^{g_i}\\ &= \{\alpha_{i-1}, \beta_{i-1}\}\\ &= \Delta(\beta_i)\\ &= \Delta(\alpha_i^{g_i})\end{aligned}$$

Similarly, we can show that this also holds for $\pi = \beta_i$.

3. $\pi \in L$ and $\alpha_i \neq \pi \neq \beta_i$. Let $\pi = \alpha_j$, where $j \neq i$. We have that

$$\begin{aligned}\Delta(\alpha_j)^{g_i} &= \{\alpha_{j-1}, \beta_{j-1}\}^{g_i}\\ &= \{\alpha_{j-1}, \beta_{j-1}\}\\ &= \Delta(\alpha_j)\\ &= \Delta(\alpha_j^{g_i}).\end{aligned}$$

Similarly, we can show that this also holds for $\pi = \beta_j$.

Now we have shown that for all $\pi \in C_i$ it holds that $\Delta(\pi)^f = \Delta(\pi^f)$. Hence, g_i is an automorphism. □

We have now found one automorphism for the pattern class C_i. But we stated that we could find a continuum of automorphisms. Therefore we define the following map:

Definition 52. Let $U \subseteq \mathbb{N}$. We define $g_U : S \to S$ as

$$\pi^{g_U} = \begin{cases} \pi & \text{if } \pi \notin L, \text{ or } \pi \in L \text{ and } |\pi| \notin U \\ \beta_u & \text{if } \pi = \alpha_u \text{ and } u \in U \\ \alpha_u & \text{if } \pi = \beta_u \text{ and } u \in U. \end{cases}$$

Lemma 53. *Let u_0 be the smallest element in $U \subseteq \mathbb{N}$ then $g_U|_{C_{u_0}}$ is an automorphism.*

Proof. This lemma can be proved using the properties of the ladder permutations and the same method that was used to proof Lemma 51. □

Now, the only thing left is to show that the growth rate of the pattern class C_i converges to 1 as i goes to infinity. But how is the growth rate of a pattern class defined? Before we can answer that question we will define the growth rate in terms of one consecutive pattern. Elizalde showed in [6, Theorem 4.1] that the growth rate exists for any consecutive pattern and it is strictly between 0 and 1:

Definition 54. Let $|\mathrm{Av}_n(\sigma)|$ denote the number of permutations of length n that avoid the consecutive pattern σ. For every consecutive pattern σ the limit

$$\lim_{n\to\infty} \left(\frac{|\mathrm{Av}_n(\sigma)|}{n!} \right)^{\frac{1}{n}} = \rho_\sigma$$

exists and $0 < \rho_\sigma < 1$. This limit is called the *growth rate* of σ.

Similarly, we have define growth rate for pattern classes.

Definition 55. Let $|\mathrm{Av}_n(C)|$ denote the number of permutations of length n that avoid all the consecutive patterns in the pattern class C. The limit

$$\lim_{n\to\infty} \left(\frac{|\mathrm{Av}_n(C)|}{n!} \right)^{\frac{1}{n}} = \rho_\sigma$$

exists and $0 < \rho_C < 1$. This limit is called the *growth rate* of the pattern class C.

Another result of Elizalde, related to the one above, is given in the following definition.

Definition 56. Let $|\mathrm{Av}_n(\sigma)|$ be the number of permutations of length n that avoid π. Then

$$a_n(\sigma) = \mathcal{O}(\rho_\sigma^n n!),$$

where ρ_σ is the growth rate for the consecutive pattern π.

This gives an upper bound for the growth rate of the avoiders of a given permutation. Now we have all the information we need to show that that the growth rate of the pattern class C_i converges to 1 as i grows.

Theorem 57. *The growth rate of the pattern class C_i goes to 1 as i goes to infinity.*

Proof. By Lemma 49 we have that if a permutation avoids α_{i-1} then it must avoid all ladder permutations of length $j > i$ so we have that $\mathrm{Av}(\alpha_{i-1}) \subseteq C_i$. Elizalde showed in [7, Theorem 2.10 and 4.1] that for all $\pi \in S_n$ it holds that

$$\mathrm{Av}(12\ldots(n-2)n(n-1)) \subseteq \mathrm{Av}(\pi) \subseteq \mathrm{Av}(12\ldots n).$$

Which gives

$$|\mathrm{Av}(\alpha_{n-1})| \geq |\mathrm{Av}(12\ldots(n-3)(n-1)(n-2))|.$$

We know that if a permutation avoids the increasing consecutive pattern $12\ldots(n-2)$ then it will also avoid the consecutive pattern $12\ldots(n-3)(n-1)(n-2)$. So

$$|\mathrm{Av}(12\ldots(n-3)(n-1)(n-2))| \geq |\mathrm{Av}(12\ldots(n-2))|,$$

which gives that

$$|\mathrm{Av}(\alpha_{n-1})| \geq |\mathrm{Av}(12\ldots(n-2))|.$$

So since $\mathrm{Av}(\alpha_{i-1}) \subseteq C_i$ we have that the growth rate of C_i is larger or equal to the growth rate of $\mathrm{Av}(12\ldots(n-2))$.

Warlimont showed in [12] that for the growth rate of the monotone permutation $12\ldots n$ we have

$$\frac{1}{\rho} = 1 + \frac{1}{n!},$$

which gives that $\rho \to 1$ as $n \to \infty$, or in other words, the growth rate of avoiding the monotone permutation $12\ldots n$ converges to 1. Hence, the growth rate of C_i must also converge to 1. □

Now we have a pattern class, C_i who's growth rate converges to 1 that has continuum of automorphisms, which is what we wanted to show.

Now we have stated our main results. In the next chapter we will conclude by stating our main results and pose some open problems and future work on the topic of this thesis, isomorphisms between consecutive pattern classes, and some related topics.

Chapter 6

Conclusions

In this thesis we showed that there are no non-trivial automorphisms of the consecutive poset. We showed that the set of non-recognizable permutations is infinite in the consecutive case and we found a consecutive pattern class, $C_i = \mathrm{Av}(\alpha_i, \beta_i) \cup L$, who's growth rate converges to one as i goes to infinity, that has uncountably many automorphisms. These are the main results of the thesis but what is the most interesting thing that we have learned and what surprised us the most, is how the study of the consecutive shadow is different form the study of the classical shadow in some aspects and then the same in other. Both in the consecutive case and the classical the only automorphisms of the full poset are the trivial ones. Regarding the non-recognizable permutations we have exactly the opposite result for the classical case and the consecutive case. In the classical case the set of non-recognizable permutations is finite while in the consecutive case it is infinite. Then when studying isomorphisms between pattern classes we found a large pattern class, as large as it can be in some sense, that has a continuum of automorphisms while in the classical case maximal pattern classes were found and they all had finitely many isomorphisms. So even though we have not fully investigated isomorphisms between consecutive pattern classes this study gave us the result in the consecutive case will differ a lot from the result in the classical case. This brings us to the discussion of future work related to this research. First I would like to mention that when we started this research we were hoping for a similar result as Albert et al. got for the classical case in [1] but instead we showed that the result in the consecutive case will be different from the classical case. Hopefully someone can find a nice result that describes isomorphisms between consecutive pattern classes.

Another open question is whether similar research can be done for other kinds of patterns. We could start by looking at quasi-consecutive patterns [3], vincular patterns [2] or perhaps mesh patterns [4]. But to do this we must be able to define the poset for these patterns. That is, if p and q are two patterns we want to define what $q < p$ means. This is easy to do for both classical and consecutive patterns: If p and q are either classical or consecutive patterns we have that $q < p$ if and only if $\mathrm{Av}(q) \subseteq \mathrm{Av}(p)$. But this is much more difficult in the case of more general patterns, see for example [11] and [5].

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