GENERALIZED STAR POLYGONS AND STAR POLYGRAMS

EIDUR SVEINN GUNNARSSON AND KARL THORLAKSSON

ABSTRACT. We extend the definition of regular star polygons and other regular star polygrams to be drawn within some non self-intersecting polygon instead of just regular polygons. Instead of only connecting corners some constant apart, we allow any two corners of the polygon to be connected to form a figure. These figures can be thought of as permutations consisting of some cycles where each cycle forms a polygon. We define new permutation patterns called interacting cycle patterns that we use to describe properties of the permutations. We describe some necessary properties for a permutation drawn within a polygon, such that its drawing forms a star. Then we describe sufficient properties of permutations, such that when drawn within any available polygon they form stars. Finally we describe sufficient properties of permutation drawings, that form a star in some polygon.

1. Introduction

Thomas Bradwardine, an English 14th-century mathematician, was the first to study star polygons in Geometria Speculativa. Later in 1619 Johannes Kepler studied them further in Harmonices Mundi. They both defined them from regular polygons in similar fashion, Kepler defined star polygons as an augmentation of regular polygons, by extending each side of a regular polygon until it meets a non-neighbouring side forming a vertex. See Figure 1.1 for an example of how a pentagon is augmented to a 5-pointed star.

![Figure 1.1. Scan from Kepler’s Harmonices Mundi](image)

Star polygons have been studied and used for tilings, tessellations and forming higher order polytopes.

Definition 1.1 (Polygon). A polygon is an ordered set of points \( \{O_0, O_1, \ldots O_{n-1}\} \), where consecutive points in the set along with the points \( O_0 \) and \( O_{n-1} \) are connected by a line segment. The line segments are called sides and the points from the set are called corners. The size of a polygon is its number of corners.

Definition 1.2 (Regular polygon). A regular polygon is a polygon where all sides are of equal length and the interior angle between each two consecutive sides are equal. An example of a regular polygon can be seen in Figure 1.2.
The *Schläfli symbol* notation is used to describe regular polygons, polyhedra, and their higher-dimensional counterparts. The Schläfli symbol \{n\} is used to denote a regular polygon of size \(n\). For example, the regular polygon in Figure 1.2 is denoted as \{5\}. *Regular star polygons* are denoted by the Schläfli symbol \{\frac{p}{q}\} where \(\frac{p}{q}\) is a fully reduced fraction. The regular star polygon \{\frac{p}{q}\} has the same vertices \(\{O_0, \ldots, O_{p-1}\}\) as the regular polygon \{p\}. However, now the line segments connect each point \(O_k\) to \(O_{k+q}\), where indices are taken modulo \(p\). Figure 1.3 shows an example of the regular star polygon \{\frac{7}{2}\}.

The Schläfli symbol \{p, q, r, \ldots, s\}, where \(p, q, r, \ldots, s\) is a list of rationals, is used to denote regular polytopes and tilings. These are regular polygons and regular star polygons (2-polytopes) amongst regular polyhedra (3-polytopes) and so on, along with tilings of these shapes. For Schläfli symbol for higher dimensional objects and tessellations refer to Coxeter’s *Regular Polytopes*. [3]

**Definition 1.3** (Boundary polygon). A *boundary polygon* is a non self-intersecting polygon, that is a polygon where its sides only meet at its corners.

**Definition 1.4** (Figure). Given a boundary polygon, a *figure* is a collection of *vertices* given by its corners. Each vertex of the figure is connected to exactly two other distinct vertices by line segments called *edges*. The edges must not lie outside of the boundary polygon and each edge connects precisely two vertices. The size of a figure is its number of vertices.
Given the boundary polygon $P$ seen in Figure 1.4, we can form a figure. The edges connecting to corner 0 can be any two of \{(0, 1), (0, 2), (0, 5)\}. However we cannot use the edge (0, 4) or (0, 3), since (0, 4) would lie outside of $P$ while (0, 3) would connect three vertices by also intersecting corner 5. So either edge will break the requirements for this to be a figure.

A permutation is a one-to-one correspondence from the set $\mathbb{Z}_n = \{0, 1, \ldots, n-1\}$ to itself. These will be written in cycle notation, for example the permutation $\pi = (021)(34)(5)$ maps

\[
\begin{align*}
0 & \rightarrow 2 \\
2 & \rightarrow 1 \\
1 & \rightarrow 0 \\
3 & \rightarrow 4 \\
4 & \rightarrow 3 \\
5 & \rightarrow 5.
\end{align*}
\]

The permutation $\pi$ has three cycles (021), (34) and (5) of lengths 3, 2 and 1 respectively. The length of a permutation is the number of elements of the set $\mathbb{Z}_n$. So the length of $\pi$ is 6. Permutations are commonly written in one-line notation, where a permutation of length $n$, $\sigma = \sigma_0\sigma_1\cdots\sigma_{n-1}$ maps $i \rightarrow \sigma_i$. So permutation the permutation in the example above is written $\pi = 201435$ in one-line notation.

We will use permutations to indicate which vertices are connected by an edge in a figure.

**Definition 1.5** (Permutation drawing). Let $P = \{O_0, O_1, \ldots, O_{n-1}\}$ be a boundary polygon and $\pi$ be a permutation of length $n$. Given the pair $(\pi, P)$, a drawing is formed.

As a basis for a drawing use the corners of $P$. Then draw line segments as indicated by $\pi$, such that when $\pi$ maps $a \rightarrow b$ then a line segment is drawn between vertices $O_a$ and $O_b$. This drawing is a figure if and only if the conditions of Definition 1.4 are met.

*Remark.* Since the edges of a drawing are not directed like the mappings in permutations, reversing any cycle of a permutation will result in the same drawing.

In the following section we will explore figures further and use them to define generalized stars. In Section 3 we look at some necessary properties of permutations, allowing us to exclude permutations that never form stars. Then in Section 4 we look at sufficient properties of permutations that, when drawn in a polygon such that they form a figure, will always be a star. In Section 5 we look at sufficient properties of permutations, forming a star in some boundary polygon, which is formed from the permutation instead of being chosen beforehand. Finally in Section 6 we look at some open problems and ways that stars can be explored further.

2. Generalized stars

What is arguably most iconic about the shape of regular star polygons is how, when drawn within polygons, their complement area is a collection of triangles. Therefore we define generalized stars as a subset of figures that have triangles between all of their arms.

**Definition 2.1** (Boundary triangles). The area bounded by a figure and its boundary polygon is a collection of polygons. Those of which are triangles are called boundary triangles.
In Figure 2.1 the boundary triangles of the figure have been shaded. The polygon formed between vertices 3 and 4 is not a boundary triangle because of the edge connecting vertices 2 and 5.

![Figure 2.1](image)

**Figure 2.1.** Shaded regions represent the boundary triangles of this figure

**Definition 2.2 (Star).** A figure is a *star (figure)* if the size of its boundary polygon is the same as its number of boundary triangles.

![Figure 2.2](image)

**Figure 2.2.** A figure that is a star

Figures that can be drawn in one continuous trace are called *unicursal*, like the figure in Figure 2.2. *Multicursal* figures are drawn with more than one trace, like the figure in Figure 2.3 which is drawn with two traces. A trace are continuously drawn line segments between the corners of a boundary polygon, where each corner is visited exactly once except the start and end corner. Unicursal figures are polygons, while multicursal figures are polygon compounds. The number of cycles a permutation has tells us how many traces we need to draw a figure.

![Figure 2.3](image)

**Figure 2.3.** A multicursal star drawn with two traces
Definition 2.3 (Regular stars). Regular stars are stars which consist of the regular star polygons denoted by the Schläfli symbol \( \{p/q\} \) and regular polygon compounds. The regular star polygons are unicursal while the regular polygon compounds are multicursal. Regular polygon compounds are denoted by the Schläfli symbol \( g\{p/q\} \) where \( p/q \) is a reduced fraction and \( g \) is an integer greater than 1. A regular polygon compound consists of the same vertices as \( \{gp\} \), where each corner \( O_k \) is connected to \( O_{k+gq} \) by an edge.

Remark. The regular star polygon given by the Schläfli symbol \( \{p/q\} \), where \( p/q \) is a fully reduced fraction, is the figure \( F = (\pi, P) \). Where \( P \) is a regular polygon of size \( p \) and \( \pi = (0, q, 2q, \ldots, (p-1)q) \), where all elements are modulo \( p \). Similarly a regular polygon compound \( g\{p/q\} \) is the figure \( (\pi, \{gp\}) \) where \( \pi = (0, gq, \ldots, (p-1)gq) \cdots ((g-1) + 0, (g-1) + gq, \ldots, (g-1) + (p-1)gq) \).

Definition 2.4 (Irregular stars). All stars that are not regular are irregular. They are further split into unicursal and multicursal irregular stars as with the regular stars.

In Figure 2.4 we see examples of one star from each class of stars defined in Definition 2.3 and Definition 2.4

If we draw the permutation \( (0362514) \) in a regular polygon we get the star \( \{7/3\} \), as can be seen in Figure 2.5a. However when the permutation is drawn in a slightly different polygon we get Figure 2.5b, where the edge connecting vertices 2 and 6 now cuts into what would have been the boundary triangle between vertices 0 and 1. This results in the second figure not being a star since it only has six boundary
triangles. So we can see that whether a permutation creates a star or not depends on the boundary polygon we draw it in.

For further exploration of figures and stars, we define some terminology and notation to ease discussion on their properties.

**Definition 2.5** (Adjacent vertices). Vertices are adjacent when their corresponding corners on their boundary polygon share a side.

**Remark.** If $a$ and $b$ are adjacent vertices this means that their indices are adjacent in value, modulo the size of their boundary polygon. They are also adjacent in value in their permutation, modulo the length of the permutation.

**Definition 2.6** (Neighbouring vertices). Vertices are neighbours when they share an edge on their figure.

**Remark.** If $a$ and $b$ are neighbouring vertices they are adjacent in the cyclic permutation of their figure. Equivalently the permutation either maps $a \rightarrow b$ or $b \rightarrow a$.

![Figure 2.6. The regular five armed star ((02413), {5})](image)

In Figure 2.6 the boundary polygon is indicated by the dotted lines and the figure is indicated by the solid lines. As we can see vertices 4 and 0 are connected by a side of the boundary polygon, meaning that they are adjacent. So are corners 0 and 1 and so on. If we look at vertices 0 and 2 we can see that they are connected by an edge of the figure meaning that they are neighbouring vertices. So are vertices 2 and 4, vertices 4 and 1 and so on.

**Definition 2.7** (Relative ordering of vertices of a figure). The comparison of the relative ordering of vertices of a figure is written $a <_c b$. This states that when traversing through the ordered set of vertices $\{O_c, O_{c+1}, \ldots, O_{c+n-1}\}$, where all indices are modulo $n$, the element $O_a$ is before $O_b$.

The comparison $a - c < b - c \pmod n$ is equivalent to $a <_c b$ for a polygon of size $n$. This subscript notation can be used for binary relations such as $<_a, >_a, \leq_a$ and $\geq_a$.

For example, in figure Figure 2.6 we have the following ordering $2 <_2 3 <_2 4 <_2 0 <_2 1$.

**Definition 2.8** (Inner and outer edges of adjacent vertices). Two adjacent and non-neighbouring vertices of a figure, $a$ and $a+1$, are each connected to two neighbouring vertices by certain edges of the figure. These are the edges $(a, b)$, $(a, c)$, $(a + 1, d)$ and $(a + 1, e)$, where $c >_a b$ and $d >_a e$. Then $(a, b)$ and $(a + 1, d)$ are the inner edges of $a$ and $a + 1$ and $(a, c)$ and $(a + 1, e)$ are the outer edges of $a$ and $a + 1$. See Figure 2.7.

An edge incident to a vertex $p$ is an inner edge to one of its adjacent vertices and an outer edge to its other adjacent vertex.
3. Necessary Properties

The permutation alone can often determine if a drawing is a star, regardless of which boundary polygon it is drawn in. We will discuss three properties of permutations that do just that in this section, thereby reducing the number of permutations to examine. To describe these properties we define a new permutation pattern called *interacting cycle patterns*.

**Proposition 3.1.** A figure exists corresponding to a permutation $\pi$ if and only if $\pi$ has neither 1-cycles nor 2-cycles.

*Proof.* Suppose $\pi$ contains either 1-cycles or 2-cycles. In the first case when $\pi$ contains a 1-cycle, there exists an element $a$ in $\pi$ which maps to itself. If $\pi$ is drawn in a boundary polygon, the vertex labelled $a$ will not be connected to any other vertices. In the second case when $\pi$ contains a 2-cycle, $\pi$ contains two elements $a$ and $b$ that map to each other that is $a \rightarrow b$ and $b \rightarrow a$. If $\pi$ is drawn in a boundary polygon, the vertices labelled $a$ and $b$ will only be connected to each other. In both of these cases we have a vertex in our drawing which is not connected to exactly two other distinct vertices, thus the permutation does not create a figure.

If all cycles of $\pi$ are of at least length three, then for each element $a$ in $\pi$ there exist two distinct elements $b$ and $c$ in $\pi$ where $b \rightarrow a$ and $a \rightarrow c$. Thus when drawing $\pi$ in a boundary polygon, $a$ will be connected to two distinct vertices $b$ and $c$. Let the length of $\pi$ be $n$, then the drawing formed by $(\pi, \{n\})$ must be a figure; since any edge drawn between two corners of a regular polygon will not go outside of the boundary polygon and the edges will only intersect corners at their end points. □

**Proposition 3.2.** If any two adjacent vertices of a figure are neighbours then the figure is not a star.

*Proof.* If two vertices of a figure are both adjacent and neighbours, then they are connected by a line segment in both the figure and its boundary polygon. Hence there will be no boundary triangle between these vertices. This means that the boundary triangles of the figure will be fewer than the size of the figure, thus the figure is not a star. □

In the figure seen in Figure 3.1 vertices 4 and 5 are adjacent and neighbouring vertices. This leads to the figure having only 5 instead of 6 boundary triangles, thus the figure is not a star.

Figures where some two adjacent vertices of a figure are neighbours is represented by the *figure pattern* seen in Figure 3.2. The relative order of the vertices on the boundary polygon is shown by where they are drawn on the circle. The filled arc represents that these vertices are adjacent while dotted arcs mean some amount of vertices are between them on the boundary polygon. The straight line segment
represents that these vertices are neighbours. We will often use these kind of figure patterns to visually represent properties of figures throughout this paper.

Pattern matching is commonly used in the study of permutations. A permutation is said to contain a pattern $p$ if there exists a subsequence of elements that order isomorphic $p$. The subsequences are normalized in value, the smallest element of the subsequence is 0, the second smallest is 1 and so on. For example the subsequence 253 would be normalized to 021.

The permutation $\pi = 0213$, written in one-line notation, has two occurrences of pattern $p = 012$. The subsequences 023 and 013, so $\pi$ contains $p$. However $\pi$ has no occurrence of $p' = 120$, so $\pi$ avoids $p'$.

Jones and Remmel [4] describe how a permutation cycle can contain or avoid a pattern. Here the subsequences are taken from cycles which can wrap around. For example let $\pi = (0342)(167)(5)$ be a permutation and $p = (012)$ be a pattern. Here the first cycle has two occurrences of $p$, 034 and 234. The second cycle has one occurrence of $p$, that is 167, and the third cycle has no occurrences of $p$.

Neighbouring vertices in a figure, are such that the corresponding elements in a permutation are adjacent in some cycle. Likewise when vertices are adjacent in a figure, the corresponding elements in the permutation are adjacent in value, modulo the length of the permutation. Thus when describing patterns of permutations, we want to be able to enforce both adjacency in cycles and adjacency in value. This notion of patterns where some elements need to be adjacent in value or in the permutation are similar to bivincular patterns, defined by Bousquet-Mélou et al. [1].

We could combine the ideas of patterns in cycles and bivincular patterns to describe patterns in our figures, this would be enough for unicursal figures where the permutation consists of a single cycle. However, these patterns would not capture interaction between the different polygons in multicursal figures, so single cycle patterns are not sufficient. Rather we need our patterns to be a collection of cycles, we will call these interacting cycle patterns, formally defined below.

**Definition 3.3** (Substring of a permutation cycle). A substring $s$ is an ordered set of consecutive elements of a permutation cycle. Let $\pi$ be a permutation of $k$ cycles $\pi = C_0C_1\cdots C_{k-1}$ where each cycle $C_i = (C_{i,0},\ldots,C_{i,p_i-1})$ is of length $p_i$. A substring from cycle $C_i$ of length $\ell$ is such that $\ell \leq p_i$ then $s = \{C_{i,j},C_{i,j+1},\ldots,C_{i,j+\ell-1}\}$ is a substring from $C_i$, where element indicies are modulo $p_i$. 
For example $\pi = (0123)(45)$ has 4 substrings of length 3,
$$S = \{\{0,1,2\}, \{1,2,3\}, \{2,3,0\}, \{3,0,1\}\}$$
all from the first cycle of $\pi$, but none from the second cycle since it is of length $2 \geq 3$.

**Definition 3.4** (Normalizing substrings in a set of substrings). Let $S$ be a set of substrings of size $k$, $S = \{s_0, \ldots, s_{k-1}\}$. Each substring $s_i = \{a_0, a_1, \ldots, a_{p_i-1}\}$ of length $p_i$ is normalized by
$$\text{norm}_S(s_i) := (f_S(a_0), f_S(a_1), \ldots, f_S(a_{p_i-1})).$$
Where $f(a_i)$ is the number of elements in the union of the substrings that are less than $a_i$,
$$f_S(a_i) = |\{s_{i,j} : s_{i,j} < a_i, \exists s_{i,j} \in S, \forall s_{i,j} \in s_i\}|.$$

Take the following set of substrings $S = \{s_0, s_1, s_2\} = \{\{1,6\}, \{2,4\}, \{7\}\}$, here the normalized substrings would be $\text{norm}_S(s_0) = (0,3)$, $\text{norm}_S(s_1) = (1,2)$, and $\text{norm}_S(s_2) = (4)$.

**Definition 3.5** (Interacting cycle patterns). An interacting cycle pattern is a pair $\tau = (\sigma, Y)$, where $\sigma$ is a permutation of length $n$ consisting of $k$ cycles $\sigma = \sigma_0 \sigma_1 \ldots \sigma_{k-1}$, where each cycle $\sigma_i$ has $p_i$ elements, and $Y \subseteq \{0,1, \ldots, n-1\}$. Each cycle $\sigma_i$ is either written in parentheses or brackets.

An occurrence of the pattern $\tau$ consists of $k$ disjoint substrings $S = \{s_0, \ldots, s_{k-1}\}$ from $\pi$, where each substring $s_i$ has length $p_i$ and is a substring from a single cycle in $\pi$, and $\text{norm}_S(s_i) = \sigma_i$. Additionally when $\sigma_i$ is written in parentheses, rather than brackets, we require that $s_i$ is an entire cycle from $\pi$.

Also for each element $y \in Y$, there exists an element $a$ from the substring that is normalized to $y$ and an element $b$ that is normalized to $y + 1 \pmod{n}$. Then $a + 1 = b$ modulo the length of $\pi$.

If there is an occurrence of $\tau$ in $\pi$, the permutation is said to contain the pattern, if the permutation $\pi$ has no occurrence of $\tau$ it is said to avoid the pattern.

Let $\tau$ be the interacting cycle pattern $\tau_1 = ([02][13], \{\}\}$ and permutation $\pi = (04)(15)(2637)$. The permutation $\pi$ has 6 occurrences of $\tau_1$, these are
1. $\{0,4\}, \{1,5\}$
2. $\{0,4\}, \{2,6\}$
3. $\{0,4\}, \{3,7\}$
4. $\{1,5\}, \{2,6\}$
5. $\{1,5\}, \{3,7\}$
6. $\{2,6\}, \{3,7\}$

If we look at $\tau_2 = ([02][13], \{0,2\})$, we have three occurrences in $\pi$. These will be a subset of the occurrences of $\tau_1$, the first, fourth and sixth item above. Here we additionally required the first element of the first substring to one less than the first element of the second substring, and the second element of the first substring to be one less than the second element of the second substring.

Finally when we have $\tau_3 = ([02][13], \{\}\}$, the second substring must match a whole cycle, so we only have the $\{0,4\}, \{1,5\}$ occurrence in $\pi$.

Since the figures we are working with are not directed unlike permutations and have circular symmetries we will expand an interacting cycle pattern to more interacting cycle patterns. All directions of an interacting cycle pattern or reflections are all combinations of either reversing a cycle or not from the pattern. Circular symmetries or rotations by $k$ of an interacting cycle pattern $\tau = (\sigma, Y)$, where the length of $\sigma$ as $n$ is found by replacing each element of $\sigma$ and $Y$ by adding $k$ to the element modulo $n$. 


For example the pattern seen in Equation (3.7) rotated by 1 is Equation (3.9). The only reflection of Equation (3.7) is Equation (3.8) that when rotated by 1 we get Equation (3.10).

**Corollary 3.6.** Permutations containing some interacting cycle pattern from Equation (3.7) to Equation (3.10) are such that their figures have two adjacent vertices that are neighbours. So their figures are never stars by Proposition 3.2.

\[
\begin{align*}
(3.7) & \quad ([01], \{0\}) \\
(3.8) & \quad ([10], \{0\}) \\
(3.9) & \quad ([10], \{1\}) \\
(3.10) & \quad ([01], \{1\})
\end{align*}
\]

**Proof.** Take the figure pattern Figure 3.2 we find an interacting cycle pattern \( \tau = (\sigma,Y) \) that corresponds to it. Fix vertex 0 on the left and vertex 1 on the right. Since these vertices are neighbours we fix that the pattern maps \( 0 \to 1 \) or \( \sigma = [01] \). Also these vertices are adjacent so the corresponding elements of the permutation should be adjacent in value modulo length of the permutation, \( Y = \{0\} \). With this we get Equation (3.7), from all possible rotations and reflections we get the remaining three equations. \( \square \)

In general when we have a figure pattern we can fix the vertices in some order and the direction of the edges to find one interacting cycle pattern for the figure pattern. Then we expand that interacting cycle pattern to account for all reflections and rotations.

Let’s take an example, in Figure 3.4b we see a figure pattern that will be discussed later in this section. To find an interacting cycle pattern \( \tau = (\sigma,Y) \) for this figure pattern, we label the vertices counter-clockwise in increasing order starting by labelling some vertex as 0, then we choose any direction for consecutive edges, see Figure 3.3. From this figure we can see that \( \sigma = [405][213] \) and \( Y = \{0\} \) since vertex 0 is adjacent to vertex 1. All other rotations can be found by starting from a different vertex and reflections from all possible ways of directing the edges.

**Proposition 3.11.** If the inner edges of two adjacent vertices do not intersect strictly within the boundary polygon of their figure or their inner edges meet in the same vertex, their figure is not a star. See patterns in Figure 3.4.

**Proof.** First we look at Figure 3.4a the inner edges of the adjacent vertices end in the same vertex such that a triangle, \( T \), is formed by the inner edges and side of the boundary polygon. This triangle splits the boundary polygon into two areas that contain additional vertices. For this to be a figure, each vertex from either area needs two incident edges, apart from the vertices at the outer edges which each just need one more.
Thus there must be an edge crossing through $T$, connecting a vertex from each area to form a figure. However, this figure will not be a star since the edge that crosses through $T$ keeps it from being a boundary triangle. Therefore the figure will be missing at least one of the necessary boundary triangles for it to be a star.

If the inner edges do not intersect within their boundary polygon we have the case shown in Figure 3.4b. Here the area formed by the inner edges is a polygon that has at least 4 corners. The only way for the remaining edges of the figure to reduce this polygon to a boundary triangle is by intersecting with the adjacent vertices, but those vertices are already connected to exactly two other distinct vertices. Hence the figure will not be a star.

An example of a figure where Proposition 3.11 holds can be seen in Figure 3.5.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure3.5.png}
\caption{A figure that is not a star, since the inner edges of 6 and 7 do not intersect within the boundary polygon.}
\end{figure}

\textbf{Corollary 3.12.} Permutations containing any pattern from Equation (3.13) to Equation (3.18) or rotations of them are such that their figures have two adjacent vertices where their inner edges do not intersect. So their figures are never stars by Proposition 3.11.

(3.13) $([40312], \{0\})$
(3.14) $([21304], \{0\})$
(3.15) $([405][213], \{0\})$
(3.16) $([504][213], \{0\})$
(3.17) $([405][312], \{0\})$
(3.18) $([504][312], \{0\})$

\textit{All rotations can be seen in Appendix 7.1.}

\textit{Proof.} Omitted, similar to proof of Corollary 3.6 above are interacting cycle patterns found from Figure 3.4 with fixed points and reflections. \hfill \Box
Definition 3.19 (Semistar). A semistar is a permutation with neither 1-cycles nor 2-cycles that avoids all patterns in Corollary 3.6 and Corollary 3.12.

Theorem 3.20. If a permutation is not a semistar then the permutation drawing in any boundary polygon is not a star.

Proof. A non-semistar is a permutation that has 1-cycles or 2-cycles, or contains some pattern in Corollary 3.6, or contains some pattern in Corollary 3.12. By Proposition 3.1, Proposition 3.2 or Proposition 3.11 the drawing formed by a non-semistar permutation is never a star figure. □

4. Novas

We have shown that the figures of non-semistar permutations will never be stars. In this section we explore a subset of semistar permutations that, when drawn in any boundary polygon where they form a figure, are always stars.

Definition 4.1 (Available boundary polygons). Given a permutation $\pi$, its set of available boundary polygons $P$ are all boundary polygons such that the permutation drawing of $(\pi, P)$ is a figure.

Definition 4.2 (Nova permutation). A permutation is a nova if it forms a star when paired with every available boundary polygon.

Definition 4.3 (Inner point). Given a figure, the intersection point of the inner edges of adjacent vertices are called inner points. The inner point of adjacent vertices $O_p$ and $O_{p+1}$ is labelled $I_p$, an example of this can be seen in Figure 4.1.

![Figure 4.1. Adjacent vertices $O_p$ and $O_{p+1}$ and their inner edges, where $I_p$ is their inner point](image)

If we consider what causes figures not to be stars. Recall that when we have figure of a semistar permutation, that the inner edges of all adjacent vertices will intersect strictly within the boundary polygon thereby forming an inner point. If we take any adjacent vertices $O_p$ and $O_{p+1}$, their inner edges will intersect in the inner point $I_p$. We want the triangle $\triangle O_p I_p O_{p+1}$ to be one of the boundary triangles of a figure. If no other edge of the figure intersects this triangle, then we know that this triangle will be a boundary triangle. Not all edges can intersect the triangle, those that can are dependent on the inner edges of the vertices regardless of their boundary polygon.

If we examine Figure 4.1 we have the adjacent vertices $O_p$ and $O_{p+1}$, and their inner edges intersecting in inner point $I_p$.

For a line segment to intersect the triangle $\triangle O_p I_p O_{p+1}$, an edge must go through two sides of the triangle. No edge can intersect the side $(O_p, O_{p+1})$, since that is also a side of the boundary polygon, and if an edge intersects with it, the drawing is not a figure. This leaves us with two sides, $(O_p, I_p)$ and $(I_p, O_{p+1})$. The only way for an edge to intersect with the triangle is if it goes through both of these sides. So if we have a semistar permutation where no edges can pass through the triangle, it must be a star for any available boundary polygon.
Definition 4.4 (Impostor edges). Let $O_p$ and $O_{p+1}$ be adjacent vertices of a figure that are not neighbours, where the neighbours on their inner edges are $O_q$ and $O_r$ respectively, see Figure 4.1. An edge between vertices $O_a$ and $O_b$, where $p + 1 < p_a < p_q$ and $r < p_b \leq p - 1$, are impostor edges to the adjacent vertices $O_p$ and $O_{p+1}$.

All possible impostor edges for adjacent vertices that are not neighbours and their incident edges intersect can be seen in Figure 4.2. Simply found by listing all ways for two adjacent vertices incident edges to intersect without being neighbours. The figures seen in Figure 4.3 for semistar permutation $(02461375)$ has one impostor edge $(6, 1)$ to the adjacent vertices 7 and 0. In this example we see an occurrence of Figure 4.2h.
Proposition 4.5. The impostor edges of adjacent vertices are the only edges that can intersect the triangle formed by the inner edges of adjacent vertices.

Proof. In Figure 4.1 the vertices \( O_p \) and \( O_{p+1} \) are adjacent and their inner edges connect them to their neighbours \( O_q \) and \( O_r \). The triangle \( \triangle O_p I_p O_{p+1} \) indicates the boundary triangle between vertices \( O_p \) and \( O_{p+1} \).

We split the remaining vertices into three sets, \( A = \{ O_k | p + 1 < p < q \} \), \( B = \{ O_k | q < k < r \} \) and \( C = \{ O_k | r < k \leq p - 1 \} \). Then we examine where the remaining edges of the figure will intersect the two edges \((O_p, O_q)\) and \((O_{p+1}, O_r)\).

Any edge between two points in the same set will intersect neither edge. Any edge going between \( O_a \in A \) and \( O_b \in B \) must only intersect the line segment \((I_p, O_q)\), similarly edges between \( O_b \in B \) and \( O_c \in C \) must only intersect \((I_p, O_r)\).

Edges between \( O_a \in A \) and \( O_c \in C \) is an impostor edge, and must intersect both the line segments \((O_p, O_q)\) and \((O_{p+1}, O_r)\).

Any other intersection is not possible when we have a figure, since such an edge would lie outside of the boundary polygon. Thus impostor edges are the only edges that can intersect a boundary triangle. \( \square \)

Definition 4.6 (Convex polygon). A convex polygon is a non self-intersecting polygon, where any line segment drawn between two points in the polygon lies inside of that polygon.

Equivalently, all interior angles between connected sides of a convex polygon are less than or equal to \( \pi \). Figure 4.4a shows an example of a convex polygon.

Definition 4.7 (Strictly convex polygon). A strictly convex polygon is a convex polygon, where every line segment between two points within the polygon is strictly within the polygon, or between two points on the boundary but not on the same side.

Equivalently all interior angles between connected sides on a strictly convex polygon are less than \( \pi \). Figure 4.4b shows an example of a strictly convex polygon.

Remark. All strictly convex polygons are available to all permutations with no 1-cycles or 2-cycles. This follows directly through from the definition of strictly convex polygons. Any line segment drawn between corners on the boundary, not on the same side, will be strictly within the polygon except at the end points. Thus not intersecting any other corner nor going outside of the polygon.

Theorem 4.8. A permutation is a nova if and only if the permutation is a semistar and its figure has no impostor edge in any available boundary polygon.

Proof. Let \( \pi \) be a semistar permutation, and its figure has no impostor edge in any available boundary polygon. Since \( \pi \) is a semistar, when drawn it will have as many potential boundary triangles as the length of \( \pi \). Proposition 4.5 tells us that impostor edges are the only edges of a figure that can intersect potential boundary
triangles. Since $\pi$ has no impostor edges, $\pi$ is a star for any of its available boundary polygons.

If $\pi$ is not a semistar, then it is not a nova by Theorem 3.20. Now let $\pi$ be a permutation that has an impostor edge when drawn in some boundary polygon. We find an available boundary polygon to $\pi$, such that $(\pi, P)$ is not a star figure. Let's say that $(O_a, O_b)$ is an impostor edge to the adjacent vertices $O_p$ and $O_{p+1}$, where $O_q$ and $O_r$ are their neighbours on their respective inner edges.

Recall that all strictly convex polygons are available boundary polygons to any permutation with no 1-cycle or 2-cycle, see Proposition 4. So we form a boundary polygon by inscribing a polygon inside a circle, thus forming a convex polygon. The basis for the corners can be seen in Figure 4.5. The remaining corners will be set in order on the arcs as noted by the dotted arcs. Clearly the figure $F = (\pi, P)$ will not be a star since the edge $(O_a, O_b)$ intersects the triangle $\triangle O_p I_p O_{p+1}$. Now since $F$ is not a star we know that $\pi$ is not a nova. □

**Corollary 4.9.** Semistar permutations which avoid impostor edge interacting cycle patterns below along with all possible rotations and reflections are novas seen in Appendix 7.2.

<table>
<thead>
<tr>
<th>(4.10)</th>
<th>((0314)[25], {0})</th>
</tr>
</thead>
<tbody>
<tr>
<td>(4.11)</td>
<td>([250314], {0})</td>
</tr>
<tr>
<td>(4.12)</td>
<td>([305214], {0})</td>
</tr>
<tr>
<td>(4.13)</td>
<td>([304125], {0})</td>
</tr>
<tr>
<td>(4.14)</td>
<td>([50314][26], {0})</td>
</tr>
<tr>
<td>(4.15)</td>
<td>([40315][26], {0})</td>
</tr>
<tr>
<td>(4.16)</td>
<td>([30415][26], {0})</td>
</tr>
<tr>
<td>(4.17)</td>
<td>([30514][26], {0})</td>
</tr>
<tr>
<td>(4.18)</td>
<td>([40513][26], {0})</td>
</tr>
<tr>
<td>(4.19)</td>
<td>([406][315][27], {0})</td>
</tr>
<tr>
<td>(4.20)</td>
<td>([306][415][27], {0})</td>
</tr>
<tr>
<td>(4.21)</td>
<td>([405][316][27], {0})</td>
</tr>
<tr>
<td>(4.22)</td>
<td>([305][416][27], {0})</td>
</tr>
<tr>
<td>(4.23)</td>
<td>([304][516][27], {0})</td>
</tr>
</tbody>
</table>
Proof. Omitted, similar to proof of Corollary 3.6, we find one interacting cycle pattern for each of figure in Figure 4.2 and expand. □

5. Existence of an Embedding

The figure of a permutation might only be a star in certain boundary polygons, as we saw in Figure 2.5. Here we show that all semistar permutations with one additional property will always have some boundary polygon such that the figure is a star. By doing so we have found a lower bound on the number of permutations that may form a star. We explore a different approach of finding stars, by finding the boundary polygon from the inner points instead of starting with a boundary polygon, similar to Bradwardine’s and Kepler’s definition of stars.

Let’s immediately take an example. With methods described in this section we will take the line segments between inner points of some figure, see Figure 5.1a for the interior of ((024153), \{6\}). Then we move the inner points such that they form a convex polygon, not needed in this example since they already do. Finally extending the line segments until they meet non-adjacent line segments in a vertex, see Figure 5.1b. What we end up with is a figure for the permutation, that is a star.

**Proposition 5.1.** If a figure of size \(n\) has \(n\) inner points \(\{I_0, I_1, \ldots, I_{n-1}\}\) where \(\{I_0, I_1, \ldots, I_{n-1}\}\) is a convex polygon and each vertex \(O_p\) is such that on its incident edges its inner points \(I_{p-1}\) and \(I_p\) are closer to \(O_p\) rather than the opposite inner points \(I_s\) and \(I_r\) (see Figure 5.2), then it is a star.

![Diagram](image)

*Figure 5.2. A vertex and the inner points on its incident edges where its inner points are closer*

Proof. Suppose that the inner points form a convex polygon and the figure is not a star. When a figure is not a star then there exists an edge that intersects a boundary triangle. The only edges that can intersect a boundary triangle are impostor edges,
by Proposition 4.5. We take any adjacent vertices $O_p$ and $O_{p+1}$ of a figure that has an impostor edge $(O_a, O_b)$, so $p + 1 < p a < q$ and $r < p b < p - 1$. This edge intersects the boundary triangle $\triangle O_p I_p O_{p+1}$, see Figure 5.3. The bad edge has two inner points, the line segment between them lies outside of the inner point polygon $\{I_0, I_1, \ldots, I_{n-1}\}$ and therefore the inner point polygon is not convex $\square$

We will use this proposition to find boundary polygons for a permutation such that the figure will become a star. However, it is not always the case we can find a convex polygon for the inner points. Lets take the permutation $(0251364)$ for example, here drawn inside of a regular polygon in Figure 5.4. Here we see that the inner edges of adjacent vertices 0 and 6 are the same as the inner edges of adjacent vertices 3 and 4. So these adjacent vertices share an inner point, that will be labelled $I_3 = I_6$. Thus we can not create a convex polygon for these inner points.

The basis for patterns where there are shared inner points, is whenever there are adjacent vertices where their intersecting inner edges end in adjacent vertices and for those adjacent vertices these edges are their inner edges also. Now to get the complete patterns we need to rearrange the vertices outer edges in all possible ways and we get the set of patterns seen in Figure 5.5.

The interacting cycle patterns for shared inner point represented by Figure 5.5 are seen here below and expanded with rotations and reflections in Appendix 7.3.

\begin{align*}
(5.2) & \quad ((032145), \{0, 3\}) \\
(5.3) & \quad ([5032146], \{0, 3\}) \\
(5.4) & \quad ([6032154], \{0, 3\}) \\
(5.5) & \quad ([3156042], \{0, 4\}) \\
(5.6) & \quad ([2156043], \{0, 4\}) \\
(5.7) & \quad ([7043\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\![2156], \{0, 4\})
\end{align*}

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure5.3}
\caption{Figure 5.3}
\end{figure}

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure5.4}
\caption{The figure $F = ((0251364), \{7\})$}
\end{figure}
Now let's take an example where we use Proposition 5.1. Take the semistar permutation $(02573614)$, that additionally avoids the shared inner point patterns. We will first find the pairs of inner points that will share an edge. The labelling of inner points of a figure can be seen from its permutation. Let the permutation map $q \rightarrow r$. We want to see what inner points will lie on the $\{O_q, O_r\}$ edge, without knowing what value maps to $q$ and what value $r$ maps to the inner points, it can be any combination of choosing one element from $\{I_q - 1, I_q\}$ and one from $\{I_r - 1, I_r\}$. So suppose a semistar permutation maps $p \rightarrow q, q \rightarrow r$ and $r \rightarrow s$ If $q < p < q$ the first inner point on the $\{O_q, O_r\}$ will be $I_q - 1$, or otherwise $I_q$. When $q < q < s < q$ the other inner point on the $\{O_q, O_r\}$ edge will be $I_r$, or otherwise $I_r - 1$. So in a figure of $(02573614)$ the inner points on the edge $\{O_0, O_2\}$ will be $I_0$ since it is not the case that $0 < 0 < 0 < 2$ and $I_1$ since it is not the case that $0 < 0 < 0 < 2$. When we complete this example we find the pairs $L = \{(I_0, I_1), (I_2, I_4), (I_5, I_6), (I_7, I_2), (I_5, I_3), (I_6, I_0), (I_1, I_5), (I_4, I_7)\}$.

These pairs correspond to parts of each edge of a figure for that permutation. Next we use Proposition 5.1 and spread inner points $\{I_0, I_1, \ldots, I_7\}$ evenly and in order on a circle, and draw line segments between inner points as indicated by $L$. As seen by the filled lines in Figure 5.6. Finally we extend pairs of line segments that are parts of incident edges of a vertex, until they meet to form said vertex. The extension of the line segments can be seen by the dotted lines in Figure 5.6. Let's take another example of a permutation that avoids the shared inner point patterns $(02573614)$. Here we find the inner point pairs $\{(I_0, I_1), (I_2, I_4), (I_5, I_6), (I_0, I_6), (I_5, I_3), (I_2, I_7), (I_6, I_0), (I_1, I_5), (I_4, I_7)\}$. In the former example we spread the inner points equally on a circle, but that will not work in this case.

Here the inner point line segments $(I_5, I_1)$ and $(I_0, I_6)$ should extend until they meet forming vertex $O_1$, along with $(I_6, I_4)$ and $(I_3, I_7)$ to $O_4$, and finally $(I_2, I_4)$ and $(I_5, I_1)$ to $O_5$. These pairs of line segments will be parallel when the inner points are spread equally on a circle, so they will never intersect. Let's examine in more detail the first pair of parallel line segments $(I_5, I_1)$ and $(I_0, I_6)$, that are on

![Figure 5.5. Shared inner point patterns](image-url)
Figure 5.6. A star formed for semistar permutation (02573614)

Figure 5.7. A star formed for semistar permutation (02516374)

the incident edges of vertex $O_1$. The vertex $O_1$ has the inner point $I_0$ and $I_1$ and the opposite inner points $I_5$ and $I_6$ that are adjacent, which is the reason for the pair of line segments being parallel when the inner points are evenly spread on a circle. If the arc length between $I_5$ and $I_6$, denoted $\overline{I_5I_6}$, is larger than $\overline{I_0I_1}$, then we can extend the line segments to form $O_1$. Similarly if $\overline{I_3I_4} < \overline{I_0I_7}$ and $\overline{I_4I_5} < \overline{I_1I_2}$ holds, then vertices $O_4$ and $O_5$ form respectively. In Figure 5.7 we have spread the inner points in order on a circle, where $\overline{I_5I_6}$, $\overline{I_6I_7}$ and $\overline{I_1I_2}$ are slightly larger than all other arc lengths between adjacent inner points. We have then extended the line segments to form a star.
So for each vertex $O_p$ with opposite inner points on its incident edges $I_r$ and $I_s$, as seen in Figure 5.8, we need to satisfy the inequality $\overline{I_{p-1}I_p} < \overline{I_rI_s}$ when spreading the inner points on a circle.

It is not clear that these inequalities can be satisfied in general. Before the culprits were when opposite inner points of a vertex are adjacent $I_q$ and $I_{q+1}$. What do we then know about the opposite inner points of a vertex $O_{q+1}$, that has the adjacent inner points $I_q$ and $I_{q+1}$. If these inner points are not adjacent it is easy to satisfy the inequalities like we did in the last example.

**Lemma 5.11.** Suppose we have a figure formed from a semistar permutation, that avoids shared inner point patterns and has a convex boundary polygon. When the opposite inner points on the incident edges of a vertex $O_p$ are adjacent, $I_q$ and $I_{q+1}$, then the opposite inner points $I_r$ and $I_s$ on the incident edges of vertex $O_{q+1}$ are not adjacent. See Figure 5.9.

**Proof.** Suppose that the opposite inner points on the incident edges of $O_p$ are adjacent $I_q$ and $I_{q+1}$, see Figure 5.9a. Now what is the distance between the opposite inner points of $O_{q+1}$ see Figure 5.9b.

![Figure 5.9](image)

There are two cases to examine either $O_{q+1}$ and $O_p$ are neighbours or not. First lets take the case when $O_{q+1}$ and $O_p$ are neighbours.

Without a loss of generality, we let $O_{q+1}$ be the end vertex on the edge that has inner points $I_p$ and $I_q$. Then the other incident edge of $O_{q+1}$ must go through $I_{q+1}$ the other inner point of vertex $O_{q+1}$. Let $O_x$ be the neighbour of $O_{q+1}$ on the other end of this edge. Now the other neighbour of $O_p$ must be $O_{q+2}$ since a inner point on that edge is labelled $I_{q+1}$, furthermore this must be the inner edge of $O_{q+1}$ and $O_{q+2}$. See Figure 5.10.

Now we find the opposite inner points on the incident edges of $O_{q+1}$, first we see that on the edge $(O_{q+1}, O_p)$ we have the opposite inner point $I_p$. Note that $O_x \neq O_{p-1}$ since otherwise adjacent vertices $O_{q+1}$ and $O_{q+2}$ would share an inner point with $O_x$ and $O_p$. So we know that $O_x \in \{O_{q+3}, O_{q+4}, \ldots, O_{p-3}, O_{p-2}\}$. Thus
the other inner point on the edge \((O_{q+1}, O_x)\) is \(I_r \in \{I_{q+2}, I_{q+3}, \ldots, I_{p-3}, I_{p-2}\}\),
the index of inner points are either one less or equal to their vertex index. Then it
is clear that \(I_p\) cannot be adjacent to any \(I_r\).

Secondly when \(O_{q+1}\) is not the neighbour of \(O_p\) then the incident edges of
\(O_{q+1}\) must intersect \(I_q\) and \(I_{q+1}\). The neighbours of \(O_p\) must now be labelled \(O_q\) and
\(O_{q+2}\) since their edges form the inner points \(I_q\) and \(I_{q+1}\). Let the neighbours of
\(O_{q+1}\) be \(O_x\) and \(O_y\). Now we have Figure 5.11

Since we do not have shared inner points we know that \(O_x \neq O_{p-1}\) and \(O_y \neq
O_{p+1}\). Then \(O_y \in \{O_{q+2}, \ldots, O_{q-1}\}\) and \(O_x \in \{O_{q+3}, \ldots, O_{p-2}\}\). So the inner
points \(I_s\) on the \((O_{q+1}, O_y)\) edge is \(I_s \in \{I_{p+1}, \ldots, I_{q-1}\}\) and \(I_r\) on the \((O_{q+1}, O_x)\)
edge is \(I_r \in \{I_{q+2}, \ldots, I_{p-2}\}\). From that we can see that the distance between \(I_s\)
and \(I_r\) is at least three, meaning that they are not adjacent.  □

Now that we know a bit more about the form of the inequalities, we find that
they can always be satisfied in a relatively simple way.

**Lemma 5.12.** The inequalities for each vertex can always be satisfied.

**Proof.** Let \(B\) be the set of adjacency arcs \(I_qI_{q+1}\) which appear in inequalities of
the form \(I_pI_{p+1} < I_qI_{q+1}\). And \(A\) be the set of adjacency arcs \(I_pI_{p+1}\) which only
appear in inequalities of the form \(I_pI_{p+1} < I_qI_s\) where \(I_q\) and \(I_r\) are not adjacent.

When \(B\) is empty then we set each arc length to be equal or \(a := \frac{2\pi}{n}\) radians.
All inequalities are satisfied since they are now all of the form \(a < k\phi\) where \(k \in \mathbb{N}\)
and \(k \geq 2\). The example of this case can be seen in Figure 5.16

Otherwise we can set the arc lengths of all arcs in \(A\) to be \(a := \frac{2\pi}{n+\ell}\) and all arcs
in \(B\) to be \(b := \frac{2\pi}{n+\ell}(1 + \frac{\ell}{k})\) where \(1 \leq \ell < k\). We have ensured that \(a < b < 2a\).

Clearly inequalities \(O_pO_{p+1} < O_qO_{q+1}\) are satisfied since they are on the form
\(a < b\). Other inequalities \(O_pO_{p+1} < O_sO_s\) where \(|s - r| > 1\) are on the form
$a < ia + jb$ or $b < ia + jb$ where $i, j \in \mathbb{N}_0$ and $i + j \geq 2$, are satisfied since $a < b < 2a$.

**Theorem 5.13.** All semistar permutations that avoid the shared inner point patterns Appendix 7.3 have an embedding such that the resulting figure is a star.

**Proof.** We spread the inner points on a circle as described by Lemma 5.12 which is possible by Lemma 5.11. Then draw line segments between inner points that should lie on the same edge, there by forming the interior of a figure. Extend each inner point line segment that is a part of the incident edges of a vertex until they meet forming that vertex. The resulting figure is then a star by Proposition 5.1. □

**Remark.** There are figures with shared inner points that are stars see Figure 5.12.

There are also semistar permutations that do not avoid shared inner point patterns, where their figures are never stars see Figure 5.4. Since either at least one of the edges of vertex 5 must intersect either the boundary triangle between 6 and 0 or 3 and 4, or the neighbours of vertex 5 are in the same place thus there is not a boundary triangle between its neighbours.

6. **Future work**

In Figure 6.1 we can see how we have classified permutations throughout this paper, the only thing left to explore are the semistar permutations that do not avoid the shared inner point patterns. As we saw in end of Section 5 some of these permutations will form stars while other will not.

In Section 5 we only explored one way of finding a boundary polygon for a permutation such that it forms a star. There are probably many ways to find the boundary polygons thereby forming different shapes of stars for the same permutation.

Finally finding permutations that form star figures from a fixed polygon, instead of the other way around as we did. Which is definitely the way onward to tile with these figures.

**References**


7. **Appendix**

Here we list all interacting cycle patterns for Figure 3.4, Figure 4.2 and Figure 5.5. Each line of the table is a rotation and within each line we have reflections.
7.1. Interacting cycle patterns for Figure 3.4

7.1.1. Figure 3.4a expanded.

(40312, {0}) (21304, {0})
(01423, {1}) (32410, {1})
(12034, {2}) (43021, {2})
(23140, {3}) (04132, {3})
(34201, {4}) (10243, {4})

7.1.2. Figure 3.4b expanded.

(405, 213, {0}) (504, 213, {0}) (405, 312, {0}) (504, 312, {0})
(510, 324, {1}) (015, 324, {1}) (510, 423, {1}) (015, 423, {1})
(021, 435, {2}) (120, 435, {2}) (021, 534, {2}) (120, 534, {2})
(132, 540, {3}) (231, 540, {3}) (132, 045, {3}) (231, 045, {3})
(243, 051, {4}) (342, 051, {4}) (243, 150, {4}) (342, 150, {4})
(354, 102, {5}) (453, 102, {5}) (354, 201, {5}) (453, 201, {5})

7.2. Interacting cycle patterns for Figure 4.2

7.2.1. Figure 4.2a expanded.

(0314, 25, {0}) (0314, 52, {0}) (0413, 25, {0}) (0413, 52, {0})
(1425, 30, {1}) (1425, 03, {1}) (1524, 30, {1}) (1524, 03, {1})
(2530, 41, {2}) (2530, 14, {2}) (2035, 41, {2}) (2035, 14, {2})
(3041, 52, {3}) (3041, 25, {3}) (3140, 52, {3}) (3140, 25, {3})
(4152, 03, {4}) (4152, 30, {4}) (4251, 03, {4}) (4251, 30, {4})
(5203, 14, {5}) (5203, 41, {5}) (5302, 14, {5}) (5302, 41, {5})

7.2.2. Figure 4.2b expanded.
7.2.5. Figure 4.2e expanded.

\[
\begin{align*}
\{250314, \{0\}\} & \{413052, \{0\}\} \\
\{301425, \{1\}\} & \{524103, \{1\}\} \\
\{412530, \{2\}\} & \{035214, \{2\}\} \\
\{523041, \{3\}\} & \{140325, \{3\}\} \\
\{034152, \{4\}\} & \{251430, \{4\}\} \\
\{145203, \{5\}\} & \{302541, \{5\}\}
\end{align*}
\]

7.2.3. Figure 4.2d expanded.

\[
\begin{align*}
\{305214, \{0\}\} & \{412503, \{0\}\} \\
\{410325, \{1\}\} & \{523014, \{1\}\} \\
\{521430, \{2\}\} & \{034125, \{2\}\} \\
\{032541, \{3\}\} & \{145230, \{3\}\} \\
\{143052, \{4\}\} & \{250341, \{4\}\} \\
\{254103, \{5\}\} & \{301452, \{5\}\}
\end{align*}
\]

7.2.4. Figure 4.2c expanded.

\[
\begin{align*}
\{304125, \{0\}\} & \{521403, \{0\}\} \\
\{415230, \{1\}\} & \{032514, \{1\}\} \\
\{520341, \{2\}\} & \{143025, \{2\}\} \\
\{031452, \{3\}\} & \{254130, \{3\}\} \\
\{142503, \{4\}\} & \{305241, \{4\}\} \\
\{253014, \{5\}\} & \{410352, \{5\}\}
\end{align*}
\]

7.2.5. Figure 4.2b expanded.

\[
\begin{align*}
\{50314, \{26\}, \{0\}\} & \{50314, \{62\}, \{0\}\} \\
\{61425, \{30\}, \{1\}\} & \{61425, \{03\}, \{1\}\} \\
\{02536, \{41\}, \{2\}\} & \{02536, \{14\}, \{2\}\} \\
\{13640, \{52\}, \{3\}\} & \{13640, \{25\}, \{3\}\} \\
\{24051, \{63\}, \{4\}\} & \{24051, \{36\}, \{4\}\} \\
\{35162, \{04\}, \{5\}\} & \{35162, \{40\}, \{5\}\} \\
\{46203, \{15\}, \{6\}\} & \{46203, \{51\}, \{6\}\}
\end{align*}
\]

7.2.6. Figure 4.2a expanded.

\[
\begin{align*}
\{40315, \{26\}, \{0\}\} & \{40315, \{62\}, \{0\}\} \\
\{51426, \{30\}, \{1\}\} & \{51426, \{03\}, \{1\}\} \\
\{62530, \{41\}, \{2\}\} & \{62530, \{14\}, \{2\}\} \\
\{03641, \{52\}, \{3\}\} & \{03641, \{25\}, \{3\}\} \\
\{14052, \{63\}, \{4\}\} & \{14052, \{36\}, \{4\}\} \\
\{25163, \{04\}, \{5\}\} & \{25163, \{40\}, \{5\}\} \\
\{36204, \{15\}, \{6\}\} & \{36204, \{51\}, \{6\}\}
\end{align*}
\]

7.2.7. Figure 4.2g expanded.

\[
\begin{align*}
\{30415, \{26\}, \{0\}\} & \{30415, \{62\}, \{0\}\} \\
\{41526, \{30\}, \{1\}\} & \{41526, \{03\}, \{1\}\} \\
\{52630, \{41\}, \{2\}\} & \{52630, \{14\}, \{2\}\} \\
\{63041, \{52\}, \{3\}\} & \{63041, \{25\}, \{3\}\} \\
\{04152, \{63\}, \{4\}\} & \{04152, \{36\}, \{4\}\} \\
\{15263, \{04\}, \{5\}\} & \{15263, \{40\}, \{5\}\} \\
\{26304, \{15\}, \{6\}\} & \{26304, \{51\}, \{6\}\}
\end{align*}
\]
7.2.8. Figure 7.2.10 expanded.

<table>
<thead>
<tr>
<th>118</th>
<th>95</th>
</tr>
</thead>
</table>
| 7.2.9. Figure 7.2.11 expanded.

<table>
<thead>
<tr>
<th>118</th>
<th>95</th>
</tr>
</thead>
</table>
| 7.2.10. Figure 7.2.12 expanded.

<table>
<thead>
<tr>
<th>118</th>
<th>95</th>
</tr>
</thead>
</table>
7.2.12. Figure 4.2m expanded.

7.2.13. Figure 4.2m expanded.
7.3.1. Figure 7.2.14 expanded.

7.3. Interacting cycle patterns for Figure 5.5

7.3.1. Figure 5.5a expanded.

7.3.2. Figure 5.5b expanded.
7.3.3. Figure 5.5c expanded.

\[
\begin{array}{c}
(5032146, \{0, 3\})
(6143250, \{1, 4\})
(0254361, \{2, 5\})
(1365402, \{3, 6\})
(2406513, \{4, 0\})
(3510624, \{5, 1\})
(4621035, \{6, 2\})
\end{array}
\]

7.3.4. Figure 5.5d expanded.

\[
\begin{array}{c}
(6032154, \{0\})
(0143265, \{1\})
(1254306, \{2\})
(2365410, \{3\})
(3406521, \{4\})
(4510632, \{5\})
(5621043, \{6\})
\end{array}
\]

7.3.5. Figure 5.5e expanded.

\[
\begin{array}{c}
(3156042, \{0\})
(4260153, \{1\})
(5301264, \{2\})
(6412305, \{3\})
(0523416, \{4\})
(1634520, \{5\})
(2045631, \{6\})
\end{array}
\]

7.3.6. Figure 5.5f expanded.

\[
\begin{array}{c}
(2156043, \{0\})
(3260154, \{1\})
(4301265, \{2\})
(5412306, \{3\})
(6523410, \{4\})
(0634521, \{5\})
(1045632, \{6\})
\end{array}
\]

7.3.7. Figure 5.5g expanded.

\[
\begin{array}{c}
(7043, \{0, 4\})
(0154, \{1, 5\})
(1265, \{2, 6\})
(2376, \{3, 7\})
(3407, \{4, 0\})
(4510, \{5, 1\})
(5621, \{6, 2\})
(6732, \{7, 3\})
\end{array}
\]

\[
\begin{array}{c}
(7043, \{0, 4\})
(0154, \{1, 5\})
(1265, \{2, 6\})
(2376, \{3, 7\})
(3407, \{4, 0\})
(4510, \{5, 1\})
(5621, \{6, 2\})
(6732, \{7, 3\})
\end{array}
\]
7.3.8. **Figure 5.5i expanded.**

\[
\begin{array}{c}
(0153, 4267, \{1, 1.5\}) (0153, 7624, \{1, 1.5\}) (3510, 4267, \{1.5\}) (3510, 7624, \{1.5\}) \\
(1264, 5370, \{2, 2.6\}) (1264, 0735, \{2, 2.6\}) (4621, 5370, \{2, 2.6\}) (4621, 0735, \{2, 2.6\}) \\
(2375, 6401, \{3, 3.7\}) (2375, 1046, \{3.7\}) (5732, 6401, \{3.7\}) (5732, 1046, \{3.7\}) \\
(3406, 7512, \{4, 4.0\}) (3406, 2157, \{4, 0\}) (6043, 7512, \{4, 0\}) (6043, 2157, \{4, 0\}) \\
(4517, 0623, \{5, 5.1\}) (4517, 3260, \{5, 1\}) (7154, 0623, \{5, 1\}) (7154, 3260, \{5, 1\}) \\
(5620, 1734, \{6, 6.2\}) (5620, 4371, \{6, 2\}) (0265, 1734, \{6, 2\}) (0265, 4371, \{6, 2\}) \\
(6731, 2045, \{7, 3\}) (6731, 5402, \{7, 3\}) (1376, 2045, \{7, 3\}) (1376, 5402, \{7, 3\}) \\
\end{array}
\]

7.3.9. **Figure 5.5j expanded.**

\[
\begin{array}{c}
(0604, 2157, \{0\}) (0604, 7512, \{0\}) (3406, 2157, \{0\}) (3406, 7512, \{0\}) \\
(7154, 0623, \{1\}) (7154, 0623, \{1\}) (4517, 3260, \{1\}) (4517, 0623, \{1\}) \\
(0265, 4371, \{2\}) (0265, 1734, \{2\}) (5620, 4371, \{2\}) (5620, 1734, \{2\}) \\
(3510, 7624, \{5\}) (3510, 7624, \{5\}) (0153, 7624, \{5\}) (0153, 4267, \{5\}) \\
(4621, 0735, \{6\}) (4621, 5370, \{6\}) (1264, 0735, \{6\}) (1264, 5370, \{6\}) \\
(5732, 1046, \{7\}) (5732, 6401, \{7\}) (1375, 2046, \{7, 3\}) (1375, 6401, \{7\}) \\
\end{array}
\]