The teaching of intuitive geometry in early 1900s Italian Middle School: Programs, mathematicians’ views and praxis

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Abstract

In 1881, intuitive geometry came to life to be taught in Italy in the first three years of the Gymnasium (corresponding to the present middle school). It was explicitly considered as an introductory subject to let students better understand the rational geometry of the Lycée. The lack of a formal definition and of a detailed tasks’ description of intuitive geometry caused continuous role changes in the Italian school programs. We discuss and analyze the reasons which led to different choices in the programs and in the books of intuitive geometry in the period between the nineteenth and the first half of the twentieth century, which are also connected with the views of important mathematicians of the time.

Introduction

By a law of 1859, two years before the Italian Union, secondary education in Italy was divided into a first and a second level. To cover classical secondary education, the law had introduced the Gymnasium and the Lycée. The Scuola Tecnica (Technical School) and the Istituto Tecnico (Technical Institute) were set up for technical secondary education, see (Menghini, 2006).

The Gymnasium and the Technical School were preceded by four years of primary school. The Technical School thus covered the same age range as the present-day middle school (11–14), while the Gymnasium lasted for five years and hence included the first two years of high school followed by three years of Lycée.

In 1896 a Complementary Course (parallel to lower Gymnasium and Technical School) was introduced – followed by the Scuola Normale - specifically for the Istruzione Magistrale, a school aimed at educating primary school teachers.

As to geometry teaching, primary school dealt with shapes, names and rules for measure of the simplest geometrical figures.

School programmes

In 1881, intuitive geometry came to life to be taught in the first three years of the Gymnasium (the “lower Gymnasium” corresponding to the present middle school for pupils in the age 11-14). Previously, geometry was not part of the school programmes for students in this age (while in 1967 a reform had brought in Euclid’s Elements as the geometry textbook aimed to teach the subject in the Gymnasium-Lycée, starting from the fifth year of the Gymnasium). An earlier intuitive experimental approach was considered a good help for students to overcome the difficulties caused by rational geometry and by the logical deduction of Euclid’s textbook. Geometrical drawing should also contribute to overcome
these difficulties. As we read in a Royal decree of 1881 (R.d.¹, 16-6-1881, n. 323) intuitive geometry had to
give to youngsters, with easy methods and, as far as possible, with practical proofs, the first and most important notions of geometry, … useful not only to access geometry, but also to let the students desire to learn, in a rational way, the subject throughout the Lycée (translated by the author).

Three years later a Royal decree (R. d., 23-10-1884, n. 2737) of the new minister Coppino abolished the study of intuitive geometry from the lower Gymnasium and moved down rational geometry to the 4th year of the Gymnasium.

This decision was due to Eugenio Beltrami (1836-1900), a mathematician known for his modelling of non-Euclidean geometry and for his theories of surfaces of constant curvature. Beltrami was a member of the scientific Accademia dei Lincei, serving as president in 1898; and was elected to the Italian Senate a year before his death.

Beltrami’s decision (Beltrami, 1885, pp. 16-17) was a consequence of a lack of clear definitions and of the fear that teachers could not emphasize in the right way the experimental-intuitive nature of geometry being tied to the traditional logical-deductive aspect of rational geometry (Vita, 1986, p. 15).

All this did not touch the Scuola Tecnica, where from 1867 a graphic-intuitive method was suggested even to produce simple deductions.

In the following years, only a few changes were introduced concerning the beginning of the study of rational geometry – which could be moved down to the third year of the Gymnasium – and the learning approach to Euclid’s books. According to Vita (1986, p. 16), “the oscillation reflects a clear didactic anxiety and the desire of finding the most psychologically adequate time to teach The Elements by Euclid, with all its logical-deductive layout, to the 13-15 year old pupils”.

In the 1900s a new program was introduced (Royal decree of 24-10-1900, n. 361) by minister Gallo: intuitive geometry was restored in lower Gymnasium, but, to prevent past problems, the programme included only elementary notions such as the names of the easiest geometrical shapes, the rules to calculate lengths, areas and volumes and also basic geometrical drawing. Some instructions specify that the new studies “were an introduction to rational geometry”. But it was not explained in which way this teaching had to be presented in order to be effectively an introduction to the study of rational geometry. Moreover, the instructions underline that these new studies were “a review and an expansion of the notions acquired by the students at the elementary school”, and required a practical approach, amplified by the teaching of geometrical drawing. Anyway, the rules that explain why the various geometrical construction work had not to be stated. With regard to rational geometry, the new programmes gave more freedom in the choice of the textbook, as long as it followed the “Euclidean method” (Maraschini & Menghini, 1992).

¹ R.d. means Regio decreto (Royal decree).
Intuitive geometry textbooks in early 1900s: Veronese and Frattini

Since the programme dated 1881 was effective for a very short period, we cannot find textbooks of intuitive geometry in those years. Instead, textbooks appeared right after 1900.

In 1901 we find a book by Giuseppe Veronese (Nozioni elementari di geometria intuitiva) and a book by Giovanni Frattini (Geometria intuitiva). Even if both authors followed the programmes, we can observe some differences in the conception of intuitive geometry. Veronese introduces the axioms as anticipation to what pupils will see later on, while Frattini presents some proofs with practical methods. He tries to involve the student “imaging” concrete materials. Let us confront in detail their work.

The authors

Giuseppe Veronese (1854-1917) was known at an international level for having developed n-dimensional projective geometry. He gave an important contribution to the study of the logical foundations of geometry with his book Fondamenti della Geometria (Veronese, 1891). In 1897 he wrote a textbook for the Lycée, which reflects his views on foundations perfectly. Veronese’s only “primitive concept” is the point, he defines a line as a linear system of points with two orders, and defines a segment as a part of a linear system of points. He belongs to that group of authors, who (like Hilbert) prefer to avoid the use of rigid movements to introduce equality and assume equality between certain objects (generally segments and angles), using the concept of one-to-one correspondence.

Giovanni Frattini (1852-1925) studied Mathematics in Rome, among his teachers there were Battaglini, Beltrami, and Cremona. He taught mathematics and descriptive geometry in an Istituto Tecnico. He was also a researcher, particularly known are the Frattini sub-groups. His idea, as a teacher, was that mathematics could be learnt only “doing it”.

Conceptions of intuitive geometry

Veronese writes in the preface of his book: “Following the ministerial recommendations, I will use mainly the pictures of the figures to name them and to mention the most obvious properties [...]” (Veronese, 1901, VII). And in the preface to the second edition “These notions should be such that any individual with a sane mind would consider them true through a mere observation and without any mathematical consideration” (preface to the second edition in 1902). Veronese wants to deal only with “those shapes that have an effective representation in the limited field of observation”. Initially, not even the straight line, the plane and unlimited space are the subject of his treatise, given that they need an abstraction process.

In the introduction to his book, Frattini underlines that a “geometrical truth” exists, and it comes from “an immediate observation of the things, which is the
essence of the intuitive method”. Intuition favours the first investigations. In Frattini’s book, lines and planes are unlimited from the beginning and the characterization of parallel lines changes to the one that everyone knows (parallel lines never meet).

**Preliminary notions**

In the Preliminary Notions, Veronese gives examples of objects (table, house..) and of their properties (colour, weight, etc.). Material points (grains of sand) lead to the abstract concept of point, and material lines (a cotton thread) lead to the abstract concept of a line, which is defined, both with practical examples (a pencil line) and as a linear set of points (as defined in his textbook for the Lycée). Like all the authors of intuitive geometry books of this period, he introduced the straight line using the idea of a stretched string, and explains later on how it can be drawn using a ruler.

In an analogous way a page of a book leads to the idea of surface. He states that there exist distinct points, that there are open and closed lines, that a straight line has two different orders, that a point X divides a line into two parts, that there are interior and exterior points of a line segment, etc. Points, lines, surfaces, solids, and every group of points are called geometric figures, and geometry is defined as the science of figures.

Frattini’s preliminary notions are quite the same, but the language is much simpler and more concise. For example, he explains that the order of the line is determined by the different positions, A, B, C, of a point moving along a line. The idea of an infinite line is given by using an elastic thread. Frattini includes in the preliminary notions also some propositions (in fact incidence axioms) as “Two points determine a straight line” and “if two straight lines have two points in common they are equal”, explaining this by means of a thread. Analogous properties are stated for straight lines and planes in the space, thinking of a piece of paper.

Frattini also introduces the idea of distance as the measure of a segment, and says that the straight line segment is the shortest distance between two points. This contested metric definition is not given by Veronese. Veronese believed it dangerous to introduce concepts that would need to be amended at some stage in higher studies.

**Intuitive evidence of axioms**

Of course none of the authors mentions the word axiom, but both introduce them by means of practical examples. As Frattini in the preliminaries, Veronese presents the incidence axioms for the plane and for the space, and adds also the congruence axioms and the parallel axiom.

Any segment of a line is not congruent to a part of itself, for example the segment AB in the picture is not congruent to CD. This can be verified visually or with a paper strip or compass (translated by the author).
Veronese ‘surrendered’ to the temptation of stating the reflexive, symmetric and transitive properties of the equality relation for the segments in a more abstract way. Afterwards, he explained that the congruence of the segments could be verified using a ruler or a compass.

Here is an example on how the classical distance axiom was interpreted from a practical point of view:

Assuming that the extension of the field of observation is appropriate, it is possible to verify that: On a straight line \( r \), given a point \( A \) and a segment \( XY \), two segments exist \( CA \) and \( AB \) having the same direction and length of \( XY \). The axiom can be proved using a piece of paper marked with a segment of the same length of \( XY \), and sliding it along the line \( r \) in the direction showed by the arrow (Veronese, 1901, p. 9) (translated by the author).

\[ \overrightarrow{A} \overrightarrow{B} \overrightarrow{X} \overrightarrow{Y} \]

With the exception of the few incidence axioms (for the plane and for the space) shown in the preliminaries, Frattini doesn’t state other axioms.

**Proofs**

Veronese’s textbook included only one simple proof. After the definition of symmetric points about a given point \( O \) (central symmetry), Veronese stated the following:

The shape symmetric to a line about a given point is another line.

Let \( ABC \) be a line and \( A'B'C' \) the shape opposite to \( ABC \) about a point \( O \). Using a compass, or copying the shape \( AOB \) on a piece of drawing paper and turning the paper up side down so that \( OA \) corresponds to \( OA' \) and \( OB \) to \( OB' \), we can verify that the point \( C' \) is on the line identified by \( B' \) and \( A' \) and so on (p. 13, translated by the author).

\[ \text{Diagram showing the symmetric points about a given point} \]

To avoid infinity, Veronese stated that two lines are parallel when they are symmetric about a point, and explained how to verify manually that two lines are parallel (p. 14). This is why he presented the previous proof. But may be that this demonstration is still considered by Veronese as a practical introduction to an “axiom”, as he states in the preface that evident properties can be initially treated as axioms, even if there exist a rigorous proof.
Frattini adds in his text some more practical proofs, giving also more weight to the properties of polygons.

*There is exactly one perpendicular line through a given point to a line on a plane.* Let us bend a plane, imagine an immense piece of paper, and shape right angles so that one folding follows the line we want to draw the perpendicular to, and the other folding must include the point where the perpendicular passes through. Let us reopen the paper, it will be possible to see the trace of the perpendicular through the point and the line (Frattini 1901, p. 21, translated by the author).

In their hands, perpendicular lines are defined basing on what can be seen in a folded paper, with a “correct” *informal* definition.

To state that “the sum of the three angles of any triangle is equal to two right angles (p. 29)”, Frattini uses the classic proof, which is based on the congruence of alternate angles. This congruence, anyway, is introduced without a proof (“the student can find a reason”). Veronese does not write about this property, not even about its consequences.

*The diagonals of a parallelogram bisect each other.* Suppose we cut out the parallelogram from a piece of paper, we would have, then, an empty space which could be filled either placing the parallelogram back in the same position or placing the angle A, marked with an arc, on top of the equivalent angle C, the side AD on the equivalent side CB and the side AB on CD. In this way the diagonals of the shape, though upside down, would be in the previous position, the same for their crossing point. The two segments OC and OA would switch their positions: this means they are the same length (Frattini, 1901, p. 33, translated by the author).

Frattini proves in the same way many properties of triangles and quadrilaterals. Veronese only shows the following figure (p. 29) and then lists elementary definitions for triangles, quadrilaterals, other polygons and for the circle without stating any property of these shapes.
About movements

The use of geometric transformations (isometries) is very effective. They were considered suitable for an intuitive introduction to geometry: as a tool. Motions can in fact be carried out experimentally. We will find this use of geometrical transformations also in other books. Both authors use them to transport a segment, or to move a figure; only Veronese explicitly defines the symmetry about a point.

The position of Veronese with respect to geometric transformations can be clarified by the appendix to his book for upper secondary school, where he finally talks about movement:

We have already said that for a practical construction of equal figures, certain instruments are used, amongst which the simplest are the ruler and compasses, and we have already seen that these are useful in solving practical problems of geometry (solved in theory with lines and circles) when making drawings. Geometrical postulates suffice for theoretical development of geometry, but are not enough for applications of geometry in the real world.

This practical means is provided by the movement of bodies. Although physical phenomena in the environment have effects on the geometrical form of bodies, by observing a body in two different positions we infer in general that the body in the second position occupies a space which is, to a good approximation, equal to that in the first position (Veronese 1897, p. 197, translated by the author).

Geometric constructions

Throughout his book, Veronese included simple drawing exercises, meant to be done by hand (to draw a dotted line, to duplicate a segment marking some corresponding points, to draw symmetric shapes using a specific point as centre of symmetry). Only at the end of the book did he introduce some geometrical constructions, “aiming to improve, with practice, the intuitive perception of geometrical shapes, whose structure will be later analyzed using logical proofs”.

The chapter, describing geometrical constructions (of a triangle given three sides, of the bisector of an angle and other more complex constructions) which are not linked to the previous chapters, tacitly used theorems never illustrated earlier in the book (especially those concerning the congruence of triangles). Some instructions precede this chapter, explaining how to execute a clear drawing and how to test the quality of rulers, squares, rubbers and pencils.

Frattini too placed geometric constructions at the end of the book. However, Frattini tries to explain them using the properties of polygons.

As an example, both authors show the construction of the perpendicular to a straight line passing through a point outside of it. Veronese doesn’t explain that construction, while Frattini explains it referring to the diagonals of a rhombus.
Further developments

Some years later a book appears by Costanzo e Negro (1905). There aren’t experimental argumentations nor proofs, but we often find the sentence “the experience teaches and elementary geometry proofs [...]” or “with the usual experimental proof...”.

In a new edition for the Scuola Complementare, Veronese (1908) adds some practical operation (to construct an equilateral triangle or a square with a sheet of paper), but presents also proper proofs (f. i. on the equality of the alternate angles).

In 1907, a book by Pisati was published. In the preface the author slightly dissented from the structure of the programmes as follows:

it seems proved that, in lower middle school, it would be a big mistake to leave the formal aspect of the subject completely apart. Pupils’ intellect, in the early years of their life, has a formal nature […]. Certainly, intuitive teaching of geometry is not easier than formal teaching (Pisati, 1907, translated by the author).

The book presents theorems and classical proofs.

We can observe that the schoolbooks are slowly shifting towards the rigour of rational geometry.

Already in 1905, the Minister Bianchi had felt the need to remind the teachers in an official note (Boll. Uff. d. P.I., 1.06.1905) to “avoid abstract statements and proofs” adding, on the other hand, to use “simple inductive reasoning” to teach the “truths required by the school programmes”. He also assigned to a commission the task of reforming secondary instruction. In the period 1905/1909 the Commissione Reale (an important member of it was Giovanni Vailati) proposed to base geometry teaching on graphic exercises and on construction of figures. But this never became part of the official programmes. In a letter to Vailati, Veronese still supports his idea that in the middle school, teachers must not speak of infinite lines to their pupils.

In 1923, the reform made by Minister Giovanni Gentile turned the clock back. In the first three years of the Gymnasium, geometry studies “must only aim to keep alive all geometrical notions that the pupils have learnt at the primary school and to fix the terminology properly in their memory”. Therefore, there are fewer requirements than in the provisions dated 1900. Amongst the books published right after the reform of Gentile, we have to mention Severi’s textbook (1928).

Francesco Severi (1879-1961) is famous for his contributions to algebraic geometry. He became the effective leader of the Italian school of algebraic geometry after Enriques and Castelnuovo. He was Rettore of the University of Rome, was involved in politics and adhered to the fascist party.

Severi’s textbook includes a preface by the Minister of Public Education. In spite of the good comments given in the preface, it is difficult to say that the book follows the school programme guidelines. Over the years, middle school geometry had lost its experimental-intuitive nature, and even its terminological function, becoming more and more rational. Textbooks were almost independent from the
school programmes – which were in fact very brief and without any particular didactic connotation. The book by Severi is surely not an exception, although his book for the high school has always been appreciated for the experimental approach (but in fact it included only an experimental approach to axioms, in line with Veronese’s book for the lower Gymnasium). Severi’s text for the middle school includes many theorems (also those regarding the angles at the centre and the angles at the circumference of a circle), with the most traditional proofs, except for using transformations (rotation and symmetry) as a support to the proofs and for avoiding the word “theorem”.

It is interesting to note what Severi had written in 1919:

In elementary and middle school the teaching of mathematics must be only intuitive. Through paper cutting, models, and other devices that can be found, for example, in the English textbooks, we can stimulate the curiosity of the pupils. Particularly geometry has to be considered, at this level, as a physical science.

No matter what will remain in the mind of the pupils; but the majority of them will not be rejected by enormous difficulties just from the beginning and will learn what is possible (Severi, 1919, translated by the author).

This change in Severi’s attitude reflects the difficulty to treat geometry in a real intuitive way.

In 1936 and 1937, a couple of reforms introduced only minor variations, which allowed some simple deductive analysis in the lower Gymnasium.

In 1940, with the reform of minister Bottai (legge 1-7-1940, n. 899), the first three-years of the Gymnasium, of the Technical school ad of the complementary school were unified to form the middle school. With reference to geometry, although its intuitive nature was confirmed, it was suggested to emphasize the evident properties “by means of several suitable examples and exercises, which, sometime, can also assume a demonstrative connotation…” So, we can find a bigger change compared to the small ones introduced in 1936: the purpose is to start from an intuitive way of thinking to go towards a more abstract logical nature.

**Ugo Amaldi and Emma Castelnuovo**

An interesting book by Ugo Amaldi (1941) followed the reform of 1940. Amaldi (1875-1957) was had been professor of algebra and geometry in different universities. His main research concerned the groups of continuous transformations. He had written many geometry textbooks, mainly for the upper secondary school, together with Federigo Enriques.

Amaldi completely stopped the process of “rationalization” of geometry. His textbook is similar to Frattini’s book, but it contains some new important changes: measurements and geometrical constructions are not illustrated in separate chapters but they are integrated with the other parts of the book, providing a useful didactic tool. We find many figures and references to real life (i.e. an opening door gives the idea of infinite planes all passing through the same straight line, paper bands illustrate congruent segments …), which had completely
disappeared in the meantime. So, given the instructions to draw the axis of symmetry of a segment using a ruler and a compass, Amaldi suggests to check the construction by folding the paper and verifying that the circumferences, used for the construction, overlap. To know the sum of the angles of a triangle, he suggests cutting the corners of a triangle drawn on paper, to place them next to each other and to check that they form a straight angle. Similarly, he suggests cutting and folding techniques to verify the properties of quadrilaterals.

At the end of the World War in 1945, a Committee, appointed by the Allied Countries, proposed new programmes which were later adopted by the Italian Minister of Education. The middle school programme reverted to practical and experimental methods, but the methodological guidelines for the higher Gymnasium are particularly interesting: it is suggested to leave more space to intuitive skills, to common sense, to the psychological and historical origin of theories, to physical reality, etc., to use spontaneous dynamic definitions which fit the intuitive method better.

Vita observes that “unfortunately these suggestions appear to be disjointed from the school programmes that do not show any peculiar innovation” (Vita, 1986). An innovation is, indeed, represented in the book on intuitive geometry by Emma Castelnuovo (1948). In her book, Castelnuovo follows in Amaldi’s footsteps, using drawings, pictures, cross-references to reality and integration of constructions and measurements. In addition to this, her book, for the very first time, interacts with the student, not only to let him follow a logical deduction or a proof but also to raise questions in his mind.

What is the meaning – you would question – of the statement that there is only one line passing through two distinct points A, B? How can the contrary be possible? It is true: it is not possible to imagine two or more distinct lines passing through A and B. It is possible, however, to draw with a compass several circles passing through two points [...] (Castelnuovo, 1948, translated by the author).

The book starts with paper folding, and goes on with ruler and square constructions. As Amaldi does, she re-uses the idea of the stretched string to introduce the properties of segments and straight lines; a method already used by Clairaut, who was Castelnuovo’s inspirator. Simple tools are made up, such as a folding meter to show how to transform a quadrilateral into a different one, and to analyze the limit situations.

With reference to her first book, Emma Castelnuovo writes (Castelnuovo, 2008, p. 37): The study of the areas, motivated by concrete examples, was introduced, as in Clairaut’s work, by drawing a variety of polygons using rulers and a compass. I believed, in this way, to start with a concrete approach. Later, I realised that the construction of a figure with a ruler and a compass limits the thought’s freedom, because you can consider only a finite number of cases: drawing is static and does not stimulate the observation nor lead to new discoveries. I understood that it is better to construct geometric figures
with concrete material that can be handled and you can do and undo it (translated by the author).

**Conclusions**

Our analysis clearly shows the difficulty of finding an equilibrium between the notions that a pupil is supposed to learn, and the notions which he can accept by means of a non-rigorous argumentation.

Methodology is linked, somehow, to the historical period: to the concept of school, to the attention to the student, etc, and to the research in mathematics education.

Mathematicians influenced the programmes, and the majority of them were in favor of rigor; but also among teachers it seems that the majority was in favor of rigor.

The very aspect that seems to be relevant for approaching geometry in an intuitive way is the active learning role of the student. Programmes tried, several times, to deny this role, and it was interpreted in different ways by authors. Emma Castelnuovo foresaw and opened the door to the use of concrete materials.

**References**


