Teaching new geometrical methods with an ancient figure in the nineteenth and twentieth centuries: the new triangle geometry in textbooks in Europe and USA (1888-1952)

Pauline Romera-Lebret
Centre François Viète, Université de Nantes. France

Abstract

Since the 1870s, new remarkable objects of the triangle were studied, first in France, then in Belgium and in Germany and finally all over Europe. This new body of knowledge would then become a new chapter of elementary geometry, mixing new objects and new methods. We suggest understanding the process of the transition in education, through the addition of a chapter dedicated to this new triangle geometry, in various French, Irish and American textbooks. Who were the authors? For what public did they write? How was this knowledge exposed? What were those new geometrical methods? Thanks to them, what do we learn about teaching geometry in these countries? We shall examine in particular the link between the presentation of the new triangle geometry and the sequel to the Elements of Euclid.

Introduction

We propose to see how an ancient figure like the triangle has been used for the teaching of new geometrical methods in Europe and in the United States between 1888 and 1952. The different uses made from this same body of knowledge will be examined as well.

My title poses a question: what theoretical qualities are required by a body of knowledge to justify its place in a textbook? The following are some reasons specific to the new triangle geometry: a coherence between the results, an increasing complexity and different levels of reading. Another question is: what knowledge is chosen to be in textbooks? In the majority of cases, the textbooks consist of knowledge required by school curricula. But textbooks often incorporate additive knowledge, which happens to be the case for the new triangle geometry. We shall see their utility.

Investigative tools for the history of education for the end of the nineteenth and the beginning of the twentieth centuries lie on the study of curriculum and textbooks (see Belhoste, Gispert, & Hulin, 1996, p. 101). This is why, to understand the transition of the new triangle geometry in education, we will study two groups of textbooks. In the first group we find three European textbooks, published between 1888 and 1896, whose authors were contemporaries of the researchers of the new triangle geometry. It will allow us to see how authors, having been witness to the research then and of the interrelating of these results, used this knowledge in textbooks. The second group include three later American textbooks written between 1916 and 1952 by a new generation of authors. These authors were born when the research about the new triangle geometry began.

Bjarnadóttir, K., Furinghetti, F., & Schubring, G. (Eds.) (2009). "*Dig where you stand*". *Proceedings of the conference "On-going research in the History of Mathematics Education"*.

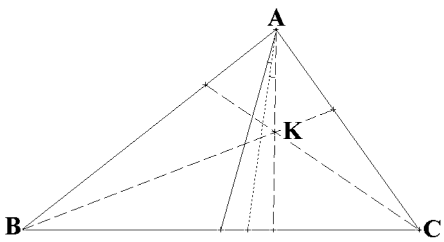
Studying these textbooks is interesting because authors only knew this body of knowledge through its completed, coherent version.

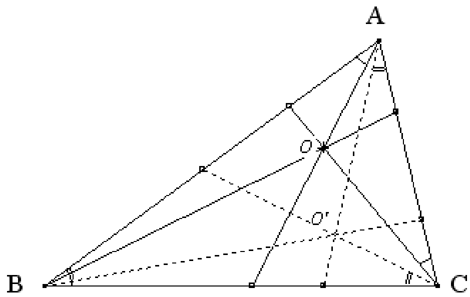
First, we are going to present an history of research about the new triangle geometry, so we can explain what the new triangle geometry is about. In the second part, we will present the French education at the end of the nineteenth century. We will specifically look at the elementary mathematics and special mathematics classes. The presentation of the two groups of textbooks is divided into four parts. We will begin the study of the three European textbooks with that of the Irishman, Casey, written in the continuing style of Euclid. Then we will study two French textbooks, one written by Rouché and Comberousse and the other by Frère Gabriel-Marie. These authors put the diverse geometric methods at the center of their textbooks. A comparison of the European textbooks will allow us to understand how the reference to Euclid or the absence of reference had an influence on the presentation of the new triangle geometry. Finally the study of the three American textbooks will show us that in the United States the new triangle geometry took part in a bigger chapter of geometry, the modern geometry.

History of research about the new triangle geometry

First, we are going to explain what the new triangle geometry is and how its definition has evolved. The evolution of this definition reflects not only the evolution of the contents but also the evolution of the perception which the researchers of this period have of it. From 1873, Frenchmen Emile Lemoine and Henri Brocard studied new properties of the triangle. They updated each from their parts, new points and new remarkable lines. Lemoine and Brocard were followed by other numerous authors whose combined discoveries took the name of new triangle geometry.

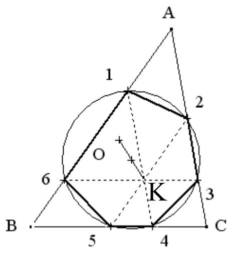
At first, the expression, 'new triangle geometry' indicated only the set of the new remarkable objects of the triangle like the Lemoine point (K, also called symmedian point) which is the point of concurrence of the three symmedians of a triangle. The symmedians of a triangle are the medians of its antiparallels. There are also the Brocard points (O and O'), which are in a triangle ABC the points for which the angles OAB, OBC, OCA in one side and the angles O'AC, O'CB, O'BA in another side are equal. This angle is named the Brocard angle.

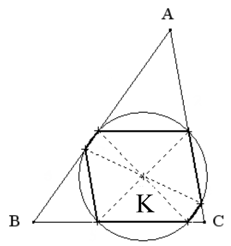


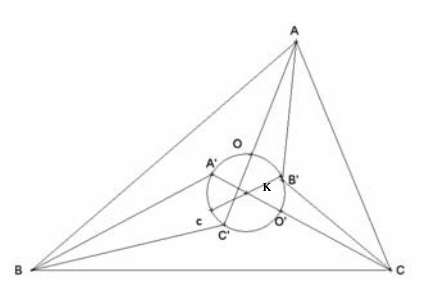


Lemoine and Brocard points appeared before 1873 in some German and French works but they were not profoundly studied.

Some remarkable circles also belong to the new triangle geometry. We can quote the first (on the left) and the second (in the middle) Lemoine circle[1] and the Brocard circle[2] (on the right). Furthermore, some remarkable lines belong to this collection of new objects of the triangle, for example, the Lemoine line[3].







From the end of the 1880's, researchers used the expression 'new triangle geometry'. This period corresponded to the moment when the researchers made the results coherent, thanks to the development of geometry of correspondences that connects the various remarkable points and lines. First they used pre-existing correspondences like the theory of isogonal conjugates[4]. The symmedian and the median passing through the same vertex are isogonal conjugates with respect to the angle. Moreover, the centroid and the Lemoine point are isogonal conjugates with respect to the triangle, as are the two Brocard points. But the authors developed new correspondences from particular cases. Given the fact that the Brocard points can be deduced from the Lemoine point by a geometrical or an analytic way, they created the brocardian points. The Brocard points are the brocardian points of the Lemoine points but it is only an example of application of this correspondence.

The geometry of correspondences comes from the ideas developed with the geometry of transformations. Thus, according to the authors, the expression 'new triangle geometry' indicated not only the new remarkable objects of the triangle but also the methods developed for the study and the classification of these objects. In the textbooks we find either the objects alone or objects and methods.

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1 The first Lemoine circle goes by the intersections of the sides of the triangle with the parallel lines to the sides going through the Lemoine point. The second Lemoine circle goes by the intersections of the sides of the triangle with the antiparallels going through the Lemoine point.

2 The Brocard circle goes by 7 remarkable points: the Brocard points, the Lemoine point, the circumcenter and the three vertices of the isosceles triangles, built on the side of the original triangle, whose sides go by the Brocard points.

3 It is the polar, by the circumcircle, of the Lemoine point.

4 Theory developed in 1865 by J.J.-A. Mathieu in his article named "Compared Geometry". Two lines passing through the vertex of an angle are isogonal conjugates, with respect to this angle, if they make equal angles with its bisector.

Emile Vigarié who was considered as the historiographer of the new triangle geometry proposed a "bibliographical and terminological study" from 1887. The introduction of this study contains his definition of the new triangle geometry:

> Being given a point M in the plan of the triangle, we can always find, and in an infinity of manners, a second point M', that corresponds to the first one according to an imagined geometrical law; these two points M, M' have between them geometrical relations whose simplicity depends on the more or less lucky choice of the law which unites them and each geometrical law gives place to a method of transformation or to a mode of conjugation which it remains then to study (Vigarié, 1887, p. 39, my transl., P. R.-L.).

It was the definition of the body of knowledge when it joined the textbooks.

The Frenchmen Emile Lemoine and Henri Brocard appear to be the initiators of a renewal of interest in the study of the properties of the triangle. Articles about the new triangle geometry mostly used recent geometrical methods. This was the case with Lemoine: if his works relied on the theory of antiparallels which dates from the seventeenth century he also used modern geometrical methods from the nineteenth century like the geometry of transformations and the trilinear coordinates. In 1873, Lemoine gave a lecture to the AFAS conference. His lecture was entitled "On some properties of a remarkable point of a triangle (Lemoine, 1873)" and it was about the Lemoine point. This work marked the renewal of interest in France's research of the new triangle geometry.

In 1881, Brocard gave a lecture called "Study of a new circle of the plan of the triangle (Brocard, 1881)" which presents the two Brocard points. On the methodological plan, Brocard used a whole set of the geometrical methods but it was the geometry of transformations of the nineteenth century that was used most to develop an innovative work on a simple mathematical object: the triangle. This lecture was essential for its mathematical density and the concern of legitimization expressed by Brocard.

At the beginning of the twentieth century, interest for research about new triangle geometry declined. Davis proposed eight different reasons that "can be given for the short life of triangle geometry as a strongly and coherently delineated corpus of results, sanctioned by the mathematical establishment (Davis, 1995, p. 207)". To name a few: the lack of recognition, "the inner exhaustion of the interest", "the increasing visual complexity" of the results or the lack of surprise.

French education

Since 1888, there has been a multitude of textbooks intended for secondary education including the new triangle geometry, either as a separate entity or by joining isolated elements. In a more general way, it corresponds to a metamorphosis of the secondary education, which would end with the important reforms of 1905.

Some of the books we are going to study were intended for elementary mathematics classes and special mathematics classes. Elementary mathematics was the last class of the French secondary education, while special mathematics was a preparatory class for entrance to *grandes Écoles* like the *École Polytechnique*. The higher education system in France was (and still is) comprised of universities and other institutions called *grandes Écoles*. In France during the nineteenth century, the scientific formation was mainly offered in grandes Écoles and not in universities. The *École Polytechnique* was in the forefront among the *grandes Écoles*. As a consequence, the others *grandes Écoles* followed its admittance program. Also the special mathematics class curriculum was based on the entrance examination program of the *École Polytechnique*. Nevertheless, special mathematics classes were materially connected with secondary schools. Thus there was a strong link between the French secondary education and the French higher education. This link was made possible by the special mathematics classes. That is why certain textbooks that were intended for elementary mathematics classes (final years of secondary school) were also for special mathematics classes. That is the case of the textbooks that we are going to study.

In 1833, the first detailed curricula of mathematics appeared, drafted by the professors of Paris Academy. Despite the numerous changes concerning the place of scientific teaching, the contents and the methods of teaching evolved little (see Belhoste, 1995, p. 36).

As Barbin (1992, p. 135) said, every mathematical period had its *Elements* of geometry. For each period, *Elements* of Euclid was a referent book, but criticisms and alternatives were proposed by the different authors. In his *New Elements of geometry*, Arnauld (1667) cast doubt on the order of Euclid's *Elements* and he adopted a methodological order for the propositions, from the easier to the composed. Moreover, he reproached Euclidean proofs as being persuasive but not enlightening to the reader. In Clairaut's *Elements of geometry* (1753), knowledge was problem solving tools (see Barbin, 1992, p. 143) and followed an innovative order. At the beginning of the nineteenth century, the authors were less critical of Euclid. With Legendre (1794), for example, it was a return to a deductive approach. Until Lacroix (1803), the mode of presentation adopted by all was based on definitions and axioms. This is different for the textbook written by Rouché and Comberousse where we find: "an image, a geometrical definition or properties, a mode of generation illustrating this property by appealing to a real experience or thought (Aspra, Marnier, Martinez, 2007, p. 116, my transl., P. R.-L.)." We can also add that during the nineteenth century, the contents of geometry textbooks and the ongoing research in geometry were diverging.

The new triangle geometry in textbooks

It was in 1888 that a textbook incorporated the new triangle geometry as an independent chapter. Before 1888, some elements of the new triangle geometry appeared in few textbooks. Rather than make an exhaustive list, we prefer to specify that the objects of the new triangle geometry were used for the analytic geometry teaching. In the *Analytic geometry and superior geometry exercises*, the Frenchman Joseph Koehler (1886) presented the symmedian, the Lemoine point and the Brocard points in the chapter dedicated to trilinear coordinates. Similar to the *Treatise of analytical geometry of the point, line, circle and conic sections* written by John Casey (1885b). The Lemoine and Brocard points were presented in the trilinear coordinates section.

Casey's Euclidean approach

In 1888, a textbook written by the Irishman John Casey (1888) incorporated a section dedicated to the new triangle geometry in the fifth edition of his *Sequel to the first six books of the Elements of Euclid.* The French authors of the new triangle geometry were more sensitive, at first, to the transition of the new triangle geometry in the English teaching than in the French teaching. In this way, Lemoine (1885, p. 44, my transl., P. R.-L.) said that "M. Casey introduced these results into the classic teaching in England". John Casey (1820-1891) had been a mathematics and mathematical physics professor at the Catholic University of Ireland since 1873. He put a lot into the development of the Irish teaching like joining the National Board of Education. He was also a member of numerous prestigious scientific institutions.

Before his *Sequel to the first six books of the Elements of Euclid,* Casey (1885) published a modern interpretation of the first six books of the *Elements* of Euclid. If in France, the authors released themselves from Euclid's Elements, the English authors stayed in a strong Euclidean tradition.

The writing of this interpretation of the first six books of the *Elements* of Euclid, intended for secondary education, was steered by an educational need. Indeed, in the preface of the third edition, Casey indicated that the double ambition of this work, sought in part by the Irish professorial world, was to give a version of the *Elements,* which respected the original book while adding the modern evolutions of the geometry.

> This edition of the Elements of Euclid undertaken at the request of the principals of some of the leading Colleges and Schools of Ireland is intended to supply a want much felt by teachers at the present day-the production of a work which, while giving the unrivalled original in all its integrity would also contain the modern conceptions and developments of the portion of Geometry over which the Elements extend (Casey, 1885a, p. i).

The first difference from Euclid's *Elements* lies in the simplification of the proofs made by Casey who thus avoided repetition. Casey kept the Euclidian propositions but gave proof corresponding to the teaching of elementary geometry of the nineteenth century. Some of the simplifications of the proofs had been possible thanks to the algebra that Casey used (in particular in Book V). The use of symbols also gave a modern tone to the statements and to the proofs too.

This wish to exceed Euclid was even more evident with the *Sequel to the first six books of the Elements of Euclid.* Having published a modern interpretation of the first six books of Euclid's *Elements*, in 1881, Casey continued Euclid's overtaking by publishing a new textbook, which constitutes *A Sequel to the first six Books of the Elements of Euclid*. This work presented "additional propositions" to the Euclidian

Books I, II, III, IV and VI.

It was his personal experience that incited Casey to write this textbook intended for secondary education. As he explained in the preface, during his higher mathematics course at university, he had "been frequently obliged, when teaching the higher mathematics, to interrupt [his] demonstrations, in order to prove some elementary propositions on which they depended, but which were not given in any book to which [he] could refer (Casey, 1888, p. V)". Casey continued by saying "the object of the present little Treatise is to supply that want". Therefore, this textbook was intended for the pupils at the end of secondary education as much as for those at the beginning of higher education.

From the fourth edition (1886), Casey introduced a supplementary chapter outside Euclidian Books, which approached the new triangle geometry. But it was only in the fifth edition (1888) that this supplementary chapter was really complete with the addition of the last section on the "general Theory of associated figures".

Seven sections made this supplementary chapter a complete and coherent corpus on the new triangle geometry. The fifth edition definitively marked this change by the evolution of the textbook title. Until the fourth edition, the entire title was: *A sequel to the first six books of the Elements of Euclid.* On the front cover of the fifth edition, the mention "containing an easy introduction to modern geometry" was added to the previous title. Casey chose to put the "recent discoveries in Geometry" of this supplementary chapter under the term "modern geometry" and he integrated this terminology into the title of his work. He clearly promoted this modern triangle geometry. He used another subtitle that we find at the beginning of the supplementary chapter: the "recent elementary geometry".

Casey used the geometry of correspondences as a coherence tool. The objects of the new triangle geometry are presented as examples of new geometrical methods use. The Lemoine point was defined in the section about the isogonal inversion and the Brocard points were presented in the situation dealing with two similar figures. The aim was not to present the diverse properties of the new remarkable objects of the triangle but to show how these objects were integrated into a geometry of correspondences inherent to the triangle.

Considering the *Sequel* as a textbook, to transpose this reflection into education amounts to say that the new remarkable objects of the triangle were useful for the teaching of new methods like the geometry of correspondences, the associated similar figures or the trilinear coordinates.

Casey belonged to the European community of authors of the new triangle geometry and wrote a lot of research articles. In 1889, a French translation of the *Sequel* was published. The preface was then signed by Joseph Neuberg who explained that he decided to publish a translation of the *Sequel* because there was no French textbook integrating these new developments of elementary geometry.

The French textbooks

The next textbook to be discussed was written by Eugene Rouché and Charles

De Comberousse. They were both graduates of the *École Polytechnique*, teachers in French *grandes Écoles* and writers of numerous textbooks. Moreover, Rouché was elected to the Academy of Sciences of Paris in 1896 and as a mathematician his research covered all mathematical areas (algebra, analysis, mechanic, probability and, above all, geometry). Comberousse contributed to educational ministerial decisions while he was a member of the *Instruction Publique* superior council and a member of the technical teaching superior council permanent committee. Through his research, Rouché had a national mathematical function whereas Comberousse occupied a national administrative function. Thus, together they combined three stages of the future of mathematical knowledge: academic research, governmental decisions and teaching experience.

Rouché and Comberousse wrote a *Treatise of Geometry* in 1864, in compliance with the official curriculum of the elementary mathematics class. The various editions thus followed the evolution of the curriculum of this class. In the edition of 1891, the authors added a sixty-page note dedicated to the new triangle geometry.

The treatise was divided into four books. Book I is devoted to the straight line, book II to the circle, book III to similar figures and book IV to area. While taking again the Euclidean *Elements*, Rouché and Comberousse sought to exceed their limits by presenting the recent advancement of elementary geometry. However, they didn't refer explicitly to Euclid in the preface or in the book. Each book has an appendix, which is essential for Rouché and Comberousse. Indeed, the authors thought that "to apply a science, it is not enough to know some parts; it is necessary to be familiarized with all its methods and to seize the unit of it (Rouché & De Comberousse, 1891, p. XXXIII, my transl., P. R.-L.)". However, the appendices presented new methods that had not yet been explored in teaching.

The four books were followed by three notes on various subjects that supplemented the knowledge presented in the first part. The third and the last note, the longest one, was related to the recent triangle geometry. Contrary to the two other parts, Rouché and Comberousse were not the authors since it was the reproduction of Joseph Neuberg's work. Joseph Neuberg was a higher education mathematics teacher in Belgian. He was also co-creator of two mathematical periodicals: the *Nouvelle Correspondance Mathématique* (1874) and *Mathesis* (1881), both of which published numerous articles about new triangle geometry.

Rouché and Comberousse did not take part in research on the new triangle geometry. Considering that the period of maturation and consistency of the new triangle geometry corresponds to the years 1882 to 1887, the transition of this body of knowledge from research to a new chapter of an elementary geometry textbook was quick. In the preface, Rouché and Comberousse expressed that they considered the new triangle geometry to be a new chapter of geometry.

The study of Neuberg's note made it possible to understand what Rouché and Comberousse meant by the term "new chapter": new remarkable elements of the triangle but also, and especially, new methods. Proportionally, the major part of

Neuberg's note consisted of a presentation of new geometrical methods.

The textbook of Rouché and Comberousse lingers particularly on the geometrical methods. For instance, the Appendices prolonged the various Books by presenting new methods. For the authors, knowing these methods was necessary for good apprehension of elementary geometry. In the same manner, the geometrical methods of the new triangle geometry were proposed in Neuberg's Note. The new remarkable objects were applications of these methods. Since the *Treatise was* a textbook, the new remarkable objects were tools for teaching new geometrical methods based, in fact, on the geometry of correspondences.

The next book we are going to study was written in 1850 by the Lasallian Frère Gabriel-Marie (F. G.-M.) member of the French congregation *Institut des frères des écoles chrétiennes*. For thirty years he was a mathematical teacher and wrote numerous textbooks. In 1882, he was elected member of the supreme council and he became general Superior of the congregation in 1893. We are interested in his *Exercises of geometry including the talk of the geometrical methods and 2000 solved questions,* which appeared for the first time in 1875. The third edition (1896) has integrated "on 118 pages a series of very interesting elementary exercises on the triangle geometry (F. G.-M., 1920, p. x, my transl., P. R. L.)". The textbook was intended for pupils of elementary mathematics classes. In theory, he was addressing secondary education pupils, but as the author specifies, it was also intended "for those who cultivate with predilection the studies of elementary geometry (F. G.-M., 1920, p. i, my transl., P. R.-L.)".

His book was divided into four parts. At first F. G.-M. exposed general methods, it was the most developed section. In the second part he proposed exercises, then thirdly, numerical problems and finally the section about the new triangle geometry, which was added in the third edition. F. G.-M. considered the report of the geometrical methods as the greatest part of his work. By the term methods, F. G.-M. specified that he calls "general solutions and examples of discussion". The approach of the Frère Gabriel-Marie was educational since he hoped that the study of these methods would allow the reader to develop not only general ideas but also a spirit of synthesis, so "attaching thousands of exercises varied to some principal types (F. G.-M., 1920, p. iv, my transl., P. R.-L.)" easy to retain.

F. G.-M. was an outsider to the author community of the new triangle geometry, but historical notes prove that he followed the research articles and that the evolution of this research towards the geometry of correspondences coincided with the introduction, by F. G.-M., to an independent chapter on the new triangle geometry in its *Exercises of Geometry.*

F. G.-M. used the geometry of correspondences without forsaking the other geometrical methods (pure geometry or analytical geometry). He did not give a simple list of new elements of the triangle, but he presented new geometrical methods inherent to the triangle whose applications lead to the objects of the new triangle geometry. The exercises distributed in the eight books, and those joined together under the title numerical problems, used the elementary methods indicated by F. G.-M. in the preface. The chapter on the new triangle geometry proposes exercises that call upon more complex methods of the nineteenth century, such as barycentric coordinates and geometry of correspondences. In this supplementary chapter the idea of F. G.-M. was to make complex methods simpler.

Comparison of the nineteenth century European textbooks

There is a basic difference between the general plan of Casey's textbook and those, identical, of Rouché and Comberousse's textbook and the one of F. G.-M. That comes from the position asserted by Casey to make a sequel to Euclid since he based the plan of his book on one of the *Elements* of Euclid. Casey gave new proposals for elementary geometry whereas the objective of Rouché and Comberousse, like F. G.-M., was to present all elementary geometry methods.

Let us look at the sections devoted to the new triangle geometry. At first, there is a common point: all authors had considered the new triangle geometry as a new chapter in elementary geometry. On a methodological level, an essential difference separates the textbook written by Casey and that of the two Frenchmen: the absence of use of coordinates in Casey's *Sequel*. According to Casey, analytical geometry does not belong in elementary geometry. The geometry of correspondences, as he proposed, relies entirely on pure geometry and geometrical constructions. In the two French textbooks, if correspondences geometrical properties are the first expressed, they are immediately followed by their translations using the coordinates. There is also a difference concerning the target public. F. G.-M. thought of his *Exercises of geometry* as a working tool for the elementary mathematics class pupils, he did not indicate higher education pupils. On the contrary, the co-authors Rouché and Comberousse, like Casey, indicated which paragraphs were dedicated to special mathematics pupils and which sections were for both levels of mathematics. Therefore, the mathematical level of F. G.-M.'s textbook is more elementary and the geometry of the correspondences is more superficial than in the two other European textbooks. Finally, French textbooks were more recently re-edited, contrary to Casey's book. The last edition of F. G.-M. *Exercises of geometry* was re-published in 1931, whereas Rouché and Comberousse's book was re-published in 1957.

The place of the new triangle geometry in the American education: College geometry, Modern Geometry

Three American books, published respectively in 1916, 1929, 1952, included new triangle geometry. The first book, *A Treatise on the circle and the sphere* was written by Julian Lowell Coolidge (1916), Assistant Professor of Mathematics at the University of Harvard. The second, called *Advanced Euclidean Geometry* (1929), was written by Roger Arthur Johnson. He was a former pupil of Coolidge and he became then a mathematics teacher at the Brooklyn College. The third book is the second edition of the *College Geometry* written by Nathan Altshiller-Court (1952). The first edition was published in 1925 and the second in 1952.

The introduction of the objects of the new triangle geometry in American textbooks made two changes. The first being on a methodological order because developments due to the analytical geometry and in particular trilinear coordinates were not imported. All the harmonization made around the geometry of correspondences is absent from American textbooks. The second one is the place occupied by the new triangle geometry within a chapter of elementary geometry: the modern geometry. This American terminology refers to the elementary geometry of the triangle and the circle. A definition of modern geometry as proposed by J. I. Tracey (1930, p. 176) in his notice of Johnson's book is: "the geometry of the triangle and the circle developed by the elementary concepts of Euclidean geometry extended to include some geometry functions and circular inversion". The introduction of Johnson's book contains another definition:

> During the second half of the nineteenth century "Modern Geometry ", in the sense of the content of the present book, aroused much interest and was prosecuted vigorously by a considerable number both in England and on the continent of Europe. Many beautiful new theorems were proved, most of them by elementary methods. Toward the end of the century this interest waned somewhat (Johnson, 1929, p. iii).

The numerous theorems of the modern geometry were obtained by the application to simple objects of the plane of methods made simple. The geometry of the transformations was used in a geometrical version where the constructions were central and not in its analytical version. The American authors wanted to unite all the elementary developments of the modern period of geometry of the triangle and the circle to constitute a sequel to Euclid. We find this idea in the formal title of Johnson's work: *Advanced Euclidean Geometry*. The claimed initiative was identical to that developed by Casey in his *Sequel to Euclid*: propose new elementary propositions with the aid of sometimes new but always simple methods.

As Court said in 1924, the new triangle geometry was the last addition to modern geometry. Those authors used the expressions triangle geometry or modern geometry of the triangle to refer to what we have called new triangle geometry. But they also speak about Brocardian geometry, Brocard configuration and Brocard figures. In fact, the only methods associated to this group of objects were the isogonal inversion in its geometrical version as well as the theory of three similar figures. So a harmonization of the results made with the elementary geometry was more natural from the Brocard points and Brocard circle in view of their respective definitions. In 1952, when Court used the geometry of correspondences, the importance of the Lemoine point become obvious and he used the expression Lemoine Geometry.

Modern geometry was taught in American colleges, which is equivalent to the university years. Therefore, it sometimes took the name of College Geometry. For

American authors, modern geometry was good training, as a research subject, for prospective high school geometry teachers.

If, in France, the new triangle geometry allowed the development of new methods which were used, among others, in the teaching of the analytical geometry, it was different in the United States where the aestheticism of the new results, obtained by simple methods, was underlined like we notice in Johnson's textbook introduction:

> Here is indeed a discipline [the modern geometry] which is a natural "Sequel" to elementary geometry, a body of propositions which may be derived by methods similar to those used in the classical plane geometry and which has all the attraction of novelty and inherent beauty (Johnson, 1929, p. iii).

The textbook written by Coolidge is called *A Treatise on the circle and the sphere*. The objective of Coolidge was to present all study methods of the circle and the sphere. The plan of his work followed the various possible methods. The presentation of the objects of the triangle was not the principal aim. But they were defined during the presentation of the circles of Brocard, Lemoine, Tucker or even Taylor which are remarkable circles of the new triangle geometry. Coolidge separated the elementary plane geometry from the Cartesian plane geometry. Each chapter contained objects of the new triangle geometry.

The second book, written by Johnson, is called *Advanced Euclidean Geometry*. The title of the work of Johnson informs us about the choices made for the presentation of the geometry of the triangle and the circle: he wished to make a presentation based on the elementary geometry, in the spirit of the Euclidean geometry. Three chapters are dedicated to the new triangle geometry: the 12th is called "symmedian point and other notable points" (the symmedian point is the American appellation for Lemoine point). According to Johnson, the symmedian point realizes the link between the classic geometrical system built around the orthocenter, the centroid and the circumcenter on one side and the geometrical system associated to the figures of Brocard on the other side (Johnson, 1929, p. 213).

Nathan Altshiller-Court, the third author, clearly identified his work as a textbook, first by the explicit title and then by the composition. The expression *College Geometry* indicated the school level for which it was drafted. The title was *College Geometry, A second course in Plane geometry for Colleges and Normal Schools* and for the second edition the title became *College Geometry an introduction to the Modern Geometry of the triangle and the circle*. In the second edition, the geometry of correspondences took a more important place in the presentation of the remarkable objects of the new triangle geometry. Court presented the geometry of Lemoine on the same plan as the geometry of Brocard and the terminology of symmedian point had been abandoned for the benefit of the Lemoine point. Nevertheless Court, like Johnson, did not use trilinear coordinates. It was an elementary geometry of correspondences that served as a tool of harmonization.

Conclusion

In France, the textbooks which incorporated the new triangle geometry as a separate part participated in the teaching of methods, a practice that was developed at the end of the twenty-first century. Furthermore, this knowledge was used for examination topics. The purpose was to teach new and complex geometrical methods through a simple figure, the triangle. On the contrary, the position of Anglo-Saxon textbooks was to continue conceptions like Euclid's. Their authors wanted to make the new geometric methods simple and elementary. Even if the European textbooks we study were known in the United States, as they appeared in the "new publications" section of the Bulletin of the American Mathematical Society, Coolidge and Johnson did not mention them as sources. For example, Johnson said that his interest for Casey's book was mainly historical (Johnson, 1929, p. vi). These two authors mainly referred, for example, to Emmerich and Fuhrmann's books. However, Court, who used an elementary version of the geometry of correspondences, quoted a lot of French and Belgian authors.

We would like to conclude with the current affairs of the new triangle geometry. In the eighties, the American Clark Kimberling, Professor of Mathematics at the University of Evansville, developed the notion of "triangle center" to replace "remarkable points of the triangle". In 1998 he published *Triangle centers and central triangles* that united and improved his research. The French couple, Sortais, published in 1987 a book of solved exercises about triangle geometry for high school pupils, which contains two chapters about the new triangle geometry. Finally, in 1992, the German Peter Baptist wrote an interesting didactic book called *Die Entwicklung der neueren Dreiecksgeometrie.* It thus remains to study the place of the new triangle geometry in the educational system at the end of the twentieth century.

References

Aspra, J., Marmier, A.-M., & Martinez, I. (2007). De l'étude des solides à la construction de l'espace. In E. Barbin & D. Bénard (Eds.), *Histoire et enseignement des mathématiques. Rigueur, erreurs, raisonnements* (pp. 109-146). Lyon: INRP.

Baptist, P. (1992). *Die Entwicklung der neueren Dreiecksgeometrie,* Band 19, N. Knoche and H. Scheid, B.I.

Barbin, E. (1992). Sur la conception des savoirs géométriques dans les *Éléments de géométrie*. In A. Gagatsis (Ed.), *Cahiers de didactique de mathématiques* (pp. 135-158). Thessalonique: Université de Thessalonique.

Belhoste, B. (1995). *Les sciences dans l'enseignement secondaire français,* textes officiels, v. 1: 1789-1914. Lyon: INRP.

Belhoste, B., Gispert, H., & Hulin, N. (Eds.) (1996). *Les sciences au lycée. Un siècle de réformes des mathématiques et de la physique en France et à l'étranger.* Paris: Vuibert, INRP.

Brocard, H. (1881). Étude d'un nouveau cercle du plan du triangle. *Compte rendu des séances des sessions de l'A.F.A.S., 10*, 138-159.

Casey, J. (1885a). *The first six books of the Elements of Euclid*, 3th edition. London: Hodges, Figgis & Co.

Casey, J. (1885b). *A Treatise of analytical geometry of the point, line, circle, and conic sections,* Dublin: University Press, London: Hodges, Figgis & Co.

Casey, J. (1888). *A sequel to Euclid,* 5th ed, Dublin: University Press, London: Hodges, Figgis & Co.

Coolidge, J. L. (1916). *A treatise on circle and the sphere.* Oxford: Clarendon Press.

Court, N. A. (1952). *College geometry: An Introduction to the modern geometry of the triangle and the circle*, 2nd ed, New York, NY: Barnes and Noble.

Davis, P. J. (1995). The rise, fall, and possible transfiguration of triangle geometry: a mini-history, *American Mathematical Monthly, 32*(3), 204-214.

F.G.–M. (1991), *Exercices de géométrie,* reprint of the 6th ed (1920. Tours: Mame, Paris: De Gigord). Paris: Jacques Gabay.

Johnson, R. A. (1960). *Advanced Euclidean geometry (Modern geometry). An elementary treatise on the geometry of the triangle and the circle,* New York: Dover Publications, unabridged and unaltered republication of the first edition published by Houghton Mifflin Company in 1929.

Kimberling, K. (1998). *Triangle centers and central triangle.* Winnipeg: Utilitas Mathematica Publishing.

Koehler, J. (1886). *Analytic geometry and superior geometry exercises,* Paris: Gauhtier-Villars & fils.

Lemoine, E. (1873), Sur quelques propriétés d'un point remarquable du plan d'un triangle, *Compte rendu des séances des sessions de l'A.F.A.S., 2*, 90-95.

Lemoine, E. (1885), Propriétés relatives à deux points ω et ω' du plan d'un triangle ABC qui se déduisent d'un point K quelconque du plan comme les points de Brocard se déduisent du point de Lemoine, *Compte rendu des séances des sessions de l'A.F.A.S.*, *14*, 23-49.

Rouché, E., & De Comberousse, C. (1891). *Traité de géométrie*, 6th ed, Paris: Gauthier-Villars.

Sortais, Y. & R., *La géométrie du triangle, exercices résolus*, Paris: Hermann.

Tracey, J. I. (1930). *Modern Geometry* by Roger A. Johnson, *Bulletin of the American Mathematical Society, XXXVI*(3), 176.

Vigarié, E. (1887). Géométrie du triangle. Étude bibliographique et terminologique, *Journal de Mathématiques Spéciales, s. 3, 1*, 34-45, 58-62, 77-82, 127-132, 154-157, 175-177, 199-203, 217-219, 248-250.