Covering Spaces for Domains in the Complex Plane

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60 ECTS thesis submitted in partial fulfillment of a
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For Stefania

Abstract

In this thesis we present a modern version of a proof published by Tibor Radó in 1922, which shows that the universal covering space for a domain G in $\mathbb{C}\setminus\{0,1\}$ is biholomorphically equivalent to the unit disk. Radó uses multivalued functions which we replace by mappings on appropriate covering spaces. We use Riemann domains for this purpose, which are essentially Riemann surfaces with global coordinates.

Útdráttur

Í þessari ritgerð fjöllum við um nútímalega útgáfu af sönnun sem Tibor Radó gaf út árið 1922. Hún sýnir að allsherjarþekjurúm sérhvers svæðis G í $\mathbb{C} \setminus \{0,1\}$ er fágað einsmóta einingarhringskífunni. Radó notaðist við marggild föll í sinni grein, en við skoðum í staðinn varpanir á viðeigandi þekjurúmum. Við notumst við Riemann-svæði, en þau eru sértilfelli af almennum Riemann-flötum, þar sem við höfum víðfeðm hnitaföll.

Contents

Þakkarorð

Ég vil þakka Reyni Axelssyni og Jón Ingólfi Magnússyni, fyrir alla þá hjálp sem þeir hafa gefið mér við vinnslu þessarar ritgerðar. Ég vil einnig sérstaklega þakka Sigurði Frey Hafstein fyrir þann mikla stuðning og þolinmæði sem hann hefur sýnt mér í gegnum árin. Síðast en ekki síst vil ég þakka Stefaniu Crotti, *grazie di cuore*.

1. Introduction

In 1922 Radó published a proof of the famous Riemann mapping theorem [10], by L.Fejér and F.Riesz.

Theorem 1.0.1 (Riemann mapping theorem)**.** *Let $G \subset \mathbb{C}$ be a simply connected domain which is not all of $\mathbb{C}$. Then there exists a biholomorphic map $f\colon G \to \mathbb{E}$, where $\mathbb{E}$ is the open unit disk $\{z \in \mathbb{C} \mid |z| < 1\}$. Furthermore we can, for any $p \in G$, choose f in such a way that $f(p) = 0$, and the map f is then uniquely determined up to a rotation.*

Their approach was a new one: they showed that the Riemann mapping theorem could be obtained as a solution of an extremal problem. The argument was very short, only about one full page. Many modern textbooks that contain a proof of the Riemann mapping theorem use similar methods, although modernized. The proof is presented in section 1.2.

In his paper, Radó goes on to prove the following theorem:

Theorem 1.0.2. *Let G be a domain in $\mathbb{C}$, such that the complement of G contains at least two points. Then there exists a holomorphic covering map $p\colon \mathbb{E} \to G$.*

Radó uses a similar idea as Fejér and Riesz used to prove Theorem 1.0.1. He starts by considering a certain class $\mathcal{G}$ of multivalued holomorphic functions from G to $\mathbb{E}$ which is defined in such a way the inverse of any function $f \in \mathcal{G}$ is a well defined single valued holomorphic function from the image of f in $\mathbb{E}$ to G. By picking a specific solution to an extremal problem Radó manages to construct a multivalued holomorphic surjection $f\colon G \to \mathbb{E}$, whose inverse is a well defined holomorphic covering map $f^{-1}\colon \mathbb{E} \to G$.

In this thesis we present a more modern version of Radó's proof. Instead of multivalued functions on G, we consider well defined holomorphic functions on appropriate covering spaces of G, satisfying the condition that they are injective with respect to the covering map (see Definition 2.2.4). This condition guarantees that a certain 'inverse' of these functions is a well defined holomorphic map (see Lemma 2.2.5). Using the same idea as Radó, we prove Theorem 3.0.1, which says that for any domain G in $\mathbb{C}$, there exists a holomorphic covering map $p\colon Y \to G$, where Y is biholomorphically equivalent to either $\mathbb{C}$ or $\mathbb{E}$. The end result is an argument pleasantly similar to a method used to prove the Riemann mapping theorem in many modern textbooks, albeit much longer.

In section 1.1 we give a brief exposition on the Uniformization theorem and its consequences, and how it relates to our thesis. In section 1.2 we present Radó's original argument to prove the Riemann mapping theorem, and we end this chapter by showing in section 1.3 how Picard's little theorem is a simple corollary of the main result of this thesis. Chapter 2 presents some topological preliminaries and our main tool, the Riemann domains, and finally, in chapter 3 we have the main proof of our thesis. We have also included appendix A which contains some basic definitions and lemmas on homotopy and covering spaces.

1.1. Background

The main theorem of this thesis, Theorem 3.0.1, mentioned in the introduction, can be viewed as a special case of the famous Uniformization theorem. The theorem, stated in the next section deals with general Riemann surfaces, but in this thesis we are working with domains of $\mathbb{C}$. We give an overview of the theorem and its consequences with simplified steps to reduce unnecessary complexities that would be outside the scope of this thesis.

1.1.1. The Uniformization Theorem

The Uniformization theorem is simple to state, but it took over a century for mathematicians to formulate it and give a convincing proof. It wasn't until the year 1907 that Henri Poincaré and Paul Koebe gave what is considered the first proofs of the theorem. The book *"Uniformization of Riemann Surfaces: Revisiting a hundred-year-old theorem"*[3] gives a detailed account of the very interesting history of the Uniformization theorem and presents proofs of various versions of it, up to the final form stated here below.

Theorem 1.1.1 (Uniformization theorem). *Every simply connected Riemann surface is biholomorphically equivalent to the Riemann sphere, the complex plane, or the unit disc.*

The modern version of the Uniformization theorem is a classification result, but historically uniformization is about parametrization. The theorem arises from the study of algebraic curves and differential equations. In the next two subsections we give a brief description of parametrization of algebraic curves as a motivation, and classification of Riemann surfaces by the Uniformization theorem.

1.1.2. Parametrization of Algebraic Curves

To explain the connection between the Uniformization theorem and parametrization, we give a simplified description of a certain class of algebraic curves called *elliptic curves*. An elliptic curve is a cubic polynomial of the form

$$y^2 = x^3 + ax + b$$

where a,b are complex numbers such that $4a^3 + 27b^2 \neq 0$. We want to parametrize this elliptic curve, and for the purpose let us assume that there exists a function $z \mapsto (x(z), y(z)) = (f(z), f'(z))$. Then f must satisfy the differential equation

$$(f')^2 = f^3 + af + b.$$

We can 'solve' this differential equation formally with the integrals

$$z = \int \frac{df}{\sqrt{f^3 + af + b}} = \int \frac{dx}{\sqrt{x^3 + ax + b}}.$$

This type of integral is know as an *elliptic integral*, since it arises when evaluating the arc length of ellipses. The integral depends not only on which branch of the square root we use, but also on the path of integration. In this sense we can regard z as multivalued function of x. The key idea to obtain the parametrization is that we look at the inverse of this multivalued function, which, as turns out, gives us x as a single valued periodic function of z, and therefore we have parametrized the elliptic curve. The above calculations can be made precise [6, Ch. 10.6].

This idea brings us to the topic of doubly periodic meromorphic functions. One example is the *Weierstrass* $\wp$ *function* which is meromorphic on $\mathbb{C}$ with periods 1 and i, that is to say $\wp(z+1) = \wp(z)$ and $\wp(z+i) = \wp(z)$ for all z, and has double poles at every point on the lattice $\Lambda = \{m + in \mid m, n \in \mathbb{Z}\}$, see [6, Ch. 10.6]. This function is given by the series

$$\wp(z) = \frac{1}{z^2} + \sum_{\substack{m,n \in \mathbb{Z} \\ (m,n) \neq (0,0)}} \left[\frac{1}{(z - m - ni)^2} - \frac{1}{(m + ni)^2} \right],$$

and there are complex valued constants $C_1, C_2 \in \mathbb{C}$ such that

$$(\wp'(z))^2 = 4\,(\wp(z))^3 - C_1\wp(z) - C_2$$

for all $z \in \mathbb{C}$.

The map $(\wp, \wp')\colon \mathbb{C} \to S$ then gives us a parametrization of the elliptic curve S where S is the curve given by $y^2 = 4x^3 - C_1x - C_2$ in the complex projective plane, and we get an identification of $\mathbb{C}/\Lambda$ and S.

Now the constants C_1 and C_2 above depend on the period lattice Λ, and by choosing a different period lattice we obtain a parametrization for a different elliptic curve.

We can think of these elliptic curves where we allow x and y to take complex values as complex tori. Now if Γ and Γ' are two lattices in $\mathbb{C}$, then the complex tori $\mathbb{C}/\Gamma$ and $\mathbb{C}/\Gamma'$ are topologically homeomorphic, but there is no biholomorphic equivalence between them unless there exists an $\alpha \in \mathbb{C} \setminus \{0\}$ such that $\Gamma = \alpha\Gamma'$. However the universal covering space for a complex torus is $\mathbb{C}$, so we get a meromorphic parametrization of the elliptic curve from the complex plane $\mathbb{C}$.

We have uniformized the parameter, in the sense that every elliptic curve can be parametrized by a function on $\mathbb{C}$, even though the underlying complex tori are not biholomorphically equivalent.

Every algebraic curve defines by desingularization a compact Riemann surface, see [4, Thm. 8.9], and every compact Riemann surface can be classified topologically by its topological genus, and therefore we can classify algebraic curves by the same genus. It can be shown that curves of genus 0 are birationally equivalent to the Riemann sphere, and are therefore rational curves. Curves of genus 1 are equivalent to a complex torus and are elliptic. All other curves, which are of genus ≥ 2, can be parametrized by a function on the unit disk $\mathbb{E}$ by the Uniformization theorem, and are called hyperbolic.

1.1.3. Classification of Riemann Surfaces

Let X be a Riemann surface and $p\colon \tilde{X} \to X$ be its universal covering. Then $G = \mathrm{Deck}(\tilde{X}/X)$ is defined as the subgroup of the holomorphic automorphism group $\mathrm{Aut}(\tilde{X})$, consisting of all f such that $p \circ f = p$. Then the group G acts without fixed points and discretely on $\tilde{X}$, that is if $\sigma \in G \setminus \{\mathrm{id}\}$ then $\sigma x \neq x$ for all $x \in \tilde{X}$ and the orbit $Gx = \{\sigma x \mid \sigma \in G\}$ is a discrete subset of $\tilde{X}$ for every $x \in \tilde{X}$. This can be seen from the fact that a covering transformation is uniquely determined by the image of any point, and the orbit Gx is equal to the fiber $p^{-1}[p(x)]$, furthermore we have that G is isomorphic to the fundamental group $\pi_1(X, x)$ for any $x \in X$.

Since $\tilde{X}$ is simply connected, then by the *Uniformization theorem* it is biholomorphically equivalent to the Riemann sphere $\overline{\mathbb{C}}$, the complex plane $\mathbb{C}$ or open unit disk

$\mathbb{E}$, and we get a classification of all Riemann surfaces, by their universal covering spaces.

By Theorem [6, Thm. 6.3.5] every automorphism of $\overline{\mathbb{C}}$ is a Möbius transformation of the form

$$\phi(z) = \frac{az+b}{cz+d} \quad a, b, c, d \in \mathbb{C},\ ad - bc \neq 0.$$

Every Möbius transformation of this form $\phi\colon \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ has at least one fixed point. So, if $\tilde{X}$ is biholomorphically equivalent to the Riemann sphere $\overline{\mathbb{C}}$, then $G = \mathrm{Deck}(\tilde{X}/X) = \{\mathrm{id}\}$ and X is simply connected. Therefore $p\colon \tilde{X} \to X$ is a biholomorphism and X is biholomorphically equivalent to the Riemann sphere $\overline{\mathbb{C}}$.

Every automorphism of $\mathbb{C}$ is of the form:

$$\phi(z) = az + b, \quad a \in \mathbb{C} \setminus \{0\},\ b \in \mathbb{C},$$

and if $a \neq 1$ this transformation has a fixed point. Thus a subgroup G of $\mathrm{Aut}(\mathbb{C})$ that acts discretely and without fixed points consists only of translations $\phi(z) = z + b$. Let Γ be the orbit of zero under G. Then Γ is a discrete additive subgroup of $\mathbb{C}$, and G is the subgroup of all translations $z \mapsto z + b$ with $b \in \Gamma$. Then by [4, Thm. 27.11], one of the following is true:

(i) $G = \{\mathrm{id}\}$.

(ii) G a cyclic group of all translations of the form $z \mapsto z + nb$, for all $n \in \mathbb{Z}$, where $b \neq 0$ is a fixed complex number.

(iii) G consists of all translations of the form $z \mapsto z + nb_1 + mb_2$, for all $n, m \in \mathbb{Z}$, where b_1, b_2 are two fixed complex numbers, linearly independent over $\mathbb{R}$.

If $\tilde{X}$ is biholomorphically equivalent to $\mathbb{C}$ then $G = \mathrm{Deck}(\tilde{X}/X)$ is one of the three types of groups above. For (i), we get that X is biholomorphic to $\mathbb{C}$, in case (ii) we get that the covering is isomorphic to $\mathbb{C} \to \mathbb{C} \setminus \{0\}$, $z \mapsto \exp\left(\frac{2\pi i}{b}\right)$. Finally for the third case X is biholomorphic to a complex torus $\mathbb{C}/\Gamma$.

The set of automorphisms of the unit disk $\mathbb{E}$ are transformations of the form [6, Thm. 6.2.3]:

$$\phi(z) = \omega \frac{z-a}{1-\overline{a}z}, \quad a, \omega \in \mathbb{C},\ |a| < 1,\ |\omega| = 1.$$

The unit disk $\mathbb{E}$ is biholomorphic to the upper half plane, via the (inverse) Cayley transform $f(z) = \frac{z-i}{z+i}$, which maps $H = \{z \in \mathbb{C} \mid Im(z) > 0\}$ to the unit disk $\mathbb{E}$. Therefore we can work in H instead of $\mathbb{E}$ and $\mathrm{Aut}(H)$ can be described as the set of all transformations of the form [6, Ch. 10.5]:

$$\phi(z) = \frac{az+b}{cz+d} \quad a,b,c,d \in \mathbb{R},\ ad-bc>0.$$

The structure of subgroups in $\mathrm{Aut}(H)$ that act discretely and without fixed points is very rich, so we will not go into details here. One example of such a group is the subgroup of translations of the form $\phi(z) = z + nb$, with n integer and b some fixed real number.

If X is a Riemann surface and its universal covering space $p\colon \tilde{X} \to X$ is not biholomorphic to $\overline{\mathbb{C}}$ or $\mathbb{C}$, then $\tilde{X}$ is biholomorphic to $\mathbb{E}$, and there exists a biholomorphic covering map $\pi\colon \mathbb{E} \to X$.

Riemann surfaces are called *elliptic*, *parabolic* or *hyperbolic* when their universal covering is isomorphic to $\overline{\mathbb{C}}$, $\mathbb{C}$, or $\mathbb{E}$ respectively, see [4, thm 27.10]. From the preceding discussion we see that the Riemann sphere $\overline{\mathbb{C}}$ is the only elliptic Riemann surface. The complex plane $\mathbb{C}$, the punctured plane $\mathbb{C}\setminus\{0\}$ and all complex tori $\mathbb{C}/\Gamma$ are parabolic, and every other Riemann surface is hyperbolic. Calling the Riemann sphere elliptic can cause some confusion as, by the classification of the algebraic curves, complex tori are sometimes called elliptic curves.

For the purpose of this thesis we are working with domains G in $\mathbb{C} \setminus \{0,1\}$, which by the main theorem of this thesis, Theorem 3.0.1, can be covered by the unit disk $\mathbb{E}$ and are therefore *hyperbolic*.

1.2. Proof of the Riemann Mapping Theorem

In his paper Radó only deals with the case when G is bounded. The general case then follows with a standard argument which is written in detail in many textbooks, see e.g. [1, Thm. VI.5.1]. We thus assume that G is bounded, fix a point $p \in G$, and define the class

$$\mathcal{F} = \{g \in \mathcal{O}(G) \mid g \text{ is bounded and injective, } g(p)=0 \text{ and } g'(p)=1\}.$$

By translating G we may take p as the origin 0. The set $\mathcal{F}$ is not empty, as the function $G \to G$, $z \mapsto z$, is in $\mathcal{F}$. Consider the uniform norm given by $g \mapsto |g|_G = \sup_{z\in G}|g(z)|$. By definition of $\mathcal{F}$, we have $|g|_G < \infty$ for every $g \in \mathcal{F}$. Define the constant $\rho = \inf\{|g|_G \mid g \in \mathcal{F}\}$, then there exists a sequence of functions (ψ_n) in $\mathcal{F}$ such that $|\psi_n|_G$ converges to ρ. The sequence (ψ_n) is bounded on G, so by Montel's theorem, see [1, Thm. IV.1.5] or Theorem 2.2.8, there exists a subsequence, also denoted by (ψ_n), that converges uniformly on compact subsets of G to a holomorphic function ψ.

As $\psi_k(0) = 0$ and $\psi_k'(0) = 1$ for all $k \in \mathbb{N}$, and since the sequence converges uniformly on compact sets, it follows that $\psi(0) = 0$ and $\psi'(0) = 1$. Therefore ψ is not constant on G and $|\psi|_G = \rho > 0$. Since ψ is holomorphic, and a limit of injective holomorphic functions, then ψ is also injective, and therefore $\psi \in \mathcal{F}$. In fact, if ψ is not injective, then there are two distinct points $x, y \in G$ such that $\psi(x) = \psi(y)$, and by Hurwitz's theorem, see Theorem 2.2.10, there are distinct points $x_k, y_k \in G$ such that $\psi_k(x_k) = \psi_k(y_k)$ for k large enough, which is a contradiction as f_k is injective.

Since $|\psi|_G = \rho$ then the image $\psi(G)$ is contained in $\rho\mathbb{E}$, where $\mathbb{E}$ is the open unit disk. The only thing left to show is that $\psi\colon G \to \rho\mathbb{E}$ is a surjective function.

Assume that ψ is not surjective onto $\rho\mathbb{E}$, then there is a complex number $re^{i\phi} \in \rho\mathbb{E}$, with $0 < r < \rho$, which is not included in the image of ψ. We use what Radó calls the *Carathéodory-Koebe square root transformation* to construct a new function $F \in \mathcal{F}$ such that $|F|_G < \rho$, which is a contradiction. Therefore ψ must be a biholomorphic map $\psi\colon G \to \rho\mathbb{E}$.

Let $\beta = \frac{r}{\rho} < 1$ and set $w = \beta e^{i\phi}$. Denote by $h_w\colon \mathbb{E} \to \mathbb{E}$, the function $h_w(z) = \frac{z-w}{z\overline{w}-1}$, then $h_w(0) = w$, $h_w(w) = 0$, and the map $z \mapsto h_w\left(\frac{1}{\rho}\psi(z)\right)$ does not take the value 0. Since G is a simply connected domain there exists a $l\colon G \to \mathbb{E}$ such that $l^2(z) = h_w(\frac{1}{\rho}\psi(z))$ and $l(0) = w_1$ where $w_1 = \sqrt{\beta}e^{i\frac{1}{2}\phi}$. Using one more transformation $h_{w_1}\colon \mathbb{E} \to \mathbb{E}$, $h_{w_1}(z) = \frac{z-w_1}{z\overline{w}_1-1}$, we can define a map $g\colon G \to \mathbb{E}$ by

$$g(z) = e^{i\frac{1}{2}\phi}h_{w_1}(l(z)).$$

We see, by definitions of the transformations h_w and h_{w_1}, that $g(0) = 0$, and

$$g'(0) = \frac{\beta+1}{2\sqrt{\beta}}\frac{1}{\rho}\psi'(0).$$

Set

$$F(z) = \frac{2\sqrt{\beta}}{\beta+1}\rho g(z),$$

then $F(0) = 0$ and $F'(0) = \psi'(0) = 1$, and since all of the maps ψ, h_w, l, h_{w_1} and g are injective and bounded, we have that $F \in \mathcal{F}$. Since $|g|_G \leq 1$, then $|F|_G \leq \frac{2\sqrt{\beta}}{\beta+1}\rho$, but since $0 < \beta < 1$ we have that $\frac{2\sqrt{\beta}}{\beta+1} < 1$, and therefore $|F|_G < \rho$, which is a contradiction.

By scaling the function ψ obtained above we get a biholomorphic map $f\colon G \to \mathbb{E}$, $f = \psi/\rho$. Full calculations of the derivatives above, and more discussion on the properties of the maps h_w can be found in Definition 3.1.4 and Lemma 3.1.5.

Now it only remains to prove uniqueness. Let f be the biholomorphic map $f\colon G \to \mathbb{E}$ constructed above, and let g be any biholomorphic map $g\colon G \to \mathbb{E}$ such that $g(0) = 0$. Then $\lambda = f \circ g^{-1}\colon \mathbb{E} \to \mathbb{E}$ is an automorphism of $\mathbb{E}$ such that $\lambda(0) = 0$, and by Schwarz's lemma, see Lemma 2.2.9, $f \circ g^{-1}(z) = az$ with $|a| = 1$. Therefore $f(z) = ag(z)$, and g is just a rotation of f.

1.3. Picard's Little Theorem

The main result of this paper, Theorem 3.0.2 is a proof of the fact that the universal covering space of a domain G in $\mathbb{C}$, of which the complement contains at least two points, is the unit disk $\mathbb{E}$. This result is just a very special case of the general Uniformization theorem for abstract Riemann surfaces, and the full general theorem needs much more theory to be proven.

One of the corollaries of this special case is Picard's little theorem, and we obtain a very short and simple proof.

Theorem 1.3.1 (Picard's little theorem)**.** *If a function $f\colon \mathbb{C} \to \mathbb{C}$ is holomorphic and not constant, then the image of f is either the whole of $\mathbb{C}$ or $\mathbb{C}$ minus a single point.*

Let G denote the image of f and assume that the complement $\mathbb{C} \setminus G$ contains at least two distinct points, then we can lift f to a holomorphic function $g\colon \mathbb{C} \to \mathbb{E}$, since $p\colon \mathbb{E} \to G$ is the universal covering space of G, by Theorem 3.0.1, but g is a bounded entire function and therefore constant, so $f = p \circ g$ must be constant.

2. Preliminaries

In this chapter we present some well known topological lemmas and constructions, with the intention of proving Theorems 2.1.17 and 2.1.18. They state that given a space G and a universal cover $p : Y \to G$, then a certain subspace of Y is the universal cover for a specific subspace of G. We also define Riemann domains, which serve as our primary tools in the proof of the Uniformization theorem for domains in $\mathbb{C}$ (see Lemma 3.0.1). We present Hurwitz's theorem in 2.2.10 and Montel's theorem for Riemann domains in 2.2.8, as they are used in later chapters.

2.1. Topological Preliminaries

2.1.1. Local Homeomorphisms and Sections

Definition 2.1.1. *Let U, V be Hausdorff spaces, $p\colon U \to V$ be a continuous map, and W be a subset of V. Then a* section *of p over W is a continuous map $\sigma\colon W \to U$ such that $(p \circ \sigma)(x) = x$ for all x in W.*

The following are well known lemmas on sections of local homeomorphisms, for a proof see [5, VII, prop 1.4-6].

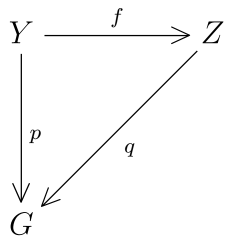
Lemma 2.1.2. *Let U and V be as in Definition 2.1.1, and let $p\colon U \to V$ be a local homeomorphism*

(i) *For any point $v \in V$ and $u \in p^{-1}[v]$ there exists a neighbourhood W around v and a section $\sigma\colon W \to U$ such that $\sigma(v) = u$.*

(ii) *A section $\sigma\colon W \to U$ of p induces a homeomorphism $W \to \sigma(W)$, $x \mapsto \sigma(x)$.*

(iii) *Let W be a connected subset of V and $\sigma_1, \sigma_2\colon W \to U$ be two sections of p. If σ_1 and σ_2 agree at a single point in W then they agree on the whole of W, that is to say $\sigma_1 = \sigma_2$.*

Lemma 2.1.3. *Let Y, Z, G be Hausdorff spaces and $p\colon Y \to G$ and $q\colon Z \to G$ be two local homeomorphisms and σ be a section p. Furthermore assume that $f\colon Y \to Z$ is a continuous map such that $p = q \circ f$, that is to say the following diagram commutes.*



Then $f \circ \sigma$ is a section of q.

Proof. Assume that the section σ is defined on a subset U of G. Then

$$q \circ (f \circ \sigma) = (q \circ f) \circ \sigma = p \circ \sigma = \mathrm{id}_U$$

so $f \circ \sigma$ is a section of q. □

Lemma 2.1.4. *Let Y, Z, G be connected and locally connected Hausdorff spaces. Let $p\colon Y \to G$ be a covering map, $f\colon Y \to Z$ surjective, and $q\colon Z \to G$ be a local homeomorphism such that $p = q \circ f$; then f and q are covering maps.*

Proof. Fix a point $x \in G$ and let U be a connected neighbourhood of x evenly covered by p. Then there exists a set I and a family of sections $\sigma_i\colon U \to Y$, $i \in I$ such that

$$p^{-1}[U] = \bigcup_{i \in I} \sigma_i(U)$$

where the sets on the right hand side are pairwise disjoint. For each $i \in I$ the map $f \circ \sigma_i\colon U \to Z$ is a section of q, by Lemma 2.1.3. Put $V_i = f \circ \sigma_i(U)$, for all $i \in I$. Now for any i, j in the set I, we have that either $V_i = V_j$ or $V_i \cap V_j = \emptyset$, since two sections that agree at a point agree on the whole of U, by Lemma 2.1.2(iii). Therefore there exists a set $J \subset I$, such that

$$\bigcup_{i \in I} f \circ \sigma_i(U) = \bigcup_{i \in J} f \circ \sigma_i(U)$$

where the sets on the right hand side are all pairwise disjoint. It remains to show that

$$q^{-1}[U] = \bigcup_{i \in J} f \circ \sigma_i(U).$$

Let $z \in q^{-1}[U]$, since f is surjective there exists a $y \in Y$ with $f(y) = z$. But then $p(y) \in U$ so $y \in \sigma_i(U)$ for some $i \in I$, therefore

$$z \in f \circ \sigma_i(U) \subset \bigcup_{j \in I} f \circ \sigma_j(U) = \bigcup_{j \in J} f \circ \sigma_j(U).$$

Then q is a covering map and it is well known that if f is a surjective continuous fiber preserving map between covering maps, then f is also a covering map, see for example [5, Ch. VII, prop. 4.3]. □

2.1.2. Chains and Step Polygons

The goal of this section is to prove Lemma 2.1.16 and Theorem 2.1.18, and for that purpose we introduce the notion of chains and step polygons. The topic is covered in depth in in [1, Ch. I] and [11, Ch. 12].

Definition 2.1.5. *A path in an open domain $U \subset \mathbb{C}$ is a continuous function τ from the interval $[0, 1] \subset \mathbb{R}$ to U. A closed path τ in U is a path such that $\tau(0) = \tau(1)$. We denote the image of the path $\tau\colon [0, 1] \to U$ by $|\tau|$,*

$$|\tau| = \{\tau(x) \mid x \in [0, 1]\}.$$

Let U be a domain in $\mathbb{C}$, we denote by $C_1(U)$ the free abelian group generated by all paths in U.

Definition 2.1.6. *A* chain *γ in a domain U of $\mathbb{C}$ is an element γ in the free abelian group $C_1(U)$. We can write γ as the finite sum*

$$\gamma = a_1\tau_1 + \cdots + a_n\tau_n, \quad a_i \in \mathbb{Z}$$

where each of the τ_i, $1 \leq i \leq n$ is a path in U.

Denote by $C_0(U)$ the free abelian group generated by all points $u \in U$. For each path τ in U we define the boundary $\partial_0(\tau) \in C_0(U)$, by $\partial_0(\tau) = \tau(1) - \tau(0)$. Note that the binary operation here is subtraction in the group $C_0(U)$ not the normal subtraction in $\mathbb{C}$. A path τ is closed if, and only if, $\partial_0(\tau) = 0 \in C_0(U)$.

By extending ∂_0 linearly, a well defined group homomorphism $\partial : C_1 \to C_0$ is obtained. It is given by

$$\partial(\gamma) = \partial\left(\sum_{j=1}^{n} a_j\tau_j\right) = \sum_{j=1}^{n} a_j\partial_0(\tau_j) \in C_0(U).$$

where $a_j \in \mathbb{Z}$ and each τ_j is a path in U.

We say a chain γ is *closed* if its boundary $\partial(\gamma)$ is $0 \in C_0(U)$. In other words, let the closed chain γ be given by the sum $\sum_{j=1}^{n} a_j \tau_j$. Then each starting point and end point of paths τ_j in γ appear equally often and with opposite signs, counted with multiplicity.

We define the *support* of the chain $\gamma = \sum_{j=1}^{n} a_j \tau_j$ by

$$\operatorname{supp}(\gamma) = \operatorname{supp}\left(\sum_{j=1}^{n} a_j \tau_j\right) = \bigcup_{j=1, a_j \neq 0}^{n} |\tau_j|,$$

which is simply the union of the images of all the closed path τ_j, and it is a compact set.

For any point $p \notin \operatorname{supp}(\gamma)$ we can define the *winding number* of a closed chain γ around p,

$$I(\gamma, p) = \frac{1}{2\pi i} \int_{\gamma} \frac{dz}{z - p}$$

see e.g. [1, Def. 1.14] or [11, Ch. 12.4]. The winding number $I(\gamma, p)$ is an integer for all $p \in \mathbb{C} \setminus \operatorname{supp}(\gamma)$ and the mapping $\mathbb{C} \setminus \operatorname{supp}(\gamma) \to \mathbb{Z}$, $p \mapsto I(\gamma, p)$, is constant on every connected component of $\mathbb{C} \setminus \operatorname{supp}(\gamma)$.

We can furthermore define the *interior* and *exterior* of the closed chain γ as the open sets

$$I(\gamma) = \{z \in \mathbb{C} \setminus \operatorname{supp}(\gamma) \mid I(\gamma, z) \neq 0\}, \quad E(\gamma) = \{z \in \mathbb{C} \setminus \operatorname{supp}(\gamma) \mid I(\gamma, z) = 0\}$$

respectively.

Definition 2.1.7. *Let p_1, p_2 be points in $\mathbb{C}$, then the segment $[p_1, p_2]$ is defined as the continuous map $[0,1] \to \mathbb{C}$, $t \mapsto (1-t)p_1 + tp_2$. A step polygon is a closed chain γ composed of k segments, $[p_1, p_2] + [p_2, p_3] + \cdots + [p_k, p_1]$, where each of the segments is parallel to either the real or the imaginary axis in $\mathbb{C}$, each of the points $p_1, p_2, \ldots, p_k$ are pairwise distinct, and the images of the segments are pairwise disjoint except for the starting and ending points of the segments.*

A step polygon $\gamma = [p_1, p_2] + \ldots + [p_k, p_1]$ can be regarded as a closed path $\tau \colon [0,1] \to \mathbb{C}$ by dividing the interval $[0,1]$ into k equal intervals $[t_i, t_{i+1}]$ with $t_1 = 0$ and $t_{k+1} = 1$, and mapping the intervals $[t_i, t_{i+1}]$ onto the segment $[p_i, p_{i+1}]$ and $[t_k, t_{k+1}]$ onto $[p_k, p_1]$. The closed path τ restricted to the half open interval $[0,1)$ is then injective.

We have the following three lemmas for step polygons, for detailed proofs see [11, Ch. 12.4], and [1, Thm. I.2.7].

Lemma 2.1.8. *Every step polygon γ divides $\mathbb{C}$ into exactly two domains,*

$$\mathbb{C} \setminus \operatorname{supp}(\gamma) = I(\gamma) \cup E(\gamma),$$

and the winding number of γ is either 1 everywhere or -1 everywhere on the domain $I(\gamma)$, and 0 on $E(\gamma)$.

Lemma 2.1.9. *Let U be an open subset of $\mathbb{C}$ and K connected compact subset of U. Then there exists a closed chain $\tau = \tau_1 + \ldots \tau_N$ in $U \setminus K$, where each τ_i is a step polygon, such that $K \subset I(\tau) \subset U$.*

The previous lemma is telling us that we can 'go around' the compact set K, in U. The idea for the proof is simple and we go through some of the details here. Pick a $\delta > 0$ such that $d(K, \partial U) > \sqrt{2}\delta$, where $d(A, B) = \inf\{|a - b| \mid a \in A, b \in B\}$. We can now tile $\mathbb{C}$ with closed squares where each side has length δ and is parallel to either the real or the imaginary axis. Since K is a compact set, it intersects only finitely many of these squares. Let us denote them by $R^1, R^2, \ldots, R^k$, each of those squares intersects K so, by our choice of δ, they are all included in U, and since they cover K we have $K \subset \bigcup_{j=1}^{k} R^j \subset U$. Let $\sigma(R^j)$ be the sum of the line segments $R^j_r, R^j_u, R^j_l, R^j_d$ where they denote the right, upper, left and lower sides of the square R^j traversed counter-clockwise around R^j. By considering the sum $\gamma = \sum_{j=1}^{k} \sigma(R^j)$ we see that γ is a closed chain. Let c be any point in K, then c intersects some square R^l. If c is included in the interior of R^l then $I(\sigma(R^l), c) = 1$ and $I(\sigma(R^k), c) = 0$ for $k \neq l$, therefore $I(\gamma, c) = 1$.

If K intersects a line segment of $\sigma(R^j)$ for some j, then that line segment is a common side for another square R^n, and it appears in $\sigma(R^n)$ with opposite orientation. By removing sides of opposite orientation we obtain another chain $\tau = \tau_1 + \tau_2 + \cdots + \tau_N$ such that $\operatorname{supp}(\tau) \subset U \setminus K$, which is equivalent to γ, and such that each of the τ_i is a step polygon, see [11, Thm. 12.4.1].

Equivalence here means that

$$\int_\gamma f(z)dz = \int_\tau f(z)dz$$

for every continuous function defined on $\operatorname{supp}(\gamma) \cup \operatorname{supp}(\tau)$. This tells us that for a constant $c \in K$ included in the interior of R^l, like above, we have that $I(\tau, c) = I(\gamma, c) = 1$. By continuity of the winding number map on the set $\mathbb{C} \setminus \operatorname{supp}(\tau)$, which contains the connected set K, we get that the winding number $I(\tau, c)$ is defined for all $c \in K$ and is equal to 1. Therefore we have that $K \subset I(\tau)$ and since the interior of τ is contained in the union of all squares which intersect K we have that $K \subset I(\tau) \subset U$.

Lemmas 2.1.8 and 2.1.9 give us what R.Remmert calls the Circuit theorem

Theorem 2.1.10 (Circuit theorem)**.** *Let K be a connected compact set in an open domain U. Then there exists a step polygon γ in $U \setminus K$ such that $I(\gamma, p) = 1$ for all $p \in K$.*

Proof. Let τ be like in Lemma 2.1.9. Fix a $c \in K$, we have that

$$1 = I(\tau, c) = \sum_{j=1}^{N} I(\tau_j, c)$$

then by Lemma 2.1.8 therefore there exists an $1 \leq m \leq N$, such that $I(\tau_m, c) = 1$. Set $\gamma = \tau_m$, and γ is precisely a step polygon in $U \subset K$ such that $K \subset I(\gamma)$. □

By Lemma 2.1.8 every step polygon divides $\mathbb{C}$ into two domains, the interior and the exterior. Let $V = E(\gamma) \cup \{\infty\}$, and let us regard it as a subset of the Riemann sphere $\overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$. Since the complement of V, $\overline{\mathbb{C}} \setminus V = E(\gamma) \cup \operatorname{supp} \gamma$ is connected then V is simply connected. Furthermore if the interior is non-empty, then by the Riemann mapping theorem the set V can be mapped biholomorphically onto the open unit disk $\mathbb{E}$. Proof of the preceding statements can be found in [1, Thm. VI.5.2-4], as that book defines simply connected subsets of the Riemann sphere as subsets such that their complement is connected, and then proves that is equivalent to the topological definition. A natural question is, can we extend the map $V \to \mathbb{E}$ to the boundary?

The following definition introduces the notion of a simple boundary point, which by Theorem 2.1.12 is precisely the condition for the map in the previous paragraph to have an extension to the boundary.

Definition 2.1.11. [1, Def. VI.5.7] *Let U be a domain in $\mathbb{C}$. We say that $z \in \partial U$ is a* simple boundary point *of U if, for every sequence $(z_n) \in U$ such that $\lim_{n\to\infty} z_n = z$, there exists a path $\gamma\colon [0,1] \to \mathbb{C}$ and a strictly increasing sequence $(t_n) \in [0,1[$, such that:*

(i) $\lim_{n\to\infty} t_n = 1$.

(ii) $\gamma(t_n) = z_n$ *for all* n.

(iii) $\gamma(t) \in U$ *for all* $t \in [0,1[$.

Theorem 2.1.12. *Let U be a bounded simply connected domain with every boundary point simple and $f\colon U \to \mathbb{E}$ be a biholomorphic mapping. Then we can extend f to a homeomorphism $f\colon \overline{U} \to \overline{\mathbb{E}}$.*

A proof of 2.1.12 can be found in [1, Thm. VI.5.10]. Let γ be a step polygon, and consider the exterior $E(\gamma)$ of γ, we want to show that every point on the boundary of $E(\gamma)$ is a simple boundary point. We start with a lemma:

Lemma 2.1.13. *Let P be the convex set*

$$P = \{z \in \mathbb{C} \mid \operatorname{Im}(z) > 0, |z| < 1\}.$$

Then $0 \in \partial P$ is a simple boundary point of P.

Proof. Let (z_n) be a sequence in P such that $\lim z_n = 0$ as $n \to \infty$. We define a continuous function $\alpha \colon [0, \infty) \to P$ by mapping every interval $[n-1, n]$ to the straight line segment $[z_n, z_{n+1}]$, that is for every $t \in [n-1, n]$, $t \mapsto (n-t)z_n + (t-n+1)z_{n+1}$. It is clear that $\alpha(t) \in P$ for all $t \in [0, \infty[$ since P is convex, and $\lim_{t\to\infty} \alpha(t) = 0$. Then the function

$$\tau \colon [0,1] \to \mathbb{C} \quad \tau(t) = \begin{cases} \alpha\left(\frac{t}{1-t}\right) & t \in [0,1), \\ 0 & t = 1, \end{cases}$$

satisfies the conditions in Definition 2.1.11. □

Lemma 2.1.14. *Let γ be a step polygon with $I(\gamma)$ not empty. Then both $E(\gamma)$ and $I(\gamma)$ are domains with every boundary point simple.*

Proof. Let x be a point on $\partial E(\gamma) = \operatorname{supp}(\gamma)$ then x lies on some segment in γ. Assume that x is an endpoint of a segment, then there are segments $[p, x]$ and $[x, q]$ in γ for some points $p, q \in \mathbb{C}$, such that p, q and x are all distinct. Then there exists a $\delta > 0$ small enough that the open disk $B(x, \delta)$ around x intersects $\operatorname{supp}(\gamma)$ only on the segments $[p, x]$ and $[x, q]$, and $p, q \notin B(x, \delta)$. The segments $[p, x]$ and $[x, q]$ are both straight and meet in the centre x of $B(x, \delta)$ and both endpoints p and q are outside the open disk $B(x, \delta)$, the open set $B(x, \delta) \setminus \operatorname{supp}(\gamma)$ divides into exactly two components U and V. Since

$$U \cup V = (\mathbb{C} \setminus \operatorname{supp}(\gamma)) \cap B(x, \delta) = (E(\gamma) \cup I(\gamma)) \cap B(x, \delta),$$

then one of the components U and V is contained in the interior of γ and the other in exterior. Assume that U is a subset of the exterior. By rotating and translating U, such that x goes to 0, we can see that U is one of the three following sets

$$P_k = \left\{ re^{i\theta} \mid 0 < r < \delta, 0 < \theta < \frac{k\pi}{2} \right\}, \quad k = 1, 2, 3.$$

There exists a well defined angle on P_3, i.e. a continuous map $\phi \colon P_3 \to]0, \frac{3\pi}{2}[$, so the maps $\psi_k(z) \colon P_k \to P$, $\psi_k(z) = \frac{|z|}{\delta} e^{2i\phi(z)/k}$, where P is the set in Lemma 2.1.13, are homeomorphisms. Furthermore for any sequence (z_n) in P_k such that $\lim z_n = 0$, then $\lim_{n\to\infty} \psi_k(z_n) = 0$.

Let (z_n) be any sequence in $E(\gamma)$ such that $z_n \to x$, then the sequence is eventually in U. We therefore get a corresponding sequence $(\psi_k(z_n))$ in P which tends to 0, but by Lemma 2.1.13, the point 0 is a simple boundary point of P. Then there exists a path τ in P, and a strictly increasing sequence $(t_n) \in [0,1[$ such that $\lim t_n = 1$, $\tau(t_n) = \psi_k(z_n)$ for all n large enough, and $\tau(t) \in P$ for all $[0,1[$.

Therefore, $(\psi_k)^{-1} \circ \tau$ is a path in $\mathbb{C}$ such that $(\psi_k)^{-1} \circ \tau([0,1[) \subset E(\gamma)$, and $(\psi_k)^{-1} \circ \tau(t_n) = z_n$ for all n large enough. Since there exists a constant $N > 0$ such that $z_n \in B_1$ for all $n > N$, and since $E(\gamma)$ is path connected, we can find a path $\tau_1\colon [0, t_{N+1}] \to E(\gamma)$ such that $\tau_1(t_n) = z_n$ for all $n \leq N+1$. Then

$$\alpha(t) = \begin{cases} \tau_1(t) & t \in [0, t_{N+1}] \\ (\psi_k)^{-1} \circ \tau(t) & t \in [t_{N+1}, 1] \end{cases}$$

is a path satisfying the conditions in Definition 2.1.11. Therefore x is a simple boundary point of $E(\gamma)$.

If x is not an endpoint of a segment, then x is on some $[p,q]$ in γ. Let $\delta > 0$ be small enough such that the open disk $B(x,\delta)$ intersects $\operatorname{supp}(\gamma)$ only on the segment $[p,q]$ in γ and $p, q \notin B(x,\delta)$. In the same way as before $B(x,\delta) \setminus \operatorname{supp}(\gamma)$ divides into two components U and V, and we can assume $U = E(\gamma) \cap B(x,\delta)$. By rotating and translating U we can see that U is the set P_2 defined above. Using the same arguments as when x is an endpoint of a segment we see that x is a simple boundary point of $E(\gamma)$.

The argument above works in the same way if we use the component of $B(x,\delta) \setminus \operatorname{supp}(\gamma)$ that lies on the interior of γ instead. □

Now we are ready to show the following lemma.

Lemma 2.1.15. *Let γ be a step polygon, then the domains $I(\gamma)$ and $\tilde{V} = E(\gamma) \cup \{\infty\}$ are both biholomorphic to $\mathbb{E}$, furthermore those biholomorphisms can be extended to homeomorphism of $I(\gamma) \cup \operatorname{supp}(\gamma)$ and $E(\gamma) \cup \operatorname{supp}(\gamma) \cup \{\infty\}$ onto the closed unit disk $\overline{\mathbb{E}}$.*

Proof. Since $I(\gamma)$ is simply connected and bounded, there exists a biholomorphism $F\colon I(\gamma) \to \mathbb{E}$, by the Riemann mapping theorem. By Lemma 2.1.14, every boundary point of $I(\gamma)$ is simple so F extends to a homeomorphism $\overline{F}\colon I(\gamma) \cup \operatorname{supp}(\gamma) \to \overline{\mathbb{E}}$.

By translation we can assume that $0 \in I(\gamma)$. Then there exists a constant $r > 0$ such that the closed disk $\overline{B(0,r)}$ is contained in $I(\gamma)$. The map $f(z) = \frac{1}{z}$ maps the domain $E(\gamma) \cup \{\infty\}$ biholomorphically onto a set W contained in $B(0, \frac{1}{r})$. By the remark preceding Definition 2.1.11 the set $E(\gamma) \cup \{\infty\}$ is simply connected so W is as well. The boundary of $E(\gamma) \cup \{\infty\}$ is $\operatorname{supp}(\gamma)$, and every boundary point is a simple boundary point of $E(\gamma) \cup \{\infty\}$, by Lemma 2.1.14. We therefore get a mapping of $\operatorname{supp}(\gamma)$ onto the boundary of W. Now for any point $z \in \partial W$, and any sequence $(z_n) \in W \setminus \{0\}$ with $\lim z_n = z$, we get a corresponding sequence $z'_n = f(z_n)$ in $E(\gamma) \cup \{\infty\}$ such that $\lim z'_n = z'$ with $z' = f(z) \in \operatorname{supp}(\gamma)$, and since every boundary point of $E(\gamma) \cup \{\infty\}$ is a simple boundary point we get a map $\beta\colon [0,1] \to \mathbb{C}$, with $\beta[0,1[\subset E(\gamma) \cup \{\infty\}$ that satisfies the conditions in Definition 2.1.11. But then the $f \circ \beta\colon [0,1] \to \mathbb{C}$ satisfies the conditions of Definition 2.1.11 for z being a simple boundary point of W.

Now W is a simply connected bounded domain of $\mathbb{C}$ so there exists a biholomorphism $G\colon W \to \mathbb{E}$ by the Riemann mapping theorem. Since every boundary point of W is a simple boundary point, the map G extends to a homeomorphism $\overline{G}\colon \overline{W} \to \overline{\mathbb{E}}$. But then

$$\overline{G} \circ f\colon E(\gamma) \cup \operatorname{supp}(\gamma) \cup \{\infty\} \to \overline{\mathbb{E}}$$

is a homeomorphism that extends the biholomorphic map

$$G \circ f\colon E(\gamma) \cup \{\infty\} \to \mathbb{E}.$$

□

The next lemma gives us the main result of this section.

Lemma 2.1.16. *Let U be a domain in $\mathbb{C}$, W be a simply connected domain in $\mathbb{C}$, $a \in U \cap W$ and $\gamma\colon [0,1] \to \mathbb{C}$ be a closed path in $U \cap W$ at a. Let h be a homotopy from γ to the constant path at a, such that $h(t,1) = \gamma(t)$, $h(t,0) = a$, $h(1,s) = h(0,s) = a$ and $h(t,s) \in U$ for all t, s. Then there exists a homotopy k, from γ to the constant path at a, fulfilling the same conditions as h, such that $k(t,s) \in U \cap W$ for all $t, s \in [0,1]$*

Proof. The image of the closed loop $|\gamma|$ is a compact set contained in $U \cap W$. By Lemma 2.1.10 there exists a step polygon τ in $(U \cap W) \setminus |\gamma|$, such that $\operatorname{supp}(\tau) \subset U \cap W$, and $|\gamma| \subset I(\tau)$. Since W is simply connected $I(\tau) \subset W$, see [11, Thm. 8.2.6]. By Lemma 2.1.8 the set $\mathbb{C} \setminus \operatorname{supp}(\tau)$ is divided up into two components $I(\tau)$ and $E(\tau)$ neither of which is empty, and they share a common boundary, $\partial I(\tau) = \partial E(\tau) = \operatorname{supp}(\tau)$. Denote by X the set $E(\tau) \cup \operatorname{supp}(\tau)$, then $\partial X = \operatorname{supp}(\tau)$, and by Lemma 2.1.15 we get a homeomorphism $f\colon X \to \overline{\mathbb{E}} \setminus \{0\}$.

The map $r\colon \overline{\mathbb{E}} \setminus \{0\} \to \partial\overline{\mathbb{E}}$, $r(z) = \frac{z}{|z|}$, satisfies $r|_{\partial\overline{\mathbb{E}}} = \mathrm{id}_{\partial\overline{\mathbb{E}}}$, so it is a retraction of $\overline{\mathbb{E}} \setminus \{0\}$ to $\partial\overline{\mathbb{E}}$.

Then the map $f^{-1} \circ r \circ f\colon X \to \partial X$ is also a retraction. Since $\partial X = \mathrm{supp}(\tau)$ we get a well defined continuous map $F\colon \mathbb{C} \to \overline{I(\tau)}$

$$F(z) = \begin{cases} z & z \in \overline{I(\tau)} = \mathrm{supp}(\tau) \cup I(\tau), \\ (f^{-1} \circ r \circ f)(z) & z \in X = \mathrm{supp}(\tau) \cup E(\tau). \end{cases}$$

Consequently, the map $k = F \circ h$ is a homotopy between γ and the constant path at a such that $k(t, s) \in U \cap W$ for all $(t, s) \in I \times I$. Note that $k(t, s) = h(t, s)$ for any (t, s) such that $h(t, s) \in \overline{I(\tau)} \cap W \cap U$. □

As a direct consequence we get the following theorem:

Theorem 2.1.17. *Let U, W and a be as in Lemma 2.1.16. Then the map*

$$\pi_1(U \cap W, a) \to \pi_1(U, a)$$

induced by the inclusion $U \cap W \hookrightarrow U$ is injective.

Proof. Pick an element $[\gamma] \in \pi_1(U \cap W, a)$ that maps to the identity element in $\pi_1(U, a)$, represented by some closed loop γ in $U \cap W$ at a. Since $[\gamma]$ maps to the identity element in $\pi_1(U, a)$, then γ is homotopic to the constant path at a in U. But by the previous Lemma 2.1.16 then there exists a homotopy to the constant path a in the set $U \cap W$, so $[\gamma]$ is the identity element. Therefore the map is injective. □

The intuitive idea behind the previous statement is that for U a subset of the plane, we can not introduce more 'holes' by intersecting with a simply connected domain W. It is worth pointing out that this is a special property of the subsets of the plane as Theorem 2.1.17 is not true for arbitrary surfaces. A simple example is the following, let U be the open subset $\overline{\mathbb{C}} \setminus \{0\}$ of the Riemann sphere, and let W be an open disk around 0 which does not include ∞. Then U is simply connected, but $U \cap W$ is not, so the map in Theorem 2.1.17 is not injective.

Theorem 2.1.18. *Let U be a domain in $\mathbb{C}$ and $p\colon Y \to U$ be a universal cover for U. Pick a base point $a \in U$ and a point $y \in Y$ such that $p(y) = a$. Fix some $r > 0$ and denote by U' the connected component of $U \cap B(a, r)$ which contains a. Furthermore, let Y' be the connected component of $p^{-1}[U']$ which contains y. Then Y' is a universal covering of U', and $p'(y) = a$, where p' is the restriction of p to the set Y'.*

Proof. Consider the following commutative diagram, where the horizontal arrows are inclusions.

$$\begin{array}{ccc} (Y', y) & \longrightarrow & (Y, y) \\ \downarrow{\scriptstyle p'} & & \downarrow{\scriptstyle p} \\ (U', a) & \longrightarrow & (U, a) \end{array} \tag{2.1}$$

If we apply π_1, we arrive at another commutative diagram.

$$\begin{array}{ccc} \pi_1(Y', y) & \longrightarrow & \pi_1(Y, y) \\ \downarrow{\scriptstyle \pi_1(p')} & & \downarrow{\scriptstyle \pi_1(p)} \\ \pi_1(U', a) & \longrightarrow & \pi_1(U, a) \end{array} \tag{2.2}$$

The spaces Y and Y' are covering spaces, therefore the maps $\pi_1(p')$ and $\pi_1(p)$ are injective. This can be seen by considering an element $[\gamma] \in \pi_1(Y', y)$ that $\pi_1(p')$ maps to the identity element. Then there exists a homotopy of the closed loop $p'(\gamma)$ at a to the constant path at a. This homotopy can be lifted, by the lifting property of covering spaces, to a homotopy of the closed loop γ at y in Y' to the constant path at y.

Theorem 2.1.17 tells us that the map $\pi_1(U', a) \to \pi_1(U, a)$ is injective, so the map $\pi_1(Y', y) \to \pi_1(U', a) \to \pi_1(U, a)$ is also injective. The space Y is simply connected, so $\pi_1(Y, y)$ is trivial. Since the diagram commutes, then $\pi_1(Y', y)$ must also be trivial. Therefore Y' is simply connected so it is a universal covering space for U'. □

2.2. Riemann Domains

Here we present the definition of a Riemann domain. We are mostly interested in the case where a Riemann domain is a covering space, which have the important property of being second countable, and we will use them to capture the multivalued functions presented in Radó's original proof [10]. We can prove that these covering Riemann domains are second countable without appealing to a famous theorem of

Radó, see [4, Thm. 23.3], and, being second countable, allows us to get a special case of Montel's theorem.

Definition 2.2.1. (i) *A Riemann domain over $\mathbb{C}$ is a connected Hausdorff space X with a local homeomorphism $p\colon X \to \mathbb{C}$.*

(ii) *Let $p\colon X \to \mathbb{C}$ be a Riemann domain over $\mathbb{C}$. A function $f\colon X \to \mathbb{C}$ is said to be holomorphic if $f \circ \sigma$ is holomorphic for every section σ of p.*

If U is an open domain in $\mathbb{C}$ and $p\colon U \to \mathbb{C}$ is the inclusion $p(z) = z$, then $p\colon U \to \mathbb{C}$ is a Riemann domain and the holomorphic functions on the Riemann domain $p\colon U \to \mathbb{C}$ are precisely the usual holomorphic functions on the open domain U in $\mathbb{C}$.

Additionally, let $p\colon X \to \mathbb{C}$ be a Riemann domain, and let G be a subset of $\mathbb{C}$ such that $p(X) \subset G$, then by shrinking the codomain of the function p we can talk about a Riemann domain $p\colon X \to G$ over G.

Definition 2.2.2. *A* morphism *$f\colon X \to Y$ between two Riemann domains, $p\colon X \to \mathbb{C}$ and $q\colon Y \to \mathbb{C}$, is a continuous map such that $p = q \circ f$.*

Lemma 2.2.3. *Let $p\colon X \to \mathbb{C}$ and $q\colon Y \to \mathbb{C}$ be two Riemann domains over $\mathbb{C}$. Let $f\colon X \to Y$ be a morphism and $\sigma\colon U \to X$ be a section of p where U is open set. Then the map $\sigma(U) \to f(\sigma(U))$, $z \mapsto f(z)$, is a homeomorphism.*

Proof. By Lemma 2.1.3 the map $f \circ \sigma$ is a section of q. But, by Lemma 2.1.2 (ii), the map $U \to f(\sigma(U))$, $z \mapsto f(\sigma(z))$ is a homeomorphism. Consequently the map $\sigma(U) \to f(\sigma(U))$, $z \mapsto ((f \circ \sigma) \circ p)(z) = f(z)$ is a homeomorphism. □

Definition 2.2.4. *Let $p\colon X \to \mathbb{C}$ be a Riemann domain. We say that a holomorphic function $f\colon X \to \mathbb{C}$ is* injective with respect to p *(or* over p*) if for any $x, y \in X$ such that $f(x) = f(y)$ then $p(x) = p(y)$.*

The condition that a holomorphic function is injective with respect to p is precisely what we need to define a local homeomorphism $q\colon f(X) \to \mathbb{C}$ such that $f\colon X \to f(X)$ is a morphism of Riemann domains.

Lemma 2.2.5. *Let $p\colon X \to \mathbb{C}$ be a Riemann domain and $f\colon X \to \mathbb{C}$ be holomorphic. The map f is injective with respect to p if and only if there is a continuous mapping*

$q\colon f(X)\to\mathbb{C}$ with $q\circ f=p$. Then q is uniquely defined, $f(X)$ is open in $\mathbb{C}$, q is a holomorphic function, and $q\colon f(X)\to\mathbb{C}$ is a Riemann domain.

Proof. Assume there exists a $q\colon f(X)\to\mathbb{C}$ such that $q\circ f=p$, and let $x,y\in X$ such that $f(x)=f(y)$. Then $p(x)=q(f(x))=q(f(y))=p(y)$ and f is injective with respect to p. Now assume that f is injective with respect to p and let y be a point in $f[X]$. Then p is constant on $f^{-1}[y]$, therefore $q\colon f(X)\to\mathbb{C}$, $y\mapsto p(x)$ where p is any point in $f^{-1}[y]$, is a well defined map such that $q\circ f=p$, and furthermore is unique.

To prove the rest, let $y\in f(X)$. We want to show that there exists a neighbourhood Y around y such that $q|_Y$ is a homeomorphism. Pick a point $x\in f^{-1}[y]$ and let $\sigma\colon U\to X$ be a section of p where U is a neighbourhood of $p(x)$ and $\sigma(p(x))=x$. By definition the map $f\circ\sigma\colon U\to f(X)$ is holomorphic and $(f\circ\sigma)(p(x))=y$. Let $z_1,z_2\in U$ such that $f\circ\sigma(z_1)=f\circ\sigma(z_2)$, since f is injective with respect to p, we have that

$$z_1=p(\sigma(z_1))=p(\sigma(z_2))=z_2$$

so the holomorphic map $f\circ\sigma$ is injective. An injective holomorphic map is a homeomorphism onto its image, so $f\left[\sigma(U)\right]$ is a neighbourhood of y homeomorphic to U, and since $q\circ f=p$ we have that $q\circ(f\circ\sigma)=\mathrm{id}_U$, and the restriction $q\colon f\left[\sigma(U)\right]\to U$ is a biholomorphism. Therefore q is a locally biholomorphic map and $q\colon f(X)\to\mathbb{C}$ is a Riemann domain. □

Lemma 2.2.6. *Let $p\colon X\to\mathbb{C}$ be a Riemann domain over $\mathbb{C}$ and $f\colon X\to\mathbb{C}$ be holomorphic and injective over p. Let $a,b\in X$ be such that $f(a)=f(b)$, $U\subset\mathbb{C}$ be a connected neighbourhood of $w\colon=p(a)=p(b)$, and $\sigma_a,\sigma_b\colon U\to X$ be sections of p such that $\sigma_a(w)=a$ and $\sigma_b(w)=b$. Then $f\circ\sigma_a=f\circ\sigma_b$.*

Proof. By Lemma 2.2.5 then $q\colon f(X)\to\mathbb{C}$ is a Riemann domain and $f\colon X\to f(X)$ is a morphism of Riemann domains. Then $f\circ\sigma_a$ and $f\circ\sigma_b$ are two sections of q which agree at a point so by Lemma 2.1.2 they agree on the whole of U. □

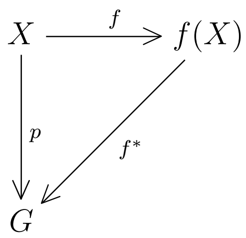
We can use Lemma 2.2.6 to define derivatives of holomorphic functions on Riemann domains. Let $p\colon X\to G$ be a Riemann domain and $f\colon X\to\mathbb{C}$ be a holomorphic function which is injective with respect to p. Pick points $g_0\in G$ and $x_0\in X$ such that $p(x_0)=g_0$. If σ_1 and σ_2 are two sections of p on a neighbourhood U of g_0, such that $\sigma_1(g_0)=x_0=\sigma_2(g_0)$, then by Lemma 2.2.6 the functions $(f\circ\sigma_1)\colon U\to\mathbb{C}$ and $(f\circ\sigma_2)\colon U\to\mathbb{C}$ are equal. Therefore the ordinary derivatives $(f\circ\sigma_1)'(g_0)$ and $(f\circ\sigma_2)'(g_0)$ are equal and independent of a choice of a section. From the preceding discussion we get the following definition:

Definition 2.2.7. *Let $p\colon X \to G$ be a Riemann domain, $f\colon X \to \mathbb{C}$ be holomorphic and injective with respect to p, $g_0 \in G$ and $x_0 \in X$. Then the* derivative *of f at x_0, is defined by*

$$f'(x_0) = (f \circ \sigma)'(g_0)$$

where σ is a section of p on some neighbourhood U of g_0 such that $\sigma(g_0) = x_0$.

Lemma 2.2.5 tells us that for a Riemann domain $p\colon X \to G$, with G open subset of $\mathbb{C}$, any holomorphic map $f\colon X \to \mathbb{C}$ which is injective with respect to p induces a local biholomorphism $q\colon f(X) \to \mathbb{C}$ such that $f(X)$ with this map is a Riemann domain. We will denote this map by f^*, and this is the unique local biholomorphism induced by f such that the following diagram commutes.



Insisting that the map $p\colon X \to G$ is a covering map, instead of being just a local homeomorphism, we have by Lemma 2.1.4 that f and f^* are covering maps as well. It is well known that $\pi_1(G, a)$ is countable ([7, Thm. 8.11]), if G is an open domain of $\mathbb{C}$. Fix an $x_0 \in X$ such that $p(x_0) = a$. For any $x \in p^{-1}[a]$, there exists a path $\gamma\colon [0,1] \to X$ with $\gamma(0) = x_0$ and $\gamma(1) = x$. Then $[p \circ \gamma] \in \pi_1(G, a)$, and we get a surjective map $\pi_1(G, a) \to p^{-1}[a]$, $[\beta] \mapsto \overline{\beta}(1)$ where $\overline{\beta}$ is the lifting of β with $\overline{\beta}(0) = x_0$. But then the fibers $p^{-1}[g]$, $g \in G$ are countable, since $\pi_1(G, g)$ is countable, and therefore X is second countable. We then get the following theorem:

Theorem 2.2.8 (Montel's theorem for Riemann domains)**.** *If (f_n) is a sequence of holomorphic functions on a Riemann domain $p\colon X \to U$, with p a covering map, and (f_n) is uniformly bounded on each compact subset of X, then there exists a subsequence of (f_n) which converges uniformly on compact subsets of X to a holomorphic function on X.*

This is a special case of more general theorem of Montel for arbitrary Riemann surfaces [9], but the general case relies on Radó's theorem to show that arbitrary Riemann surfaces are second countable.

For the sake of completeness we present here two results from classical complex analysis that will be used in later sections.

Lemma 2.2.9 (Schwarz's lemma). *Let $f\colon \mathbb{E} \to \mathbb{C}$ be a holomorphic map such that $f(0) = 0$ and $|f(z)| \leq 1$. Then $|f(z)| \leq |z|$ for all $z \in \mathbb{E}$ and $|f'(0)| \leq 1$. Moreover if $|f'(0)| = 1$ or if $|f(z)| = |z|$ for some $z \neq 0$, then there exists a constant $a \in \mathbb{C}$ with $|a| = 1$ such that $f(z) = az$.*

Theorem 2.2.10 (Hurwitz's theorem). *Let (f_k) be a sequence of holomorphic functions on a domain $U \subset \mathbb{C}$ that converges uniformly on compact subsets of U to a holomorphic function f on U that is not identically zero. If f has a zero of order m at z_0, then for every small enough $\rho > 0$ there exists a natural number $N \in \mathbb{N}$ such that f_n has exactly m zeroes, counted with multiplicity, in the disk defined by $|z - z_0| < \rho$, for all $n > N$. Furthermore these zeroes converge to z_0 as $n \to \infty$.*

For proofs of Lemma 2.2.9 and Theorem 2.2.10 see [2, Thm. VI.2.1] and [2, Thm. VII.2.5].

3. Uniformization Theorem for Domains in $\mathbb{C}$

In this chapter we present the main results of this thesis, proving the following theorem:

Theorem 3.0.1 (Uniformization theorem for domains in $\mathbb{C}$)**.** *Let G be a domain in $\mathbb{C}$, then there exists a holomorphic covering map $p : X \to G$, where X is either $\mathbb{C}$ or the unit disk $\mathbb{E}$.*

Note that if $G = \mathbb{C}$ or $G = \mathbb{C}^* := \mathbb{C} \setminus \{0\}$, then G can be covered by $\mathrm{id} : \mathbb{C} \to \mathbb{C}$ or by $\exp : \mathbb{C} \to \mathbb{C}^*$, respectively. We only have to consider the case when the complement of G contains at least two points. That is to say, we prove the following:

Theorem 3.0.2. *Let G be any domain in $\mathbb{C} \setminus \{0, 1\}$, and $p \colon Y \to G$ be a universal covering space for G. Then Y is isomorphic to the open unit disk $\mathbb{E}$.*

The preceding theorems give us a classification of domains, the plane $\mathbb{C}$ is a universal covering space for itself or the punctured plane $\mathbb{C} \setminus \{z\}$, for any $z \in \mathbb{C}$. All other domains in $\mathbb{C}$ have the unit disk $\mathbb{E}$ as a universal cover.

The strategy used to prove 3.0.2 is similar to the method presented by Radó in [10], shown in section 1.2, to prove the Riemann mapping theorem. The natural way to extend this method for a general domain G is to consider a family of functions $f_n : G \to \mathbb{E}$ and construct a surjective function $G \to \mathbb{E}$ by solving an extremal problem as in the proof for the Riemann mapping theorem. Since we are trying to construct a covering map $p : \mathbb{E} \to G$, we will have to deal with some multivalued functions. This is precisely what Radó does in his paper [10].

Here we look at a universal covering space $p : Y \to G$, and work with maps $f_n : Y \to \mathbb{E}$, injective with respect to p, and use the same kind of extremal argument as Radó does in his paper, to construct a surjective map $f : Y \to \mathbb{E}$, which is injective with respect to p. In light of Lemma 2.2.5 and Lemma 2.1.4, the map

$f : Y \to \mathbb{E}$ is a covering map, but Y and $\mathbb{E}$ are both simply connected so they are biholomorphic, that is to say, G can be holomorphically covered by $\mathbb{E}$.

3.1. The Set $\Sigma(G, y_0)$

As we said above, we are only considering the case when the complement of G contains at least two points. Without loss of generality we can assume that $G \subset \mathbb{C} \setminus \{0, 1\}$.

Let $p\colon Y \to G$ be a universal covering space of G, $g_0 \in G$ and $y_0 \in Y$ such that $p(y_0) = g_0$. The space Y is then a Riemann domain. We are interested in the space of holomorphic functions on the Riemann domain Y, which are injective with respect to p.

Definition 3.1.1. *Let G be a domain of $\mathbb{C}\setminus\{0, 1\}$, $p\colon Y \to G$ be a universal covering space of G, and pick points $g_0 \in G$ and y_0 in Y such that $p(y_0) = g_0$. We denote by $\Sigma(G, y_0)$ the set of holomorphic functions $f\colon Y \to \mathbb{E}$ injective with respect to p such that $f(y_0) = 0$, and such that the derivative $f'(y_0) = (f \circ \sigma)'(g_0)$, see Definition 2.2.7, is a real number in the interval $]0, \infty[$, where σ is some section p with $\sigma(g_0) = y_0$. Furthermore we define $\lambda(G, y_0)$ as*

$$\lambda(G, y_0) = \sup \{f'(y_0) \mid f \in \Sigma(G, y_0)\}.$$

We now have to show that the set $\Sigma(G, y_0)$ is not empty. Fortunately, when G is bounded it is easy to find a function $f \in \Sigma(G, y_0)$.

Lemma 3.1.2. *If G is bounded then $\Sigma(G, y_0)$ is non-empty.*

Proof. Let C be a large enough positive real number such that $f(G) \subset \mathbb{E}$, where $f(z) = \frac{z-g_0}{C}$. Then $f'(g_0) > 0$, and if $p\colon Y \to G$ is a universal cover such that $p(y_0) = g_0$, we see that the function $f \circ p\colon Y \to \mathbb{E}$ is in $\Sigma(G, y_0)$. □

Lemma 3.1.3. *If $\Sigma(G, y_0)$ is not empty then $\lambda(G, y_0)$ is finite.*

Proof. Since p is a local homeomorphism there exists a section $\sigma\colon U \to Y$, with U open disk of radius ρ centered at g_0, such that $\sigma(g_0) = y_0$. Now for any function $f \in \Sigma(G, y_0)$, we have that $h = f \circ \sigma$ is a holomorphic function from U to $\mathbb{E}$, and $h'(g_0) > 0$. Schwarz's lemma gives us that $h'(g_0) \leq \frac{1}{\rho}$, and since f was arbitrary, it follows that $\lambda(G, y_0) \leq \frac{1}{\rho}$. □

Definition 3.1.4. *For any point $w \in \mathbb{E}$, $w \neq 0$, we define the map $h_w \colon \mathbb{E} \to \mathbb{E}$ by*

$$h_w(z) = \frac{z - w}{z\overline{w} - 1}.$$

The map h_w is the unique automorphism of $\mathbb{E}$ such that $0 \mapsto w$ and $w \mapsto 0$. This can be seen by letting g be any automorphism of $\mathbb{E}$ such that $g(0) = w$ and $g(w) = 0$. Then the map $\psi = g^{-1} \circ h_w$ is an automorphism of $\mathbb{E}$ and $\psi(0) = 0$ and $\psi(w) = w$. By the Schwarz lemma we have that $\psi \colon \mathbb{E} \to \mathbb{E}$ is the identity map, and therefore $g = h_w$.

Taking the derivative of the map h_w we get

$$h_w'(z) = \frac{(z\overline{w} - 1) - \overline{w}(z - w)}{(z\overline{w} - 1)^2} = \frac{|w|^2 - 1}{(z\overline{w} - 1)^2},$$

so $h_w'(0) = |w|^2 - 1$ and $h_w'(w) = \frac{1}{|w|^2 - 1}$.

The next lemma shows that if we can find a map $f \in \Sigma(G, y_0)$ with $f'(y_0) = \lambda(G, y_0)$, then it must be surjective onto $\mathbb{E}$. In light of Lemma 2.1.4 then a surjective morphism $f \colon Y \to \mathbb{E}$ is a covering map. The method used in the proof is what Radó refers to as the *Carathéodory-Koebe square root transformation.*

Lemma 3.1.5. *If $f \in \Sigma(G, y_0)$ is not surjective onto $\mathbb{E}$, then there exists a mapping $F \in \Sigma(G, y_0)$ such that $F'(g_0) > f'(g_0)$.*

Proof. Let $w \in \mathbb{E}$ be a value that f does not take, and write

$$w = \beta e^{i\phi}, \quad 0 < \beta < 1.$$

Let $h_w \colon \mathbb{E} \to \mathbb{E}$ be the automorphism in Definition 3.1.4, by composing h_w and f we get a map $h_w \circ f \colon Y \to \mathbb{E} \setminus \{0\}$. Since Y is simply connected, there exists a function $r \colon Y \to \mathbb{E} \setminus \{0\}$ such that $r^2 = h_w \circ f$ and $r(y_0) = w_1$, where $w_1 = \sqrt{\beta} e^{i\frac{1}{2}\phi}$.

Define a function $F \colon Y \to \mathbb{E}$ by

$$F(y) = e^{i\frac{1}{2}\phi} \left[h_{w_1}(r(y)) \right].$$

To show that F is injective with respect to p assume that there exist $c_1, c_2 \in Y$ such that $F(c_1) = F(c_2)$. Then $h_{w_1}(r(c_1)) = h_{w_1}(r(c_2))$ and since h_{w_1} is bijective, $r(c_1) = r(c_2)$. By definition of r,

$$h_w(f(c_1)) = (r(c_1))^2 = (r(c_2))^2 = h_w(f(c_2))$$

and since h_w is bijective, $f(c_1) = f(c_2)$. But f is injective with respect to p so $p(c_1) = p(c_2)$, and therefore F is injective with respect to p.

We see that $F(y_0) = 0$, and by the chain rule

$$F'(y_0) = e^{i\frac{1}{2}\phi} h'_{w_1}(r(y_0))r'(y_0) = e^{i\frac{1}{2}\phi} h'_{w_1}(w_1)r'(y_0) = e^{i\frac{1}{2}\phi} \frac{r'(y_0)}{|w_1|^2 - 1}.$$

To evaluate $r'(y_0)$ we note that

$$2r(y_0)r'(y_0) = h'_w(f(y_0))f'(y_0) = h'_w(0)f'(y_0)$$

so

$$r'(y_0) = \frac{(|w|^2 - 1)f'(y_0)}{2w_1}.$$

We gather all the terms and we have

$$\begin{aligned} F'(y_0) &= e^{i\frac{1}{2}\phi} \frac{1}{2} \frac{|w|^2 - 1}{|w_1|^2 - 1} \frac{1}{w_1} f'(y_0) \\ &= e^{i\frac{1}{2}\phi} \frac{1}{2} \frac{\beta^2 - 1}{\beta - 1} \frac{1}{\sqrt{\beta} e^{i\frac{1}{2}\phi}} f'(y_0) \\ &= \frac{\beta + 1}{2\sqrt{\beta}} f'(y_0) \end{aligned}$$

Then $F'(y_0)$ is a real number and greater then 0, so $F \in \Sigma(G, y_0)$. Since $0 < \beta < 1$, it follows that $\frac{\beta+1}{2\sqrt{\beta}} > 1$, which implies $F'(y_0) > f'(y_0)$. □

As said before the previous Lemma 3.1.5 shows that if $f \in \Sigma(G, y_0)$ and $f'(y_0) = \lambda(G, y_0)$, then $f\colon Y \to \mathbb{E}$ is a surjective function. Since f induces a locally biholomorphic map $f^*\colon \mathbb{E} \to G$, such that $p = f^* \circ f$, Lemma 2.1.4 tells us that both f and f^* are covering maps.

Lemma 3.1.6. *If $f \in \Sigma(G, y_0)$ and $f'(y_0) > \frac{4}{5}\lambda(G, y_0)$, then $\frac{1}{4}\mathbb{E} \subset f(Y)$.*

Proof. If f is a surjective function onto $\mathbb{E}$ there is nothing to prove. Assume f is not surjective and let $w = \beta e^{i\phi}$ be a value which f does not take. Define F as in the proof for Lemma 3.1.5. We get the following inequalities

$$\lambda(G, y_0) \geq F'(y_0) = \frac{\beta + 1}{2\sqrt{\beta}} f'(y_0) > \frac{\beta + 1}{2\sqrt{\beta}} \frac{4}{5} \lambda(G, y_0)$$

so that

$$\frac{\beta + 1}{2\sqrt{\beta}} < \frac{5}{4}.$$

Then $\beta > \frac{1}{4}$ and $w \notin \frac{1}{4}\mathbb{E}$. □

Let f be as in Lemma 3.1.6 and fix $\rho > 0$ so $U = B(g_0, \rho)$ is evenly covered by p. Then $f^* \colon f(Y) \to G$ is a function such that for any section $\sigma \colon U \to Y$ of p, we have the equality $f^* \circ (f \circ \sigma) = \mathrm{id}_U$. Therefore $(f^*)'(0)$ is a real number and $(f^*)'(0) > 0$.

3.2. Constructing a Surjective Function

In this section we are going to show that there exists a function $f \in \Sigma(G, y_0)$ with $f'(y_0) = \lambda(G, y_0)$ for any domain G in $\mathbb{C} \setminus \{0, 1\}$, and the condition on the derivative shows that it is a surjective function.

Let G_n be the connected component of $G \cap B(g_0, n)$ which contains g_0, for $n = 1, 2, \ldots$. Let g be any point in G, since G is a domain there exists a path $\gamma \colon [0, 1] \to G$ such that $\gamma(0) = g_0$ and $\gamma(1) = g$. The image of γ is a compact set $|\gamma|$ in $\mathbb{C}$ so there exists an $N > 0$ such that $|\gamma| \subset B(g_0, N)$. Then g is in G_N, and since g was arbitrary we see that

$$G = \bigcup_1^\infty G_n.$$

Let $p \colon Y \to G$ be a universal cover of G and let $y_0 \in Y$ be such that $p(y_0) = g_0$. Denote by Y_n the connected component of $p^{-1}[G_n]$ that contains y_0. We have by Lemma 2.1.18 that Y_n is a universal cover of G_n and

$$Y_1 \subset Y_2 \subset \ldots \subset Y.$$

Using a similar argument as before we can show that the union of all the Y_n is Y. Let $y \in Y$ be an arbitrary point. Then there exists a path $\gamma \colon [0, 1] \to Y$ such that $\gamma(0) = y_0$ and $\gamma(1) = y$. Then $p \circ \gamma$ is a path in G from g_0 to $p(y)$, and the image of $p \circ \gamma$ is compact so there exists an $N > 0$ such that $|p \circ \gamma| \subset G_N$. But since Y_N is a universal covering space of G_N, there exists a lifting $\tilde{\gamma}$ of $p \circ \gamma$ to Y_N such that $\tilde{\gamma}(0) = y_0$ and $p|_{Y_N} \circ \tilde{\gamma} = p \circ \gamma$. Since $Y_N \subset Y$ then $\tilde{\gamma}$ is also lifting in Y, but by the lifting property $\tilde{\gamma}$ and γ are equal and therefore $y = \gamma(1) = \widetilde{\gamma(1)} \in Y_N$ and we get:

$$\bigcup_1^\infty Y_n = Y.$$

Since every G_n is bounded, Lemma 3.1.2 tells us that $\Sigma(G_n, y_0)$ is not empty, for any $n \geq 1$. Let $n, m \in \mathbb{N}$ such that $n \geq m$. Since $Y_m \subset Y_n$ we can restrict any function $f \in \Sigma(G_n, y_0)$ to the subspace Y_m, and we get a function $f|_{Y_m} \in \Sigma(G_m, y_0)$. Therefore we have that

$$\lambda(G_m, y_0) \geq \lambda(G_n, y_0) \geq \lambda(G, y_0) \quad n \geq m.$$

For the final piece of the puzzle we need to obtain a lower bound on $\lambda(G, y_0)$, and for that purpose we use Schottky's theorem.

Theorem 3.2.1 (Schottky's theorem)**.** *Let $a \in \mathbb{C} \setminus \{0,1\}$, then there exists a constant $M > 0$ such that for any holomorphic function $f\colon \mathbb{E} \to \mathbb{C}\setminus\{0,1\}$ with $f(0) = a$, $|f(z)| < M$ for all $z \in \frac{1}{2}\mathbb{E}$.*

Schottky's theorem can be proved using the covering space of the set $\mathbb{C} \setminus \{0, 1\}$ which is precisely what we are trying to establish. There exist however proofs, using elementary methods, so we avoid circular reasoning. See for example [11, Ch. 10.3] for a proof using Bloch's theorem.

Using this theorem, and the same notation we have used through out this section, we get the following lemma:

Lemma 3.2.2. *There exists a constant $C > 0$ such that $\lambda(G_n, y_0) \geq C$ for all $n \geq 1$.*

Proof. Let $0 < \rho < 1$ such that the open disk $U = B(g_0, \rho)$ is evenly covered by $p\colon Y \to G$, and $U \subset G_1$. Then there exists a section $\sigma\colon U \to Y$ of p, such that $\sigma(g_0) = y_0$ and the image of σ is contained in Y_1, and therefore in Y_n for all $n \geq 1$. Fix $n_0 \geq 1$, since G_{n_0} is bounded the set $\Sigma(G_{n_0}, y_0)$ is not empty and therefore $\lambda(G_{n_0}, y_0) > 0$. Let $f \in \Sigma(G_{n_0}, y_0)$ be as in Lemma 3.1.6, then $\frac{1}{4}\mathbb{E} \subset f(Y_{n_0})$. According to Lemma 2.2.5, f induces a local biholomorphism $f^*\colon f(Y_{n_0}) \to G$. By restricting f^* we get a map $h\colon \frac{1}{4}\mathbb{E} \to G$, $z \mapsto f^*(z)$, and since $G \subset \mathbb{C} \setminus \{0,1\}$, Schottky's theorem tells us there exists a constant $M > 0$ such that $|h(z)| < M$ for all $z \in \frac{1}{8}\mathbb{E}$, and this constant M is independent of n_0. Schwarz's lemma 2.2.9, tells us that $|h'(0)| \leq 8M$, but h is just a restriction of f^* so $|(f^*)'(0)| \leq 8M$. By the remarks after Lemma 3.1.6, $(f^*)'(0)$ is a positive real number so $(f^*)'(0) < 8M$.

Since $\sigma(U) \subset Y_n$, for all n we get that $f^* \circ (f \circ \sigma) = \mathrm{id}_U$. Therefore, finally, by the chain rule:

$$f'(y_0) = \frac{1}{(f^*)'(0)} > \frac{1}{8M}.$$

□

It is important to note that the lower bound obtained in Lemma 3.2.2 depends only on the map $p\colon Y \to G$, the constant g_0, and the open disk U used in the proof. This constant is independent of which $n \geq 1$ is used. Next we will show simultaneously that there exists a function $F \in \Sigma(G, y_0)$, $F\colon Y \to \mathbb{E}$, and that $F'(y_0) = \lambda(G, y_0)$, which implies that $F\colon Y \to \mathbb{E}$ is a covering map.

Theorem 3.2.3. *There exists a function $F \in \Sigma(G, y_0)$ such that $F'(y_0) = \lambda(G, y_0)$.*

Proof. Since $\Sigma(G_n, y_0) \neq \emptyset$ by 3.1.2, we can choose a sequence (F_n) such that $F_n \in \Sigma(G_n, y_0)$, and

$$\lambda(G_n, y_0) - F_n'(y_0) \leq \frac{1}{n} \quad \text{for all } n = 1, 2, \ldots$$

The sequence of functions $F_n\colon Y_n \to \mathbb{E}$ is uniformly bounded, but have different domains, so we can not apply Montel's theorem directly. By restricting all of the functions (F_n) to Y_1 we get a sequence of functions $F_n|Y_1\colon Y_1 \to \mathbb{E}$, which is uniformly bounded. By Montel's theorem we get a subsequence $(F_{1,n})$ of (F_n) such that the restrictions to Y_1 converge uniformly on compact subsets of Y_1 to a holomorphic function $F\colon Y_1 \to \mathbb{E}$. In the same way as before we can restrict the functions in the sequence $(F_{1,n})$ to the set Y_2, and by Montel's theorem we get a subsequence $(F_{2,n})$ such that the restrictions to Y_2 converge uniformly on compact subsets of Y_2 to a holomorphic function $Y_2 \to \mathbb{E}$. By continuing this process we obtain for any $i > 1$ a subsequence $(F_{i,n})$ of $(F_{i-1,n})$, which converges uniformly on compact subsets of Y_i to a holomorphic function $Y_i \to \mathbb{E}$.

Consider the 'diagonal' sequence (f_n) defined by $f_n = F_{n,n}$ for all $n \geq 1$. We get a corresponding increasing sequence $m_1, m_2, \ldots$ of natural numbers such that $f_n\colon Y_{m_n} \to \mathbb{E}$. The sequence (f_n) converges uniformly on compact subsets of Y to a holomorphic function $f\colon Y \to E$. Indeed, let K be a compact subset of Y, by definition of (Y_n) there exists a constant N such that $K \subset Y_N$, and therefore the domains of (f_n) eventually contain Y_N. By restricting the functions to the set Y_N, we get that these restrictions converge uniformly on compact subsets of Y_N, and therefore on K. Since K was arbitrary we have that (f_n) converges uniformly on compact subsets of Y to a holomorphic function $f\colon Y \to \mathbb{E}$.

By Lemma 3.2.2 there exists a constant $C > 0$ such that $\lambda(G_n, y_0) \geq C$ for all $n \geq 1$, and it follows that

$$f'(y_0) = \lim_{n\to\infty} f_n'(y_0) = \lim_{n\to\infty} \lambda(G_n, y_0) \geq C.$$

Therefore the holomorphic function f is not constant on Y. To conclude that $f \in \Sigma(G, y_0)$ we need to show that f is injective with respect to p.
Assume there are two points $z_1, z_2 \in Y$ with $p(z_1) \neq p(z_2)$ such that $f(z_1) = f(z_2)$. Let $N > 0$ be such that $p(z_1), p(z_2) \in G_n \subset G_{n+1}$ for all $n \geq N$. Let $V, W \subset Y_N$

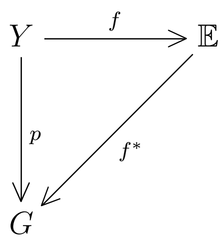
be neighbourhoods of z_1 and z_2, respectively, such that p induces homeomorphisms $V \to p(V)$, $W \to p(W)$, with $p(V) \cap p(W) = \emptyset$, and denote by σ_V and σ_W the inverses of these homeomorphisms, respectively. The functions $f_n \circ \sigma_V$ and $f_n \circ \sigma_W$ converge to the functions $f \circ \sigma_V$ and $f \circ \sigma_W$ neither of which is identically zero. By Hurwitz's theorem 2.2.10 there are points $\alpha_k \in p(V)$ and $\beta_k \in p(W)$ with $f_k(\sigma_V(\alpha_k)) = f_k(\sigma_W(\beta_k))$ for all k sufficiently large, but $p(V) \cap p(W) = \emptyset$ which is a contradiction since the functions f_n are injective with respect to $p|_{Y_n}$.

The only thing left is to show that $f'(y_0) = \lambda(G, y_0)$. Since f is in the set $\Sigma(G, y_0)$ we have the following inequality

$$\begin{aligned}\lambda(G, y_0) \geq f'(y_0) &= \lim_{n\to\infty} f_n'(y_0)\\ &= \lim_{n\to\infty} \lambda(G_n, y_n) \geq \lambda(G, y_0).\end{aligned}$$

So $\lambda(G, y_0) = f'(y_0) > C$. □

In conclusion, by the last Theorem 3.2.3 there exists a surjective holomorphic function $f\colon Y \to \mathbb{E}$ which is injective with respect to $p\colon Y \to G$. So $f(Y) = \mathbb{E}$ is a Riemann domain, $f\colon Y \to \mathbb{E}$ a morphism of Riemann domains and we have the following commutative diagram.



Since f is surjective, we have by Lemma 2.1.4 that $f^*\colon \mathbb{E} \to G$ is a covering map. And since $\mathbb{E}$ is simply connected, $f^*\colon \mathbb{E} \to G$ is a universal covering space of G.

We have now proved Theorem 3.0.1 and have obtained the main results of our thesis, a *Uniformization theorem for domains in* $\mathbb{C}$.

A. Appendix

In this chapter we are presenting theorems and definition concerning the fundamental group and covering spaces. These are all well known results but stated here for the sake of completeness. The definitions are taken from [8] and the proofs of the theorems can be found in [8] or [4].

A.1. Path Homotopy and the Fundamental Group

Definition A.1.1. *Let $f_0, f_1 \colon X \to Y$ be continuous maps and $H \subset X$.*

(i) We say that f_0 is homotopic to f_1, denoted $f_0 \approx f_1$, if there is a continuous map $F \colon X \times I \to Y$ such that

$$F(x,0) = f_0(x), \qquad \textit{and} \qquad F(x,1) = f_1(x),$$

We call F a homotopy between f_0 and f_1.

(ii) We say that f_0 is homotopic to f_1 relative to H, $f_0 \approx f_1(\mathrm{rel}\, H)$, if $f_0 \approx f_1$ and there exists a homotopy between them with the further condition that:

$$F(x,s) = f_0(x) \qquad \textit{for all } x \in H, s \in I.$$

Both homotopy and relative homotopy are equivalence relations on the set of functions continuous functions $X \to Y$, and we denote by $[f]$ the equivalence class of a continuous function $f : X \to Y$.

We have a special case for paths.

Definition A.1.2. *Two paths $f_0, f_1 \colon I \to X$, where I is the interval $[0,1]$, are said to be path homotopic if they are homotopic* $\mathrm{rel}\{0,1\}$.

Definition A.1.3. *If $f, g\colon I \to X$ are paths such that $f(1) = g(0)$ then we can define the product $h = f * g$ by*

$$h(t) = \begin{cases} f(2t) & \text{for } t \in [0, \frac{1}{2}], \\ g(2t-1) & \text{for } t \in [\frac{1}{2}, 1]. \end{cases}$$

The function h is well defined and a continuous path in X. The product also induces a well defined operation on the path homotopy equivalence classes which are also denoted by $[\cdot]$:

$$[f] * [g] = [f * g].$$

The operation is associative on path homotopy classes. Let $c_x\colon I \to X$ denote the constant path at x, that is $c_x(s) = x$ for all $s \in I$. This acts as an identity, that is if f is a path with $x_0 = f(0)$ and $x_1 = f(1)$, then $[c_{x_0}] * [f] = [f]$ and $[f] * [c_{x_1}] = [f]$. For any path f, we can define a path $f^{-1}(s) = f(1-s)$ for $s \in [0, 1]$ and $[f] * [f^{-1}] = [c_{f(0)}]$, that is $[f^{-1}] = [f]^{-1}$. A closed loop at $x \in X$ is a path $f\colon I \to X$ such that $f(0) = f(1) = x$. With the operation $*$ the classes of closed loops at x will form a group.

Definition A.1.4. *Let $x \in X$, the fundamental group of X at x (or relative to the base point x) is the group of path homotopy classes of closed loops at x with $*$ acting as the group operation. The fundamental group of X at x is denoted by $\pi_1(X, x)$.*

Here we are mostly interested in the fundamental group of domains in $\mathbb{C}$ (or $\mathbb{R}^2$), and it is well known that those fundamental groups are countable. It is simple to get a more general theorem which we state here. For a proof see [7, Thm. 8.11].

Theorem A.1.5. *The fundamental group of a second countable manifold is countable.*

Definition A.1.6. *Let X be a topological space and $x \in X$. We say X is* simply connected *if it is path connected and $\pi_1(X, x)$ is the trivial group.*

Definition A.1.7. *Let $h\colon (X, x_0) \to (Y, y_0)$ be a continuous map. Define*

$$h_*\colon \pi_1(X, x_0) \to \pi_1(Y, y_0),$$

by

$$h_*([f]) = [h \circ f].$$

The map h_* is called the homomorphism induced by h relative to the base point x_0. It is a well defined group homomorphism.

A.2. Covering Spaces

Definition A.2.1. *Let $p\colon X \to Y$ be a continuous surjective map. The open set U of Y is said to be evenly covered by p if the inverse image can be written as the union of pairwise disjoint open sets V_x in X such that p induces a homeomorphism of V_x onto U, for each x.*

Definition A.2.2. *Let $p\colon X \to Y$ be a continuous and surjective map, X and Y be Hausdorff spaces. If every point y of Y has a neighbourhood U that is evenly covered by p, then p is called a covering map, and X is said to be a covering space, or a cover, of Y.*

Theorem A.2.3. *Let $p\colon X \to Y$ be a covering map and Y_0 be a subspace of Y. Then the induced map $p_0\colon X_0 \to Y_0$, where $X_0 = p^{-1}[Y_0]$, is a covering map.*

Definition A.2.4. *Let $p\colon X \to Y$ and $f\colon H \to Y$ be continuous maps. We say that $\tilde{f}$ is a lifting of f, if $\tilde{f}\colon H \to X$ is a continuous map such that $p \circ \tilde{f} = f$.*

Definition A.2.5. *A continuous map $p\colon X \to Y$ is said to have the curve lifting property if the following condition holds. For every curve $u\colon [0,1] \to Y$ and for every point $x_0 \in X$ with $p(x_0) = u(0)$, there exists a lifting $\tilde{u}\colon [0,1] \to X$ of u such that $\tilde{u}(0) = x_0$.*

Lemma A.2.6. *Let $p\colon X \to Y$ be a covering map. Then it has the curve lifting property.*

Lemma A.2.7. *(Homotopy lifting lemma) Let $p\colon X \to Y$ be a covering map and let $p(x_0) = y_0$. Let the map $F\colon I \times I \to Y$ be continuous, with $F(0,0) = y_0$. There is a unique lifting of F to a continuous map $\tilde{F}\colon I \times I \to Y$ such that $\tilde{F}(0,0) = x_0$. If F is a path homotopy, then $\tilde{F}$ is a path homotopy.*

Let $p\colon X \to Y$ be a covering map, and let $\alpha, \beta\colon [0,1] \to Y$ be closed curves at $y_0 \in Y$. Let $x_0 \in X$ such that $p(x_0) = y_0$, and $\tilde{\alpha}, \tilde{\beta}$ be liftings to X with $\tilde{\alpha}(0) = \tilde{\beta}(0) = x_0$. We see from Lemma A.2.7 that if α, β are path homotopic then $\tilde{\alpha}(1) = \tilde{\beta}(1)$. If X is simply connected we have that α,β are path homotopic if and only if $\tilde{\alpha}(1) = \tilde{\beta}(1)$.

The Homotopy lifting lemma has the following stronger form:

Lemma A.2.8. *(General lifting lemma) Let X, Y, H be connected and locally path connected spaces. Let $p\colon X \to Y$ be a covering map with $p(x_0) = y_0$ and $f\colon H \to Y$ be a continuous map with $f(h_0) = y_0$. The map f can be lifted to a map $\tilde{f}\colon H \to X$ such that $\tilde{f}(h_0) = x_0$ if and only if*

$$f_*(\pi_1(H, h_0)) \subset p_*(\pi_1(X, x_0)).$$

Furthermore, if such a lifting exits, it is unique.

We see that Lemma A.2.8 implies Lemma A.2.7 since I is simply connected.

Definition A.2.9. *(Universal covering) Suppose X and Y are connected topological spaces and $p\colon X \to Y$ is a covering map, p is called the universal covering of Y if it satisfies the following property: For every covering map $q\colon Z \to Y$ with Z connected, and every choice of points $x_0 \in X$, $z_0 \in Z$ with $p(x_0) = q(z_0)$, there exists exactly one continuous fiber-preserving mapping $f\colon X \to Z$ such that $f(x_0) = z_0$.*

Let Y be connected and locally simply connected. If $p\colon X \to Y$ is a covering map and X is simply connected then p is a universal covering of Y.

Theorem A.2.10. *(Existence of the Universal covering) Suppose Y is a connected manifold. Then there exists a connected, simply connected manifold X and a covering map $p\colon X \to Y$. (This implies that X is a universal covering, by the remark above)*

Theorem A.2.11. *Suppose Y is a manifold, X is a Hausdorff space and $p\colon X \to Y$ is a local homeomorphism with the curve lifting property. Then p is a covering map.*

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