Patent Valuation

Þórdís Jensdóttir

Thesis of 30 ECTS credits submitted to the School of Science and
Engineering at Reykjavik University in partial fulfillment of the
requirements for the degree of
Master of Science (M.Sc.) in Financial Engineering

May 2021

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Abstract

Patents are intangible assets that provide an exclusive right to invest in the underlying invention. They protect the innovator from others duplicating that same invention over the life of the patent. Patents can be very valuable, but in the attempt to assign a monetary value to a granted patent, there is no common methodology of valuation. The biggest challenge is the uncertainty concerning the future value of the underlying invention. Standard valuation approaches tend to underestimate uncertain investment opportunities by ignoring the value of managerial flexibility. This flexibility contains the option to respond to different events and observe new possibilities as time goes by. An advanced way to capture the value of flexibility is the option pricing framework. Under this approach investment opportunities are valued as options on real assets, known as real options. This approach has increasingly been gaining acceptance in the valuation of uncertain investment opportunities such as patents. There are various real options that may be appropriate in the valuation of a patent. The options that have mostly been applied in the literature are the option to abandon and the option to invest or wait for more information to be obtained. Our concentration will be on the solution methods that can be applied in real options problems, along with their features. The aim is to find an applicable approach that can assign a reasonable value to a patent.

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Útdráttur

Einkaleyfi eru óefnislegar eignir (*e.* intangible assets) sem veita handhafa sínum einkaréttinn til þess að fjárfesta í undirliggjandi uppfinningu. Eigandi einkaleyfis er varinn fyrir því að aðrir framleiði eftirmynd af sömu vöru yfir líftíma einkaleyfisins. Einkaleyfi sem slík geta verið mikils virði en erfitt hefur reynst að verðmeta þau. Engin sameiginleg skoðun er á því hvaða aðferðafræði skal beita. Stærsta áskorunin er hin mikla óvissa sem bundin er við framtíðarvirði undirliggjandi uppfinningar. Hefðbundnum verðmats aðferðum hættir til að vanmeta virði áhættusamra fjárfestingartækifæra með því að líta framhjá virði sveigjanleika. Þessi sveigjanleiki felst í því að geta brugðist við mismunandi uppákomum og uppgvötað ný tækifæri eftir því sem tímanum líður. Ein leið til þess að meta virði þessa sveigjanleika er að nota sömu aðferðafræði og notuð er til þess að verðleggja afleiður. Með þessari nálgun eru fjárfestingartækifæri verðmetin sem valkostur til að fjárfesta í efnislegri eign, betur þekkt sem raunvilnanir (*e.* real options). Þessi aðferðafræði er öflugt tól til þess að verðmeta áhættumikil fjárfestingartækifæri líkt og einkaleyfi. Ýmsar raunvilnanir geta átt við þegar kemur að verðmati á einkaleyfi. Þær raunvilnanir sem helst hafa verið skoðaðir í þessum tilgangi er valkosturinn til þess að stöðva fjárfestingarverkefnið og valkosturinn til þess að bíða þar til frekari upplýsinga hefur verið aflað. Þessi greinargerð mun leggja áherslu á þær aðferðir sem notaðar eru við verðlagningu raunvilnana og eiginleika þeirra, með raunhæft verðmat á einkaleyfi að leiðarljósi.

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I dedicate this to my son.

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List of Abbreviations

BSM	Black-Scholes-Merton
CAPM	Capital Asset Pricing Model
CCA	Contingent Claim Analysis
CEO	Chief Executive Officer
DCF	Discounted Cash Flow
DP	Dynamic Programming
DTA	Decision Tree Analysis
FDA	Food and Drug Administration
GBM	Geometric Brownian Motion
IPRs	Intellectual Property Rights
NPV	Net Present Value
PV	Present Value
ROV	Real Option Valuation
R&D	Research and Development

1 Introduction

Innovation plays a big role for businesses in today's competitive market place. To keep this competitive advantage, enterprises or individuals need to protect their innovative ideas with intellectual property rights ($IPRs$) e.g. copyrights, brands, trademarks, and patents. The importance of $IPRs$ has been continuously increasing in many fields of business in recent years. Their value is essential if one is to make well founded management decisions [1] [2].

In this thesis the focus will be on the valuation of patent rights. Patents are intangible assets which can be extremely valuable but often it can be difficult to assign a monetary value to them. A patent provides the exclusive right of limited duration (most often 20 years) to produce a new, useful and non-obvious invention (product or service). It protects the innovator from competitors using that same invention with the right to sue others for infringement. There is a need to distinguish between the underlying invention which is often called the underlying asset and the patent itself. Patents can be valued in their application process where the aim is to decide if it is worth continuing with the application to the next stage. On the other hand, if a patent has been granted, the valuation process focuses on decisions regarding whether to pay the renewal fees to keep the patent alive, if the patented product should be commercialized, or what it's monetary value should be when it comes to licensing, financing, litigation, and sale. The uncertainty concerned with the future value of the underlying asset is a big challenge in the attempt to value a patent. Technical and commercial challenges can arise such as uncertainties regarding the future expected cash flow, market conditions, the effect of competition, and the volatility of the patented project. Because of these and other uncertainties there is a need to engage in research to find objective and realistic methods for patent valuation [3] [4].

Some methods are available in order to value patents but there is no common standard or methodology of valuation. The methods that have been used in practice can be divided into two groups, qualitative and quantitative valuation methods. The qualitative methods provide a value reference mainly through rating and scoring based methods and are generally not expressed in monetary terms. This valuation approach is often considered to be interpretative and subjective [5] [6]. The quantitative methods differentiate from the qualitative methods as they are objective and assign a monetary value to the patent. The most common method is the economic analysis method which has three different approaches cost, market, and income. It has been argued that the cost and market based methods are not satisfactory when it comes to patent valuation. The cost based method measures the value of the patent by looking at the development costs behind the patented product. Since it is based on historic costs it fails to take into account the future benefits that might arise from the patent, and therefore should not be used in making rational decisions. The aim of the market based method is to value a patent by comparing the underlying asset to a similar patented asset that has been traded recently in the active market. The problem with this method is that it can be quite a challenge to find a comparable asset where other assets often differentiate from the patented product in one way or another [3] [7] [8] [9].

The income based method attempts to value patents by calculating the present value (PV) of future expected cash flow from the patented product or project. The main method used in practice is the standard Discounted Cash Flow (DCF) approach including the Net Present Value (NPV) valuation. The difficulty of this approach is related to the forecasting of future cash flow and the estimation of the discount factor that reflects the risk concerned with the cash flow. The main drawback of this approach is that it fails to take into account the various decisions that are open to managers over the lifetime of the patent. It has been shown that this managerial flexibility to respond to different events and observe new possibilities as time goes by, can be of great value. A patent is an example of an asset that derives it's value from its potential to become valuable in the future, contingent on the occurrence of events. These assets will exceed their DCF value with the difference coming from the option component. In order to capture the value of flexibility contingent valuation approaches have been applied. In Decision Tree Analysis (DTA), each decision node allows for managerial decisions after some uncertainty has been resolved, and more information has been obtained. Another more advanced way to capture the value of flexibility is the option pricing model e.g. the discrete time binomial tree approach and the continuous time Black-Scholes-Merton (BSM) model [3] [8] [10] [11].

In theory, the option pricing approach is more accurate than the DTA approach. The option pricing model was introduced in 1973 by three economists Fischer Black, Myron Scholes, and Robert Merton [12]. The aim of the model was to value options on financial assets such as stocks. It was a major breakthrough in finance and economics at the time. In 1977, Myers [13] found that this approach could be extended to non-financial assets. He proposed the term "Real Options", where investment opportunities are valued as options on real assets. The research literature on real options has gradually been increasing since the 1980s [14]. The motivation for using option pricing analysis in capital budgeting arises from its potential to capture the value of flexibility, which is nothing more than a collection of options associated with an investment opportunity. Both the DCF and the DTA approaches have their limitations when the expected costs and cash flow are uncertain. In this case the real option approach has helped academics in assessing the probability of financial success of an uncertain project [10] [11] [15].

An option is the right but not the obligation to buy (call) or sell (put) an underlying asset at a predetermined future price. Patents are similar to options since they provide the owner of the patent with an exclusive right but not the obligation to produce or invest in the patented product. Therefore, it has been argued that patents should be valued as options. This approach has increasingly been gaining acceptance in the valuation of patents [3]. Some scholars have considered the value of a patent as an option to delay the commercialization of the patented project, similar to a call option on the PV of future cash flow [8]. Others have emphasized the option to abandon the patented project [15] [16]. Valuing a patent as the option to abandon has been found valuable in the pharmaceutical industry where drugs usually need to go through many phases in research and development ($R\&D$) before a product is accepted by the Food and Drug Administration (FDA). This process is expensive as well as uncertain and therefore the probability of an abandonment is large [17]. Another approach has been emphasized in patent valuation where the value of the patent is obtained as the difference between a patent protected project and the same project without patent protection [9].

The valuation of patents can be approached from different perspectives as in whether to value a single patent or a portfolio of patents. As well there is a distinction between a patent that is in it's application process or one that has already been granted. The aim of this thesis is to develop a general model that can be used to assign a monetary value to a single granted patent. The concentration will be on the quantitative valuation methods available, focusing on the income based methods and real options [18] [19]. In order to account for cost uncertainty as well as uncertainty over future expected cash flow, a simulation approach using real options will be developed and implemented. The baseline for the model presented is the model originally proposed by Schwartz (2004) [16], later developed by Ernst, Legler, and Lichtenthaler (2010) [9], and simplified by Hernández, Güemes, and Ponce (2018) [15]. These researchers concentrated on the option to abandon the underlying project. As an extension of previous research we will also include the option to expand. The value of a patent will be examined as a sequential option to invest in development and commercialization where the expected cash flow and investment costs follow a stochastic process. If the value of the project is marginal, we will have the option to expand. However, if conditions are unfavorable we will default.

This thesis is organized as follows. Section 2 will outline the traditional DCF valuation approach including NPV. Under this approach we will briefly describe how risk can be measured as well as implementing the Capital Asset Pricing Model ($CAPM$) which is widely used to estimate the risk adjusted discount rate. In Section 3 we will introduce the importance of flexibility in investment analysis and explain the DTA approach that can capture the value of flexibility. In Section 4 we will proceed to the option pricing framework discussing the main properties of both European and American options. The assumptions underlying option pricing and the discrete time binomial model will be explained. We will compare the application of DTA to Real Option Valuation (ROV) and introduce the different types of real options that can be applicable in the valuation of a patent. Section 5 will cover the option to invest and it's characteristics. In Section 6 we will move from discrete time analysis to continuous time valuation and introduce the underlying mathematics. We will discuss the BSM model that can be used to value a patent as a call option on the commercialization of the underlying asset. The mathematical theory underlying real option models will be examined in Section 7, comparing the two analytical solution methods, Dynamic Programming (DP) and Contingent Claim Analysis (CCA). The proposed model of this study and the result from the simulations will be explained in Section 8, along with sensitivity analysis. In Section 9 we will conclude our findings.

To increase the reader's understanding of the different quantitative valuation approaches and to make their comparison clearer, we will start by making some base assumptions about a patent protected project in Section 2. We will proceed with those assumptions through the thesis where additional factors will be added as the valuation methods get more advanced. The goal is to shed a light on the improvement of the valuation as we move from the traditional DCF valuation method to ROV. Results from the calculations are presented in million dollar values and rounded to two decimal points.

2 Discounted Cash Flow (DCF) Valuation

The starting point in our journey to calculate the value of a granted patent is to address some important topics in corporate finance. These are DCF valuation methods, PV and NPV, including $CAPM$ used to determine the risk adjusted discount rate. It has been argued that DCF valuation is fundamental to all other quantitative valuation methods. For example, to use option pricing models for real assets we often need to begin with a DCF valuation in order to measure the value of the underlying asset. There are many different DCF methods available but the conventional method values an asset/project by discounting its expected future cash flow. This is done after tax in the case of real assets at a discount rate which is a measure of the risk included in the project. The equation for calculating the PV of an asset is given by Eq.1 [8].

$$PV = \sum_{t=1}^{t=n} \frac{CF_t}{(1+r)^t} \tag{1}$$

where

n = The life of the project
CF_t = The expected cash flow at time t
r = The discount rate

As illustrated in Section 1 we will begin with some base assumptions of a patent protected project presented in the following example.

Example. Suppose you are a chief executive officer (CEO) in a company. Your management is faced with an internal opportunity to accept or reject a 5 year $R\&D$ project aimed at discovering if a marketable product can be produced from a new innovation. This innovation just got granted a patent with a lifetime of 20 years. It is estimated that the patented product will generate an annual cash flow of $\$0.70million$ from the 6th year when the $R\&D$ phase will be completed until the patent expires. Since this is a long term investment (20 years) the risk free rate should be the rate on a risk free government bond with maturity in 20 years. Because you are in the middle of the Covid-19 pandemic, risk free rates on such bonds are extremely low. The long term average for a nominal risk free rate will vary between markets but has averaged 4.5% in the United States [20]. Thus it is suitable to estimate a risk free rate of 5% in this case. From Eq. 1 the PV can be calculated as follows

$$PV = \sum_{t=6}^{t=20} \frac{\$0.70}{(1.05)^t} \approx \$5.69million$$

The requirements for this approach are the expected future annual cash flows ($CF_t = \$0.70million$) and the discount rate which in this case is assumed to be the risk free rate ($r = 5\%$).

Whereas the PV only accounts for cash inflows, the NPV approach is a more comprehensive indicator of a project's potential profitability, since it takes the initial investment outlay (the cost of funding the project) into account.

$$NPV = -I + \sum_{t=1}^{t=n} \frac{CF_t}{(1+r)^t} \tag{2}$$

where

$n =$ The life of the project
$CF_t =$ The expected cash flow at time t
$r =$ The discount rate
$I = PV$ of investment costs

Example. We will proceed with our previous example. Now it is known to the management that the immediate investment needed to start the $R\&D$ process will be $I_0 = \$0.20 million$. After the first phase of the $R\&D$ program which is assumed to be completed in two and a half years, an additional investment of $I_{2.5} = \$0.55 million$ will be needed to finish the product's development. In the 5th year a cost of $I_5 = \$0.85 million$ is estimated to produce and commercialize the product. Therefore, the NPV of this investment opportunity becomes

$$NPV = -\$0.20 - \frac{\$0.55}{(1.05)^{2.5}} - \frac{\$0.85}{(1.05)^5} + \sum_{t=6}^{t=20} \frac{\$0.70}{(1.05)^t} \approx \$4.34 million$$

In traditional investment analysis, a project should be undertaken if its NPV is positive such that future cash flow exceed it's investment costs discounted to the present. Thus, this would be considered a profitable investment opportunity [8] [21].

2.1 Estimating the Discount Factor

In principle the discount factor (the opportunity cost of capital) is the expected return investors demand from an investment. This rate reflects the investment's level of risk which increases when uncertainty is large. Investors therefore demand higher returns from risky investments than from projects that are safer. It can be hard to capture all risk factors of an investment in one discount rate. When there is no risk the expected return should be the same as the risk free rate. In the examples above we assumed that the expected return was the risk free rate but in reality it is unlikely that investment opportunities are free from risk. In fact, they are risky, since they face uncertainty regarding the future [21] [22] .

2.1.1 Measuring Risk

In finance, risk can be measured by using statistical measures of spread, variance (σ^2) and standard deviation (σ). The variance of returns tells us how far the expected return ($\bar{r}$) is from the actual return at time t (r_t), where the expected return is the mean of all the r_t values. Each measurement is squared to eliminate negative numbers. The sum of all the deviations is divided by the numbers of observed returns minus one ($N-1$) to account for the loss of degree of freedom (which is the one parameter that accounts for the intermediate step, the mean), see Eq.3.

$$\sigma^2 = \frac{\sum_{t=1}^{t=N}(r_t - \bar{r})^2}{N-1} \tag{3}$$

where

r_t = Return in period t
$\bar{r}$ = The mean return
N = Number of observed returns

The standard deviation is simply the square root of the variance as shown in Eq.4.

$$\sigma = \sqrt{\sigma^2} \tag{4}$$

This variability is a measure of risk and an indicator of how far in either direction the returns are moving [21]. It can be hard to measure the variance for a new invention that has no history of returns. In this case, the variance can be estimated from other similar investments or obtained from the average variance of a firm in the same industry. It is also possible to assign different probabilities to different market scenarios and calculate the PV of the expected cash flow for each scenario. The variance is then found across those different PVs. Probability distributions can be estimated for the inputs in the PV calculations (e.g. the market size). Then simulation can be used to estimate the variance between the PVs that are obtained [8] [23].

In finance, uncertainty of future returns is called volatility. Thus, when volatility is high the variance or the gap between investors expectations and their actual return is large. Volatility is the standard deviation multiplied by the square root of time. The average volatility of the market portfolio in the United States is approximately 20%. Most individual securities have a higher standard deviation because of an extra variability in their returns due to specific risk [21] [24].

2.1.2 Specific Risk and Market Risk

There are two types of risk, specific risk and market risk. Every firm or project has it's own specific risk related to it's operation e.g. outcomes from $R\&D$, employee competence or consumer demand. When an investor holds a portfolio of many stocks he/she can diversify away the specific risk by choosing stocks from many different sectors. The more assets one has in a portfolio the better diversified he/she can be. The saying: "Don't put all your eggs in one basket" is convenient in this case. The market risk on the other hand is the type of risk that can not be diversified away. It is the type of risk that faces the market as a whole e.g. financial crisis or currency/interest rate shifts. Therefore, the dominant risk factor for an investor with a well diversified portfolio is the rise/fall of the market [10] [11] [21] [25].

To compensate for this market risk investors require a risk premium. The market risk premium can be estimated by subtracting the average historical return on risk free assets (r) from the average historical return on common stocks (r_m) as illustrated in Eq.5.

$$\text{Market risk premium} = r_m - r \tag{5}$$

The market risk premium can not be measured with precision because we can never be sure that the future will be like the past. Neither can we go inside the heads of investors and find a probability distribution of their expectations. Therefore, we make the assumption that the average historical risk premium is stable. The historical risk premium over the last century between stocks and Treasury bonds has averaged 6.26% in the United Sates. The estimation of the risk premium will always differ between markets and depend on the approach used (possible range from 3% to 12%) [21] [26].

2.1.3 Capital Asset Pricing Model (CAPM)

The $CAPM$ can be used to estimate the discount rate for an investment opportunity. The model was proposed in 1964 by three economists William Sharpe, John Lintner, and Jack Treynor [27]. It states that in a competitive market the expected risk premium of an investment should vary in proportion to it's beta (β) which measures how sensitive an investment is to the rise/fall of the market. The higher the beta the more risky the investment. The beta of a risk free asset is 0 since movements in the market do not affect its return. The average beta of the market is 1. An asset that has a beta larger than 1 is more sensitive to market movements than one that has a beta lower than 1. A beta value of 1.5 means that when the market rises/falls the change in the asset will exceed the market movement by 50%, while an asset with a beta value of 0.5 will only change by half of the movement in the market. To measure beta we begin by calculating the covariance between the investment (r_i) and the market returns as illustrated in Eq.6.

$$\sigma_{im} = \frac{\sum_{t=1}^{t=N}(r_{i_t} - \bar{r}_i)(r_{m_t} - \bar{r}_m)}{N-1} \tag{6}$$

where

r_{i_t} = Return from investment at time t
$\bar{r}_i$ = Mean value of investment's returns
r_{m_t} = Market return at time t
$\bar{r}_m$ = Mean value of market returns
N = Total number of periods

By dividing the covariance by the variance of the market returns (${\sigma_m}^2$) we obtain the security's beta as follows

$$\beta_i = \frac{\sigma_{im}}{{\sigma_m}^2} \tag{7}$$

The covariance can also be denoted as $\sigma_{im} = \rho_{im}\sigma_m\sigma_i$ where ρ_{im} is the correlation coefficient between the security and the market. Hence, beta can also be written as $\beta_i = \frac{\rho_{im}}{\sigma_m}\sigma_i$. Estimates of β may be distorted if there are extreme returns in some periods but these errors tend to cancel out when beta is estimated for a portfolio of assets. Thus, financial managers often go to industry betas which is the mean beta value for all stocks in a particular industry [21].

Since beta is a measure of an investment's risk relative to the market, an investment with a beta of 1.5 should have a 50% higher risk premium than the market. Similarly a beta of 0.5 should have a 50% lower risk premium [27].

$$\mu_i - r = \beta_i(r_m - r)$$

where

$\mu_i - r =$ Expected risk premium of investment i
$\beta_i =$ Beta value of investment i
$r_m - r =$ Expected market risk premium

Thus, the expected return on an investment (μ_i) should equal the risk free rate plus the product of the investment's beta and the expected market risk premium (see Eq.8).

$$\mu_i = r + \beta_i(r_m - r) \tag{8}$$

When risk increases, beta increases, and an investor will require a higher return from the investment [21].

Example. When looking at the investment opportunity in more detail you realize that it is not a risk free investment. You notice that there is an uncertainty regarding it's future and you need to measure a discount rate that reflects this risk. The market risk premium can be calculated from historical data as mentioned in Section 2.1.2, assuming a market risk premium of 6.26%. The risk free rate is 5% as discussed in Section 2. We will also assume that you find a perfectly correlated security in the market, that has the same cash flow and risk as your project with a beta of 1.60, 60% more volatile than the market. When such security exists you can use its beta as an estimate for risk. After careful analysis you can obtain the risk adjusted discount rate using the $CAPM$ model from Eq.8.

$$\begin{aligned} &\mu = 0.05 + (1.60)(0.0626) = 0.15 \\ &\Rightarrow \mu = 15\% \end{aligned}$$

Now we can recalculate the PV and NPV of the investment opportunity under uncertainty as follows

$$PV = \sum_{t=6}^{t=20} \frac{\$0.70}{(1.15)^t} \approx \$2.04 million$$

$$NPV = -\$0.20 - \frac{\$0.55}{(1.05)^{2.5}} - \frac{\$0.85}{(1.05)^5} + \sum_{t=6}^{t=20} \frac{\$0.70}{(1.15)^t} \approx \$0.68 million$$

The investment outlays are discounted at the risk free rate because for now, we will assume that those are known for certain. The cash flow is uncertain and therefore discounted at the risk adjusted discount rate. It appears that the NPV of this investment opportunity is positive and that the management should invest immediately [10] [21].

3 Decision Tree Analysis (DTA)

The standard NPV approach is based on today's expectation of future information and demands that an immediate decision is made at the time the valuation takes place. This approach can be convenient when the underlying asset to be valued is currently generating positive cash flow and the future can be estimated with some reliability. In the case of young patented assets this is usually not the case. These assets face large uncertainty in the future that can not be predicted with certainty. More importantly, the underlying asset may not be generating any positive cash flow at the time the valuation takes place nor in the upcoming years thereafter. The cash flow is generally dominated by expenses in the first years and the standard DCF valuation may result in a misleading negative NPV [3] [8].

The standard NPV fails to take into account the future opportunities and the value of the various decisions a management might want to take over the life of the project. This managerial flexibility is an option to respond to future unexpected events. When uncertainty is large such as in the case of new inventions, the probability of unforeseen events is high and the option to respond to these events increases in value. On the other hand, when there is little uncertainty, the probability of unexpected future events is low and the option becomes less valuable [8] [10] [11] [28].

Flexibility can be valued by assigning probabilities to different future events such that success has a probability of q while failure has a probability of $(1-q)$. The two main approaches used in practice to capture the value of flexibility are DTA and the more accurate ROV approach. DTA captures the value of managerial flexibility by taking various discrete decision points into account. It is useful in helping managers visualize the possible risk events that might arise over the life of the project. In the following example we will assume that the project is only affected by specific risk namely the outcomes from the $R\&D$ program. To account for the market risk we will need to apply ROV discussed under the option pricing framework in Section 4.2 [10].

Example. Lets proceed with our previous example from Section 2.1.3. The management is aware of future events and wants to model the flexibility to make decisions as time unfolds. They realize that there is a $q_1 = 0.5$ probability that in two and a half years the first phase of the $R\&D$ program will result in a favorable project. This gives the management the option to invest $I_{2.5} = \$0.55 million$ and start the next phase of $R\&D$. If the first phase fails which will happen with a probability of $(1-q_1) = 0.5$, they will not invest, and halt the project, resulting in a NPV of \$0. If they proceed to the second $R\&D$ phase there is a probability of $q_2 = 0.7$ that they will end up with a marketable product. In that case they have the option to invest $I_5 = \$0.85$ in the 5th year and start production and marketing. There is a probability of $(1-q_2) = 0.3$, that the second phase of the $R\&D$ will fail and the management will cease the project, resulting in a NPV of \$0. Now the standard NPV of this project becomes

$$\text{Standard } NPV = -\$0.20 - \frac{\$0.55}{(1.05)^{2.5}} - \frac{\$0.85}{(1.05)^{5}} + (0.5)(0.7)\sum_{t=6}^{t=20}\frac{\$0.70}{(1.15)^{t}}$$
$$\approx -\$0.64 million$$

The NPV is negative, thus under traditional investment analysis this project would not be considered profitable. By solving the decision tree presented in Figure 1 we are able to value the managerial flexibility to respond to the different risk events. The squares in the tree represent decision nodes where the management needs to make up its mind whether to invest. The circles represent the $R\&D$ risk events and the probabilities of them occurring. To calculate today's NPV of the project, we work from the right (t_5) to the left (t_0) in the tree.

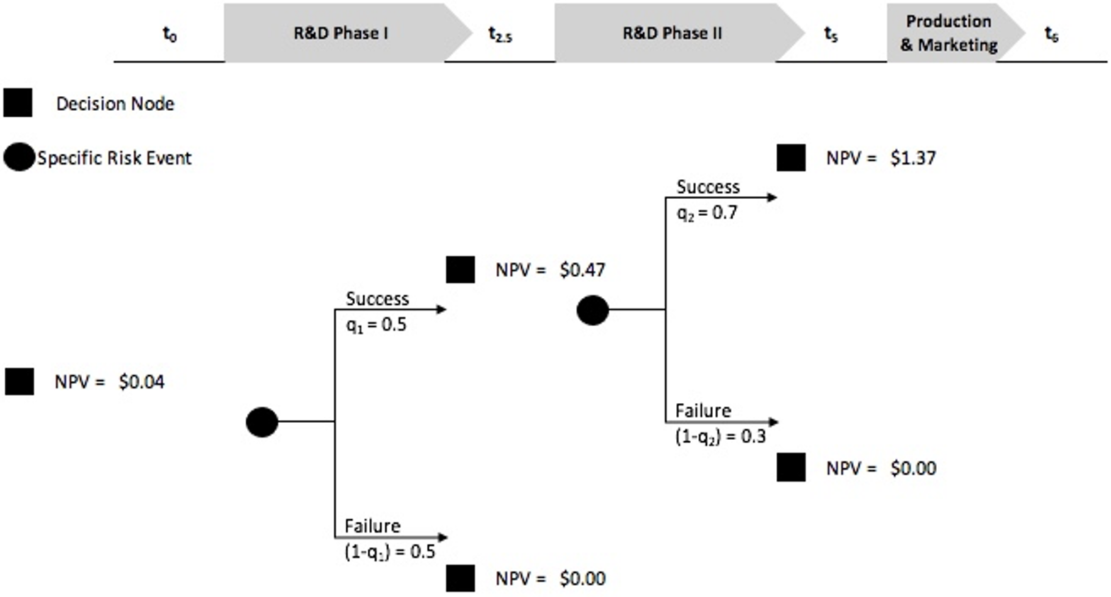

t0
R&D Phase I
t2.5
R&D Phase II
t5
Production & Marketing
t6
Decision Node
Specific Risk Event
NPV = $1.37
Success
q2 = 0.7
NPV = $0.47
Success
q1 = 0.5
NPV = $0.04
Failure
(1-q2) = 0.3
NPV = $0.00
Failure
(1-q1) = 0.5
NPV = $0.00

Figure 1: DTA with Specific Risk Events ($ in millions).

When solving the DTA, we use today's PV for both the expected cash flow and the investment costs at each stage. In the 5th year the PV of the cash flow is \$2.04$million$ as calculated in Section 2.1.3. If the second phase of the $R\&D$ program is successful we have the option to invest \$0.85$million$ with a $PV(I_5) = \$0.67 million.$ If it fails we will not invest. The value of the option to invest at t_5 becomes

$$NPV = max\Big(PV - PV(I_5), 0\Big)$$
$$= max\Big(\$2.04 - \$0.67, 0\Big)$$
$$\approx \$1.37 million$$

The PV of cash flow in $t_{2.5}$ equals the probability weighted payoffs in t_5. Hence, the value of the option to invest for \$0.55$million$ with a $PV(I_{2.5}) = \$0.49 million$ becomes

$$\begin{aligned} NPV &= max\Big(\big(0.7(\$1.37)+0.3(\$0)\big)-PV(I_{2.5}),0\Big) \\ &= max\Big(\$0.96-\$0.49,0\Big) \\ &\approx \$0.47million \end{aligned}$$

The PV of cash flow in t_0 equals the probability weighted payoffs from $t_{2.5}$ where there is an equal probability of success and failure. Therefore, the value of the option to invest $\$0.20million$ today is

$$\begin{aligned} NPV &= max\Big(\big(0.5(\$0.47)+0.5(\$0)\big)-I_0,0\Big) \\ &= max\Big(\$0.24-\$0.20,0\Big) \\ &\approx \$0.04million \end{aligned}$$

We can see that the value of the investment opportunity is significantly higher than the standard NPV of $-\$0.64million$ when the flexibility to default, if $R\&D$ outcomes are unfavorable, is included in the valuation. Even though the cumulative probability of success in the $R\&D$ phases is only 35% $\big((0.5)(0.7)=0.35\big)$, the project should be undertaken. The value of the flexibility or the option to respond to different risk events is especially important when investments demand some detailed decision making. This is essential when there is uncertainty associated with $R\&D$, production, or marketing [10].

DTA can help managers to analyze sequential investment decisions where uncertainty is resolved at discrete points in time. It forces them to use strategic thinking, comparing immediate decisions to subsequent ones. It is useful when the probabilities of expected cash flow can be quantified at an initial decision node. On the other hand, it can be difficult to estimate future cash flow from the underlying asset when the quality and validity of the final product as well as its market demand is unknown. In practice, it is likely that there are more then two possible outcomes from every decision node such that the decision tree can quickly expand.

When estimating the correct risk adjusted discount rate it can be helpful if a proxy for risk can be used in obtaining the appropriate rate. Assuming that a perfectly correlated twin security exists - like we did in our previous example. Unfortunately it might be hard or even impossible to find a comparable asset in the market in the case of a new invention. In the presence of managerial flexibility the main drawback of the DCF valuation is the use of a single constant discount rate over the life of the project. This assumes that the risk increases at a constant rate through time or that the same uncertainty is resolved in each period.

In realistic investment settings, variance is likely to change from time to time. A new invention in its final state of $R\&D$ is probably less variable then one that is in its first stages of research. In reality investments are made, implemented, and revised continuously through time. Events may not occur at discrete points in time, but rather, uncertainty resolves continuously. Therefore, it is a simplification of reality to assume a constant discount rate of PVs. The real option approach changes the nature of risk and invalidates the use of a single discount rate [3] [11] [28].

4 Option Pricing Framework

An option is a claim that pays off only under a certain future event or circumstance that is possible but cannot be predicted with certainty. An asset can be valued as an option if it's payoff is a function of the underlying asset. The option framework sets a precedent for the valuation of a patent because the payoff from a patent is a function of the underlying investment opportunity. A patent gives a firm/individual the right to develop and commercialize a product. This right is analogous to an option where its holder has the right but not the obligation to exercise the option. For this reason it has been argued that patents should be valued as options.

Option pricing is widely used to value traded financial assets known as financial options. Lately the option pricing model has been extended to the valuation of real assets such as investment projects. Those types of options are known as real options and provide a way to capture the value of managerial flexibility. The main difference between financial options and real options is that most financial assets are traded in the market while very few real assets are traded. Therefore, when valuing a financial option, the variables needed as an input for the model can be obtained from the market. On the other hand, in the case of most real options, the value of these variables needs to be estimated. Table 1 illustrates the difference between the variables of financial and real options used in option valuation, along with their symbols [3] [8].

Table 1: Comparison of the Variables used in Real vs. Financial Option Pricing.

Real Options	**Financial Options**
$V = PV$ of the Expected net Cash Flow	S = Stock Price
$I = PV$ of Investment Costs	K = Strike Price
T = Time left to Invest	T = Time to Maturity
r = Risk Free Rate	r = Risk Free Rate
σ = Volatility of the Expected Cash Flow	σ = Volatility of the Stock's Returns
δ = Cost of Maintaining the Option	δ = Dividend Payment

There are two types of options: a call option and a put option. The holder of a call has the right but not the obligation to buy the underlying asset at a predetermined price at a certain time in the future. The holder of a put option has the right but not the obligation to sell the underlying asset at a predetermined price at a certain time in the future. The predetermined price is called the strike price of the option contract. The date when the contract expires, is called the expiration date or maturity. The payoff from a call option is positive if the underlying asset exceeds the strike price when the option is exercised. If this is not the case the option expires worthless with a payoff of zero. The holder of a call option breaks even when the payoff from the option equals the price of the option (c), (see Eq.9).

$$\text{Net Payoff from a Call Option} = max(S - K, 0) - c \tag{9}$$

The payoff from a put option is positive if the strike price exceeds the underlying asset when the option is exercised. Otherwise the option expires worthless with a payoff of zero. Therefore, put options work as an insurance against the decline in the underlying asset's value. The holder of a put option breaks even when the payoff from the option equals the price of the option (p), (see Eq.10).

$$\text{Net Payoff from a Put Option} = max(K - S, 0) - p \tag{10}$$

The net payoff profiles for both a call and put option can be seen in Figure 2 [8] [24].

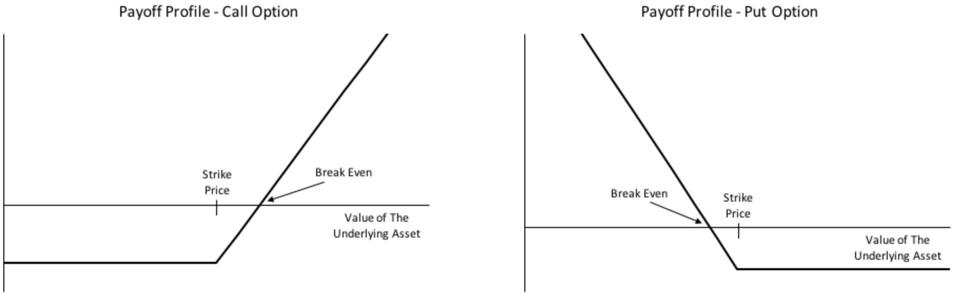


Figure 2: The Net Payoff Profiles for a Call and Put Option.

Options can be either European or American. European options can only be exercised at maturity while American options can be exercised at any time prior to maturity. Table 2 summarizes the change in the price of financial options, when one variable changes and others are held constant. The + sign indicates that an increase in a variable causes the price of the option to increase. The − sign indicates that an increase in a variable causes the price of the option to decrease [24].

Table 2: The Changes in Option Prices when a Variable's Value Increases.

Variable	*European Call*	*European Put*	*American Call*	*American Put*
Current Stock Price	+	-	+	-
Strike Price	-	+	-	+
Time to Maturity	Uncertain	Uncertain	+	+
Risk Free Rate	+	-	+	-
Volatility	+	+	+	+
Dividend	-	+	-	+

It can be seen that American options become more valuable as the time to maturity is extended. This is because the holder of a long life option has more exercising opportunities than the holder of a short life option. European options usually become more valuable as the time to maturity increases. However, this is not always the case. For example, when there are dividends, the stock price will decline after a dividend has been paid, and a long life call option might be less valuable than a short life one [24].

An option is said to be "in the money" if its payoff from today's exercise is positive (analogous to $NPV > 0$) and "out of the money" when its payoff from today's exercise is negative ($NPV < 0$). When an option is "at the money" the value of the underlying asset equals the cost of exercising the option and its payoff from today's exercise is zero ($NPV = 0$) [24] [28].

4.1 Option Valuation

When pricing derivatives a simple and widely used tool is the binomial tree approach introduced by Cox, Ross, and Rubinstein in 1979 [29]. It values options in discrete time manner and assumes that the underlying asset follows a random walk. In a random walk, given the current position, we always have equal probabilities of going up or down. Hence, if a random variable has the value x_0 at time t_0, it will either go up over a time period Δt, to $x_0 + \Delta x$ with probability $\frac{1}{2}$, or down to $x_0 - \Delta x$ with probability $\frac{1}{2}$. The probability that the random variable goes up/down j times over n periods is found from the binomial distribution as follows

$$P\Big(X\big(n\Delta t\big) = j\Delta x\Big) = \left(\frac{1}{2}\right)^n \binom{n}{j}$$

This process can be generalized by letting p be the probability of an up movement and $(1-p)$ be the probability of a down movement. As the time steps get smaller the probability distribution of a random walk converges to the standard normal distribution. The process which is obtained in the limit is the Wiener process discussed in Section 6.2 [28]. It is the foundation to the BSM model, which assumes that the change in the price of the underlying asset is lognormally distributed and continuous in time (see Section 6.5). The BSM model was introduced and later the binomial model was presented in order to simplify the continuous time approach to a discrete valuation process.

These two option pricing models build on the assumption that the market is complete. This means that there are no transaction costs or taxes, no restrictions on short sales, borrowing and lending at the same rate is allowed, and arbitrage opportunities do not exist. As well it's assumed that the risk free rate is constant over the life of the option. When these assumptions hold we can construct a replicating portfolio composed of Δ units of the underlying asset (e.g. stock) and a dollar amount B in a risk free loan. This is done in a way that it will not matter whether the price of the underlying asset goes up or down in one period. To avoid arbitrage opportunities, the price of setting up the portfolio should equal the current price of the option. The reason that a riskless portfolio can be set up this way is that the option and its underlying asset are both affected by the same source of uncertainty. That source of uncertainty is the price movements in the underlying asset. In a short period of time the price of the option is perfectly correlated to the change in the price of the underlying asset [11] [24] .

4.1.1 Replicating Portfolio

Assume we want to know the price of a call option that expires in 1 year. The current price of the underlying non dividend paying stock is $S = \$100$. In one period it is known that the stock's price will either move up to $uS = S_u = \$150$ or down to $dS = S_d = \$50$. The size of the up movement is $u = \frac{S_u}{S} = 1 + r_u = 1.5$, and the size of the down movement is $d = \frac{S_d}{S} = 1 + r_d = 0.5$ (where r_u is the rate of an up movement and r_d is the rate of a down movement). For there to be no arbitrage opportunities we need the inequality to hold $u > 1 + r > d$ (assuming a risk free rate of 5%, as before). With probability q the price will rise over a given time period T, and with probability $(1-q)$ it will fall. The price of the call option is contingent on the price of the stock. With a strike of $K = \$110$, the value of the call option in the up state is $f_u = max\left(S_u - K, 0\right) = \$40$, and in the down state it is $f_d = max\left(S_d - K, 0\right) = 0$ (see Figure 3).

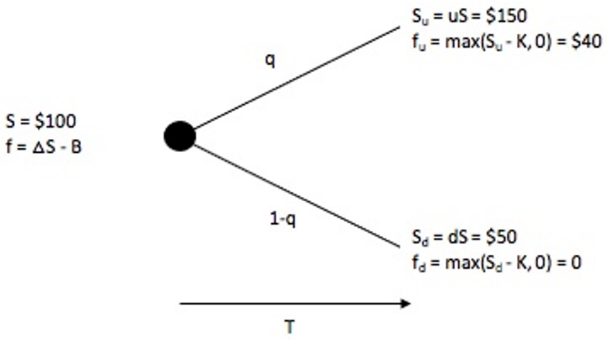


Figure 3: A One Step Binomial Tree.

For the portfolio to have the same return as the option in time T (1 year), we let the value of the portfolio in both the up and down states equal the value of the option in each state.

$$\begin{aligned} \Delta uS - (1+r)^T B &= f_u \\ \Delta dS - (1+r)^T B &= f_d \end{aligned} \tag{11}$$

By solving for the two unknowns we can find how many units of the underlying stock Δ we need to hold, as well as the dollar amount of loan B we will need, to retain a perfectly hedged portfolio.

$$\Delta = \frac{f_u - f_d}{(u-d)S} \qquad \text{and} \qquad B = \frac{df_u - uf_d}{(u-d)(1+r)}$$

Now $\Delta = \frac{\$40-1.5}{(1.5-0.5)(\$100)} = 0.4$ and $B = \frac{0.5(\$40)-1.5(0)}{(1.5-0.5)(1+0.05)} \approx \$19.05$. Therefore, we need to hold 0.4 shares in the underlying stock and take a short position for approximately $19.05 to replicate one call option over this particular period. Hence, the price of the call option today, that gives it's holder the right to buy the underlying stock at time T, is

$$f = \Delta S - B = 0.4(\$100) - \$19.05 = \$20.95$$

The number of shares we need to hold (Δ) is called the hedge ratio or the option's delta. The equation obtained above is the discrete value of delta. In reality there is a need to adjust a portfolio from time to time in order to remain a perfectly hedged portfolio. This is known as dynamic hedging. Of course it is not possible to buy 0.4 units of shares, but usually investors do not construct a portfolio to hedge only one call option. It would be more realistic to have 100 calls on a stock. In that case, one would need to buy 40 units of the underlying stock [11] [24] [29].

4.1.2 Risk Neutral Valuation

We can price an option by assuming a risk neutral world, although we know that the real world is not risk free. This is an important assumption in option valuation where we assume that investors are risk neutral and do not require higher expected rate of return to compensate for an increased risk. When we move from a risk neutral world to reality two things happen. The expected growth rate in the underlying asset changes and the discount rate changes. These two changes always offset each other. Thus by assuming risk neutrality we can obtain the right price of a derivative in the world we live in, not only in the risk neutral world [24].

In risk neutral valuation we assume that the expected rate of return on the investment is the risk free rate and that the present value of the expected cash flow can be discounted at the risk free rate. Risk neutral valuation builds on the principles of a replicating portfolio discussed in the previous section. Now we rearrange the terms in Eq.11, such that we can construct a portfolio with a long position in an option and a short position in Δ units of the underlying asset (see Figure 4).

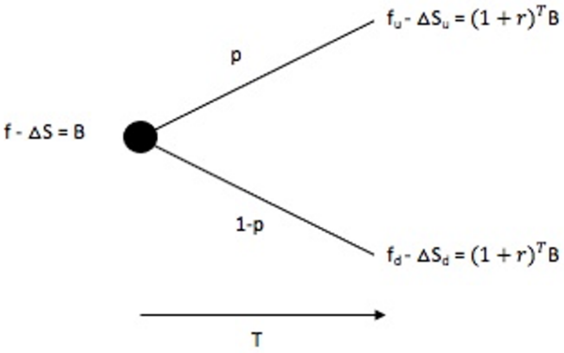

$f_u - \Delta S_u = (1+r)^T B$
p
$f - \Delta S = B$
1-p
$f_d - \Delta S_d = (1+r)^T B$
T

Figure 4: A Riskless Hedge.

Furthermore, to avoid arbitrage opportunities the portfolio must earn the risk free rate. If the portfolio's return were higher than the risk free rate one could borrow money at the risk free rate and buy the portfolio earning a higher rate of return. If its return were lower, one could short the portfolio and invest its money at the risk free rate. Thus, the value of the portfolio today should equal it's value at time T, discounted to the present at the risk free rate. In the previous section, we found delta (Δ) such that it does not matter whether the price goes up or down in one period. Thus we can let the value of the portfolio today equal the value of either of the two portfolios at time T.

$$
\begin{aligned}
f-\Delta S &= (f_u-\Delta uS)(1+r)^{-T}\\
f &= f_u(1+r)^{-T}+\Delta S\left(1-u(1+r)^{-T}\right)\\
f &= f_u(1+r)^{-T}+\left(\frac{f_u-f_d}{S(u-d)}\right)S\left(1-u(1+r)^{-T}\right)\\
f &= \frac{f_u(1+r)^{-T}(u-d)}{(u-d)}+\frac{(f_u-f_d)\left(1-u(1+r)^{-T}\right)}{(u-d)}\\
f &= \frac{f_u u(1+r)^{-T}-f_u d(1+r)^{-T}+f_u-f_u u(1+r)^{-T}-f_d+f_d u(1+r)^{-T}}{(u-d)}\\
f &= \frac{f_u\left(1-d(1+r)^{-T}\right)+f_d\left(u(1+r)^{-T}-1\right)}{(u-d)}\\
f &= \frac{f_u\left(1-d(1+r)^{-T}\right)+f_d\left(u(1+r)^{-T}-1\right)}{(u-d)}
\end{aligned}
$$

By taking $(1+r)^{-T}$ outside the parentheses we obtain

$$f=f_u(1+r)^{-T}\left(\frac{(1+r)^T-d}{u-d}\right)+f_d(1+r)^{-T}\left(\frac{u-(1+r)^T}{u-d}\right)$$

And by letting

$$p=\frac{(1+r)^T-d}{u-d}\quad\text{and}\quad(1-p)=\frac{u-(1+r)^T}{u-d}\tag{12}$$

We get

$$f=\left[pf_u+(1-p)f_d\right]\left(1+r\right)^{-T}\tag{13}$$

This result of risk neutral valuation gives the correct value for the price of an option. Hence, the price of an option is the PV of its expected payoff over some time period T, where p is the riskless probability of an up movement, and $(1-p)$ is the riskless probability of a down movement. The reader can verify that the risk neutral valuation approach will result in the same price of the option calculated under the replicating portfolio in Section 4.1.1.

In a risk neutral world, p is the riskless probability of an up movement and the growth rate of the underlying asset (e.g. expected return from a stock) is the risk free rate. Solving the equation for the expected value of a stock price at time T, we get

$$
\begin{aligned}
E[S_T] &= puS+(1-p)dS\\
&= puS+dS-pdS\\
&= pS(u-d)+dS
\end{aligned}
$$

By substituting $p = \frac{(1+r)^T - d}{u-d}$, we obtain

$$
\begin{aligned}
&= \Big(\frac{(1+r)^T - d}{u-d}\Big) S(u-d) + dS \\
&= S(1+r)^T
\end{aligned}
$$

Thus we have shown that the expected value of the stock at some future time T is $E[S_T] = S(1+r)^T$ with an expected growth rate of $\frac{S_T}{S} = (1+r)^T$, in a risk neutral world [24] [29] [30].

4.1.3 Extending the Number of Periods

The valuation approach in the previous section considers a one period binomial tree. This can be extended to an increased number of periods. If we divide the option's time to maturity into n intervals of equal length, we have obtained a binomial tree with n periods. A multiple period binomial tree is solved in the same way as the one step tree, starting from the end date and working backwards towards the present. The general multiplicative binomial formula for pricing options with n periods (here a call option), is given by

$$
f = \frac{\sum_{j=0}^{j=n} \frac{n!}{j!(n-j)!} p^j (1-p)^{n-j} max\big(u^j d^{n-j} S - K, 0\big)}{(1+r)^n} \tag{14}
$$

The first component of the equation $\frac{n!}{j!(n-j)!} p^j (1-p)^{n-j}$ is simply the binomial distribution. It calculates the probability that the stock's price will move up j times over n periods, where each up movement happens with probability p. The other component of the equation, $max\big(u^j d^{n-j} S - K, 0\big)$, represents the payoff from a call option at maturity. It takes into account how many times the price S moves up by the amount u, and down by the amount d, over the n periods. Finally, the expected payoff at maturity is summed up for all the possible option values, multiplied with their probabilities, and discounted by the risk free rate. In this way, we are able to calculate the value of the option today (f). When the number of periods approaches infinity, the discrete time binomial distribution converges to the cumulative standard normal distribution and the above formula converges to the continuous time BSM formula (see Section 6.5) [11].

When $n \Rightarrow \infty$, the discrete rate $\big(1 + \frac{r}{n}\big)^n$, where n is the number of periods, converges to the continuously compounded rate e^r. When using continuous compounding the parameters u, d, p, f and $E[S_T]$ become [24]

$$
u = e^{\sigma\sqrt{\Delta t}} \quad \text{and} \quad d = e^{-\sigma\sqrt{\Delta t}} = \frac{1}{u} \tag{15}
$$

$$
p = \frac{e^{rT} - d}{u-d} \tag{16}
$$

$$
f = \big[p f_u + (1-p) f_d\big] e^{-rT} \tag{17}
$$

$$
E[S_T] = S e^{rT} \tag{18}
$$

4.1.4 Estimating the Up/Down Movements

The parameters u and d should be chosen so that they match the volatility of the underlying asset. As mentioned in Section 2.1, the volatility of an asset is a measure of the variance in it's returns in a small time interval ($\sigma^2 \Delta t$). The variance of a variable X is defined as $E(X^2) - [E(X)]^2$, where E is the expected value. Therefore, when the return from the underlying asset is $(u-1)$ with probability p, and $(d-1)$ with probability $(1-p)$, we can mach the volatility of the underlying asset as follows

$$\begin{aligned} &E(X^2) - [E(X)]^2 = \sigma^2 \Delta t \\ &\Rightarrow p(u-1)^2 + (1-p)(d-1)^2 - [p(u-1) + (1-p)(d-1)]^2 = \sigma^2 \Delta t \end{aligned}$$

Substituting for $p = \frac{e^{rT} - d}{u-d}$, this equation simplifies to

$$e^{r\Delta t}(u+d) - ud - e^{2r\Delta t} = \sigma^2 \Delta t$$

Using Taylor series expansion for the exponential function $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} ...$ ignoring Δt^2, and higher powers of t, the above equation simplifies to

$$(1 + r\Delta t)(u+d) - ud - (1 + 2r\Delta t) = \sigma^2 \Delta t$$

Now we can assume that $u = e^x$ and $d = e^{-x}$, so that $ud = 1$, and

$$\begin{aligned} (1 + r\Delta t)(u+d) - 1 - (1 + 2r\Delta t) &= \sigma^2 \Delta t \\ (u+d) &= \frac{\sigma^2 \Delta t}{1 + r\Delta t} + 2 \end{aligned}$$

By using the Taylor series for u and d, and ignoring x^3 and higher powers of x, we get

$$\begin{aligned} u = e^x &= 1 + x + \frac{x^2}{2!}..... \\ d = e^{-x} &= 1 - x + \frac{x^2}{2!}..... \\ (u+d) \approx 2 + 2\frac{x^2}{2!} &\approx 2 + x^2 \approx \frac{\sigma^2 \Delta t}{1 + r\Delta t} + 2 \\ \Rightarrow x &\approx \frac{\sigma\sqrt{\Delta t}}{\sqrt{1 + r\Delta t}} \end{aligned}$$

Now we can use the Maclaurin series for $f(x) = \frac{1}{\sqrt{1+x}}$ up to the second derivative, $f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0)...$ to obtain an approximation of the following function

$$\frac{1}{\sqrt{1 + r\Delta t}} = 1 - \frac{1}{2} r\Delta t + \frac{3}{8}(r\Delta t)^2$$

Hence, the approximated value of x becomes

$$\begin{aligned} x &\approx \frac{\sigma\sqrt{\Delta t}}{\sqrt{1+r\Delta t}} \approx \sigma\sqrt{\Delta t}(1-\frac{1}{2}r\Delta t+\frac{3}{8}(r\Delta t)^2) \\ &\approx \sigma\sqrt{\Delta t}-\frac{1}{2}\sigma r(\Delta t)^{3/2}+\frac{3}{8}\sigma r^2(\Delta t)^{5/2} \\ &\Rightarrow x \approx \sigma\sqrt{\Delta t} \end{aligned}$$

Therefore, the values for the up and down movements of the underlying asset become a function of volatility and time as follows

$$u = e^{\sigma\sqrt{\Delta t}} \quad \text{and} \quad d = e^{-\sigma\sqrt{\Delta t}} = \frac{1}{u}$$

Note, when we move from a risk neutral world to the real world the volatility (σ) stays the same. The only thing that changes is that $r = \mu$. Thus, we don't use the risk neutral probabilities but the objective probabilities, given by

$$p^* = \frac{e^{\mu\Delta t}-d}{u-d}$$

Working through the same steps as before with $p*$ as the probability of an up movement, we end up with the same equations for u and d. Therefore, we can conclude that the size of the up and down movements are independent of the expected return and will be the same in all worlds [24].

4.2 Comparing DTA and ROV

The DTA is applicable when risk is mainly firm specific and market risk is insignificant. On the other hand, DTA tends to undervalue the investment opportunity when market risk is significant. The outcome from a $R\&D$ program is an example of specific risk since it is not affected by what happens in the overall economy. The market risk is the commercial risk factor of the potential future cash flow of a fully developed and marketable product. To obtain the correct value for an investment opportunity when commercial risk is significant we need to use ROV, assuming risk neutrality [10]. To illustrate this point we will proceed with our previous example in Section 3.

Example. Now we assume that the annual volatility in the expected cash flow is $\sigma = 47\%$ (the average volatility in firm value of a publicly traded biotechnology firm [8]). The expected annual cash flow from the 6th year until the patent expires in twenty years is \$$0.70 million$. The expected cash flow is discounted at a 5% risk free rate with a $PV = \$5.69 million$, as calculated in Section 2. From Eq.15 we can calculate the sizes of the up and down movements as follows

$$u = e^{\sigma\sqrt{\Delta t}} = e^{0.47\sqrt{2.5}} \approx 2.10 \quad \text{and} \quad d = e^{-\sigma\sqrt{\Delta t}} = \frac{1}{u} \approx 0.48$$

Since this risk over the expected cash flow is not diversifiable we calculate the risk neutral probabilities as presented in Eq.12.

$$p = \frac{(1+r)^t - d}{u-d} = \frac{(1+0.05)^{2.5} - 0.48}{2.10 - 0.48} = 0.4 \quad \text{and} \quad 1-p = 0.6 \tag{19}$$

The probability of an up movement in the expected cash flow is $p = 0.4$, and the probability of a down movement is $(1-p) = 0.6$. The probabilities of success and failure in the $R\&D$ phases do not need to by adjusted to risk neutral probabilities since their risk is diversifiable. Therefore, we have the same probabilities as in our previous example, $q_1 = 0.5$ and $(1-q_1) = 0.5$ for the first phase of $R\&D$, and $q_2 = 0.7$ and $(1-q_2) = 0.3$ for the second phase. When working through the binomial tree we start with the future value of the underlying asset at t_5 and work back, discounting towards the present. The evolution of the cash flow can be seen in Table 3.

Table 3: The Evolution of the Cash Flow ($ in millions).

$PV_u = uPV = \$11.97$ $\quad PV_d = dPV = \$2.71$

$PV_{uu} = u^2PV = \$25.17$ $\quad PV_{ud} = udPV = \$5.69$

$PV_{du} = duPV = \$5.69$ $\quad PV_{dd} = d^2PV = \$1.29$

In the 5th year (t_5) the management has the option to invest $I_5 = \$0.85 million$ in production and marketing. If the expected cash flow at each node exceeds the cost of this investment, the option is kept alive. Otherwise, it expires worthless. The payoffs at each node are calculated as follows

$$\begin{aligned} f_{uu} &= max\left(PV_{uu} - I_5, 0\right) = max\left(\$25.17 - \$0.85, 0\right) = \$24.32 million \\ f_{ud} &= max\left(PV_{ud} - I_5, 0\right) = max\left(\$5.69 - \$0.85, 0\right) = \$4.84 million \\ f_{du} &= max\left(PV_{du} - I_5, 0\right) = max\left(\$5.69 - \$0.85, 0\right) = \$4.84 million \\ f_{dd} &= max\left(PV_{dd} - I_5, 0\right) = max\left(\$1.29 - \$0.85, 0\right) = \$0.44 million \end{aligned}$$

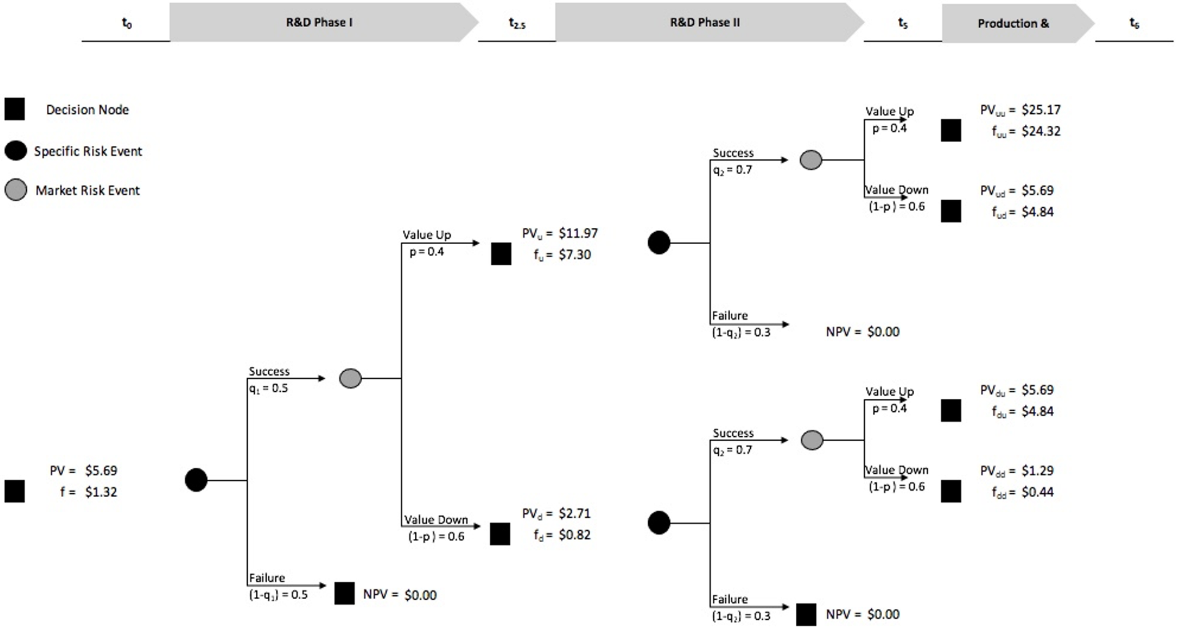
At $t_{2.5}$ the management has the option to invest $I_{2.5} = \$0.55 million$ and begin the second $R\&D$ phase. The value of the option is the probability weighted expected payoff from the 5th year, calculated with the risk neutral probabilities. The total payoff is then multiplied with the objective probabilities that apply to the second $R\&D$ phase.

$$\begin{aligned} f_u &= max\Big(\big(q_2[pf_{uu} + (1-p)f_{ud}] + (1-q_2)(\$0)\big)\big(1+r\big)^{-t} - I_{2.5}, 0\Big) \\ &= max\Big(\big(0.7[0.4(\$24.32) + 0.6(\$4.84)] + 0.3(\$0)\big)\big(1+0.05\big)^{-2.5} - \$0.55, 0\Big) \\ &= \$7.30 million \end{aligned}$$

$$\begin{aligned} f_d &= max\Big(\big(q_2\left[pf_{du}+(1-p)f_{dd}\right]+(1-q_2)(\$0)\big)(1+r)^{-t}-I_{2.5},0\Big) \\ &= max\Big(\big(0.7\left[0.4(\$4.84)+0.6(\$0.44)\right]+0.3(\$0)\big)(1+0.05)^{-2.5}-\$0.55,0\Big) \\ &= \$0.82 million \end{aligned}$$

When we have calculated the values of the options, f_u and f_d in $t_{2.5}$, we can obtain the value of the option to invest $I_0 = \$0.20 million$ today. We do this in the same manner as before but now using the objective probabilities that apply to the first phase of the $R\&D$ program (see Figure 5).

$$\begin{aligned} f &= max\Big(\big(q_1\left[pf_u+(1-p)f_d\right]+(1-q_1)(\$0)\big)(1+r)^{-t}-I_0,0\Big) \\ &= max\Big(\big(0.5\left[0.4(\$7.30)+0.6(\$0.82)\right]+0.5(\$0)\big)(1+0.05)^{-2.5}-\$0.20,0\Big) \\ &\approx \$1.32 million \end{aligned}$$



t0
R&D Phase I
t2.5
R&D Phase II
t5
Production &
t6
Decision Node
Specific Risk Event
Market Risk Event
PV = $5.69
f = $1.32
Success
q1 = 0.5
Failure
(1-q1) = 0.5
NPV = $0.00
Value Up
p=0.4
PVu = $11.97
fu = $7.30
Value Down
(1-p) = 0.6
PVd = $2.71
fd = $0.82
Success
q2 = 0.7
Failure
(1-q2) = 0.3
NPV = $0.00
Value Up
p=0.4
PVuu = $25.17
fuu = $24.32
Value Down
(1-p) = 0.6
PVud = $5.69
fud = $4.84
Success
q2 = 0.7
Value Up
p=0.4
PVdu = $5.69
fdu = $4.84
Value Down
(1-p) = 0.6
PVdd = $1.29
fdd = $0.44
Failure
(1-q2) = 0.3
NPV = $0.00

Figure 5: ROV with Market Risk of $\sigma = 47\%$ ($ in millions).

Now we have obtained the value of the option to invest in the project. This value of $\$1.32 million$ is significantly higher than the standard NPV of $-\$0.64 million$. As well, it is greater than the NPV of $\$0.04 million$, calculated contingent on the success of the $R\&D$ phases modeled with the DTA (see Section 3). Therefore, we can see that when there is an uncertainty (σ) over the expected cash flow, the value of the option to invest in the project increases. Furthermore, DTA tends to undervalue the investment opportunity while ROV results in a correct value [10].

4.3 Examples of Real Options

There are various types of real options that can be used to value an investment opportunity. In this section we will introduce the options that might be applicable in the valuation of a young patented asset.

The Option To Invest. The option to invest gives it's holder the right to wait or delay an investment until uncertainty has been resolved and only invest if conditions turn out favorable. It is similar to an American call option with a strike price (I) equal to the cost of the investment and the underlying asset (V) equal to the PV of expected cash flow, $f = max(V - I, 0)$. This option is embedded in many projects but it is most valuable in projects that have an exclusive right to invest, such as in the case of a patent [8] [11].

The Option to Expand. The option to expand provides its holder the right to scale up it's project by a certain percentage amount $x\%$, for a certain additional cost I_x. It can be exercised e.g. if market conditions turn out more favorable than expected. This option can be viewed as a call option where the underlying asset is the value of the expansion ($x\%V$) and the strike price is the cost of expanding (I_x). The expansion value needs to be compared with the pre-expansion value (V) and the value of the option becomes $f = max(x\%V - I_x - V, 0)$ [31]. This option can be valuable when one wants to take advantage of introducing a new product into an uncertain developing market. It allows for a cautious beginning and expanding if conditions turn out favorable [11].

The Option to Contract. Another option that allows for altering the scale of production is the option to contract if market conditions turn out weaker than expected. This option can be valuable in the case of a patented project since it gives its holder the right to reduce the scale of production by a certain percentage amount $c\%$. By doing this one can save a part of the estimated investment cost I_c. This option is similar to a put option where the underlying asset is the size of the contraction and the strike is the cost that will be saved $f = max(I_c - c\%V, 0)$. Like the option to expand, this option can be valuable when a new product is introduced into an uncertain market [11].

The Option to Abandon. The option to abandon is valuable if operations go poorly and the firm might want to abandon the project for some salvage value. An example is to sell equipment, or sell the patent itself, to another party that is more capable of finishing the process of production and commercialization. This option is similar to an American put option on the value of the underlying project (V) with a strike price (A) equal to the salvage value, $f = max(A - V, 0)$. The value of the investment opportunity with an abandonment option becomes $V + max(A - V, 0)$ or $f = max(A, V)$. Note that we would only abandon if the salvage value exceeds the value of the underlying asset [11].

The Option to Default. This option is valuable in most real life $R\&D$ projects. It is seldom that an investment is made in a single up front payment. Most often investments are made in stages and if the first stage turns out successful this option gives the right to invest and move on to the next stage. If the first stage fails on the other hand, we will default. Therefore, this option can be valued as a compound option or as a sequential option. If a project has ten stages the 5th option would depend on the first four stages and the 10th option would depend on the previous nine stages. Whether you reach the last stage, is a function of whether you make it through the previous stages [8] [11].

The Option to Switch. The option to switch offers a range of flexibility. An example is to maintain a relationship with a couple of suppliers and switch between them as their prices change. Another example would be to switch between production facilities in different countries to respond to changes in costs, exchange rates, or other market conditions. In the case of a patented product that faces a volatile market the ability to switch between inputs or outputs in the companies production may be of great value. An increased flexibility may be worthwhile in responding to changing market demand and being able to differentiate product's mix, shape, or size [11].

Growth Options. Growth options are found in all $R\&D$ programs. When a new invention is in its first stages of research, testing, and development, the project involves high initial costs and often insufficient expected cash inflows. The NPV is often negative and the investment opportunity appears unattractive. In such cases, clever managers observe future growth opportunities. If the product is successfully developed and commercialized it might open access to new markets, increase the company's competitive advantage, be the first in a series of other higher quality products, or the foundation of a new invention. If an initial investment is not made these future opportunities may be lost to competitors. Therefore, growth options are call options on the future potential commercial project [11].

5 The Option To Invest

The option to invest or wait is embedded in many projects but is most valuable in projects where there is an exclusive right to invest, such as in the case of patents. The option to delay an investment project can be valuable when a project has a negative NPV today but might have a positive NPV in the future. It might also be worth waiting even though the current NPV is positive [8]. Most investment opportunities have the following three characteristics in common. Firstly, they are partly or completely irreversible. If you change your mind you can not fully recover the initial investment, it is at least partially sunk cost. Secondly, their future return is uncertain, and finally, their timing can be controlled. Thus every investment opportunity comes with an option, namely the option to invest. When an investment is completely irreversible and the level of uncertainty is high, this option to control the timing of your investment increases in value [28].

In Section 2.1.3, we had an estimate of beta by assuming that a twin security existed in a market. In reality it's hard to find a twin security for a patented product especially if it's a new unique invention. Hence, in this section we will take advantage of the risk neutral assumption of the option pricing framework. We will assume that all risk is diversifiable (no market risk, $\beta = 0$) such that the future expected cash flow can be discounted at the risk free rate.

Example. Now five years have passed from the initial valuation of the investment opportunity where the management decided to go ahead with the project. The first phase of the $R\&D$ program was successful but several problems have arisen in the second phase. Hence the final product won't be completed until in the 6th year. Recall, it was estimated that the product would be completed in the 5th year and production and marketing could start in the same year. The first revenues from the product sales were estimated in the 6th year and until the patent expired. Unfortunately, there will be a one year delay of completing the final product.

The management needs to decide whether to invest in production and marketing without having the final product completed. There is a need to decide if it is worth waiting for the $R\&D$ outcomes and delay market entry by one year (to the 7th year from start). The management has also realized that the cost required to produce and introduce the product into the market will be much higher than initially estimated or $\$4.70 million$. There is a probability of $q = 0.5$ that the final product will be successful generating an annual cash flow of $\$1.05 million$. There is also an equal probability that the product will not be favorable with an annual cash flow of $\$0.35 million$. At this stage, the investment costs that have already been paid I_0 and $I_{2.5}$, are sunk costs (irreversible) and therefore, should not be taken into the valuation. The expected cash flow today is $C_0 = \$0.70 million$ and will be the same until the patent expires $\big(0.5(\$1.05) + 0.5(\$0.35) = \$0.70\big)$. The NPV from today's investment as opposed to delaying the investment for one year are presented in Figure 6.

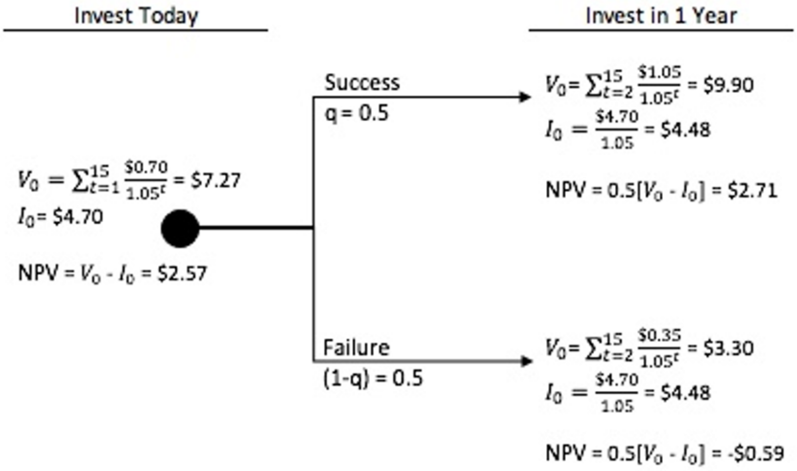

Invest Today
Invest in 1 Year
Success
q = 0.5
Failure
(1-q) = 0.5
$V_0 = \sum_{t=1}^{15} \frac{\$0.70}{1.05^t} = \$7.27$
$I_0 = \$4.70$
NPV = V_0 - I_0 = $2.57
$V_0 = \sum_{t=2}^{15} \frac{\$1.05}{1.05^t} = \$9.90$
$I_0 = \frac{\$4.70}{1.05} = \$4.48$
NPV = 0.5[V_0 - I_0] = $2.71
$V_0 = \sum_{t=2}^{15} \frac{\$0.35}{1.05^t} = \$3.30$
$I_0 = \frac{\$4.70}{1.05} = \$4.48$
NPV = 0.5[V_0 - I_0] = -$0.59

Figure 6: The NPV of Investing Today and of Investing in 1 Year.

The NPV of the project, if the management decides to invest today, before knowing the outcome of the $R\&D$ phase, is $\$2.57 million.$ In one year the management has the choice to invest $\$4.70 million,$ if the annual expected cash flow has risen to $\$1.05 million.$ The probability of this happening is $q = 0.5$ and results in a NPV of $\$2.71 million.$ If the cash flow falls, the NPV becomes $-\$0.59 million$ and the management will not invest. We can see, that even though the NPV of the project today is positive, it is worth waiting under the assumption of investing if the cash flow rises. By investing today we kill the opportunity cost of waiting, which equals the difference between the NPV of waiting and the standard $NPV : \$2.71 - \$2.57 = \$0.14 million.$ This means that we are willing to pay $\$0.14 million$ for receiving an investment opportunity that is flexible, instead of one that only allows for an immediate investment.

Another way to look at the value of the flexibility is to consider the investment cost we would be willing to pay, for a flexible investment, rather than an immediate one.

$$NPV = 0.5\Big(\frac{-\$I}{1.05} + \sum_{t=2}^{t=15} \frac{\$1.05}{(1.05)^t}\Big) = \$2.57 million$$

By solving for I we get $I \approx \$5.00 million.$ Therefore the opportunity to invest for $\$4.70 million$ immediately, has the same value as the opportunity to invest now or next year for $\$5.00 million.$ Thus we are willing to pay more for an investment opportunity that is flexible.

If we didn't have the option to wait and our only choice was to invest today or never invest, the standard NPV approach would have been satisfactory and we would have invested in the project today with a NPV of $\$2.57 million.$ If our investment would have been reversible, there would have been no reason to wait. In that case we could have recovered our investment if the cash flow would have fallen to $\$0.35 million.$ However, when we have the ability to control the timing of our investment and our investment is irreversible, there is an additional opportunity cost to investing [28].

5.1 Analogy to Financial Options

The option to invest is similar to a financial call option on a common stock. A call option gives the holder the right to buy the underlying asset (e.g. stock) at a predetermined exercise price. In comparison, when you hold an investment opportunity you have the right to invest in the project for an investment cost. In both cases if you exercise your option you will receive the underlying asset (stock/investment opportunity) for which the future price/value is uncertain [28].

In this section we will look at the relationship between a financial call option and the real option to invest. We will recalculate the value of our investment opportunity ($NPV = \$2.71 million$) using the option pricing framework (see Section 4.1.2). In reality, it is unlikely that the underlying asset, in the case of a young patented asset, is traded in the market such that one could take a short position and construct a replicating portfolio. Taking a short position means that we could borrow the asset from another party and sell it in the market. However, we will assume that this would be possible and that we could construct a risk free portfolio with an option in our investment opportunity (F_t) and a short position in Δ units of the underlying asset. At time t the value of this portfolio is

$$\Pi_t = F_t - \Delta C_t$$

where C_t is the expected cash flow at time t. Now if the annual cash flow rises to $\$1.05 million$ the PV of the option to invest in one year will be

$$F_1 = -\$4.70 + \sum_{t=1}^{t=14} \frac{\$1.05}{(1.05)^t} \approx \$5.69 million$$

If the expected cash flow declines to $\$0.35 million$ we will not invest, so the value of the option in that case, is zero. In order to calculate the value of the option today F_0, we can construct a replicating portfolio. For the portfolio to be risk free we need to make the assumption that it earns the risk free rate as discussed in Section 4.1.2. The value of the portfolio today is $\Pi_0 = F_0 - \Delta \$0.70$. The value of the portfolio in one year is either $\Pi_1 = F_1 - \Delta \$1.05$ in the favorable state or $\Pi_1 = 0 - \Delta \$0.35$ in the unfavorable state. We need to find Δ such that we have an equal outcome whether the value of the underlying asset goes up or down in one year.

$$\begin{aligned} \Pi_1 &= F_1 - \Delta \$1.05 = 0 - \Delta \$0.35 \\ &\Rightarrow \Delta\big(\$1.05 - \$0.35\big) = \$5.69 \\ &\Rightarrow \Delta = 8.13 \end{aligned}$$

Taking a short position in 8.13 units of the underlying project the value of our portfolio in one year becomes

$$\begin{aligned} \Pi_1 &= F_1 - \Delta \$1.05 \\ &= \$5.69 - 8.13(\$1.05) \\ &= -\$2.85 \end{aligned}$$

Because the expected rate of capital gain in the cash flow is zero (the expected cash flow is $\$0.70 million$ every year) no investor would hold a long position without expecting to earn at least the risk free rate. Therefore, the cost of holding the short position equals the risk free rate and the long position receives an annual payment of $0.05\Delta C_0 = 0.05\big(8.13(\$0.70)\big) = \$0.28 million$. The portfolio's return is the capital gain $(\Pi_1 - \Pi_0)$ minus the payment required for holding the short position.

$$\begin{aligned} Return &= \Pi_1 - \Pi_0 - \$0.28 \\ &= \Pi_1 - \big(F_0 - \Delta\$0.70\big) - \$0.28 \\ &= -\$2.85 - F_0 + \big(8.13(\$0.70)\big) - \$0.28 \\ &= \$2.56 - F_0 \end{aligned}$$

Since the portfolio must earn the risk free rate, it's return needs to equal $0.05\Pi_0$. Thus the value of our investment opportunity today becomes

$$\begin{aligned} \$2.56 - F_0 &= 0.05\Pi_0 \\ \$2.56 - F_0 &= 0.05\big(F_0 - \Delta\$0.70\big) \\ 1.05F_0 &= \$2.56 + 0.05\big(8.13(\$0.70)\big) \\ F_0 &= \frac{\$2.85}{1.05} = \$2.71 million \end{aligned}$$

The value of the option to invest is $F_0 = \$2.71 million$. We have been able to value our investment opportunity, in the same way as a financial call option on a common stock. By assuming that the underlying asset is traded we were able to construct a risk free portfolio so that it would not matter whether the expected cash flow went up or down in one year.

When the underlying asset of an investment opportunity is not traded, or there is no perfectly correlated asset in the market we can not follow the option pricing theory. However, the contingent NPV calculation under the assumption of risk neutrality resulted in the same value of the option to invest (see Figure 6). The approach of choosing the higher NPV from investing today or waiting under the strategy of only investing if the cash flow rises, is essentially dynamic programming. Hence when the assumption holds that risk is diversifiable and the expected cash flow can be discounted at the risk free rate, option pricing and dynamic programming yield the same answer [28].

In context to the net payoff profile from a call option illustrated in Figure 2, F_0 is the price you would need to pay to receive the option (earlier this cost was noted as c for a call and p for a put). Therefore, to break even the value of the underlying asset V_0 needs to exceed both the strike price I and the cost of the option $c = F_0$. On the other hand, the standard NPV approach states that when V_0 exceeds the cost of the investment, $NPV > 0$ and an investment should be made. It ignores the opportunity cost of waiting that is incorporated in the option value F_0, indicating that one should not invest unless $V_0 > I + F_0$. Hence, the full cost of investing today should be $I + F_0 = \$4.70 + \$2.71 = \$7.41 million$, which results in a totally different NPV of $\$7.27 - \$7.41 = -\$0.14 million$, in comparison to the one calculated before $\$7.27 - \$4.70 = \$2.57 million$ (see Figure 6).

5.2 Characteristics of the Option to Invest

Now when we have shown that an investment opportunity can be treated as the option to invest, and that this option is analogous to a financial call option, we can examine how each variable affects the value of this option. We will proceed with the assumptions given in Section 5. The investment needed for production and commercialization is $I = \$4.70 million$. The expected cash flow today is $C_0 = \$0.70 million$ (in the 6th year from initial start) and the risk free interest rate is $r = 5\%$. There is an uncertainty of 50% over the cash flow. There is a probability of $q = 0.5$ that the cash flow will rise to $\$1.05 million$ $\left(uC_0 = 1.5(\$0.70) = \$1.05\right)$ and a probability of $(1-q) = 0.5$ that it will fall to $\$0.35 million$ $\left(dC_0 = 0.5(\$0.70) = \$0.35\right)$. Hence, the expected cash flow until the patent expires is $\$0.70 million\left(0.5(\$1.05) + 0.5(\$0.35) = \$0.70\right)$. We will use the same replicating portfolio as introduced in Section 5.1 to see how changing the investment cost (I), the expected cash flow from the underlying product (C_0), the magnitude of the up/down movements in the cash flow next period, and the probability of those movements $\left(q \text{ and } (1-q)\right)$ will affect the value of the option to invest [28].

5.2.1 Changing the Investment Cost

When the cost of the investment (I) is not constant but can vary, the PV of our option in one year in the favorable state (cash flow rises to $\$1.05 million$), becomes a function of I.

$$F_1 = -I + \sum_{t=1}^{t=14} \frac{\$1.05}{(1.05)^t} \approx \$10.39 - I$$

Recall, if the expected cash flow declines to $\$0.35 million$ we will not invest, so the value of the option in that case, is zero. Hence, the value of the portfolio in one year is either $\Pi_1 = F_1 - \Delta\$1.05$ in the favorable state or $\Pi_1 = 0 - \Delta\$0.35$ in the unfavorable state. To maintain the risk free portfolio as earlier stated, we need to find how many units of the underlying project (Δ) we would need to hold such that it will not matter whether the expected cash flow rises or falls.

$$\begin{aligned}
\Pi_1 &= F_1 - \Delta\$1.05 = 0 - \Delta\$0.35 \\
&\Rightarrow \Delta\left(\$1.05 - \$0.35\right) = \$10.39 - I \\
&\Rightarrow \Delta = \frac{\$10.39}{0.7} - \frac{I}{0.7} \\
&\Rightarrow \Delta = \$14.85 - \frac{I}{0.7}
\end{aligned}$$

The value of the portfolio today therefore becomes $\Pi_0 = F_0 - \Delta\$0.70 = F_0 - \left(\$10.39 - I\right)$ and since we need to take a short position in $\Delta = \$14.85 - \frac{I}{0.7}$ units of the product the value of our portfolio in one year is

$$\begin{aligned}
\Pi_1 &= F_1 - \Delta\$1.05 \\
&= \left(\$10.39 - I\right) - \left(\$14.85 - \left(\frac{I}{0.7}\right)\right)\$1.05 \\
&= \$10.39 - I - \$15.59 + 1.5I \\
&= 0.5I - \$5.20
\end{aligned}$$

The cost of holding the short position equals the risk free rate, so the long position receives an annual payment of $0.05\Delta C_0 = 0.05(\$14.85 - \frac{I}{0.7})\$0.70 = \$0.52 - 0.05I$. The portfolio's return therefore becomes

$$\begin{aligned}
Return &= \Pi_1 - \Pi_0 - (\$0.52 - 0.05I)\\
&= \Pi_1 - (F_0 - (\$10.39 - I)) - (\$0.52 - 0.05I)\\
&= (0.5I - \$5.20) - (F_0 - (\$10.39 - I)) - (\$0.52 - 0.05I)\\
&\approx \$4.68 - 0.45I - F_0
\end{aligned}$$

Since the portfolio must earn the risk free rate, the return should equal $0.05\Pi_0$ and we can find the value of the investment opportunity today (F_0), as a function of I.

$$\begin{aligned}
\$4.68 - 0.45I - F_0 &= 0.05\Pi_0\\
\$4.68 - 0.45I - F_0 &= 0.05(F_0 - \Delta\$0.70)\\
\$4.68 - 0.45I - F_0 &= 0.05(F_0 - (\$10.39 - I))\\
1.05F_0 &= \$5.20 - 0.5I\\
F_0 &= \$4.95 - \left(\frac{0.5}{1.05}\right)I
\end{aligned}$$

We have obtained the equation for the investment opportunity as a function of the direct investment cost. When $I = \$4.70 million$, the value of the option is $F_0 = \$2.71 million$, and it becomes better to wait than invest today. Now there has to be some value of I that will justify an immediate investment. By solving the following inequality that justifies investing today we can find this value of I. The PV of the expected cash flow if we invested today was $V_0 = \$7.27 million$ (see Figure 6).

$$\begin{aligned}
&V_0 > I + F_0\\
&\quad\Rightarrow \$7.27 > I + \left(\$4.95 - \left(\frac{0.5}{1.05}\right)I\right)\\
&\quad\Rightarrow \$7.27 - \$4.95 > I\left(1 - \frac{0.5}{1.05}\right)\\
&\quad\Rightarrow I < \$4.42 million
\end{aligned}$$

Thus, when $I < \$4.42 million$, lets say $I = \$4.41 million$, the value of the option to wait becomes

$$\begin{aligned}
F_0 &= \$4.95 - \frac{0.5}{1.05}I\\
F_0 &= \$4.95 - \frac{0.5}{1.05}\$4.41\\
F_0 &\approx \$2.85 million
\end{aligned}$$

which is less than the payoff from an immediate investment $V_0 - I = \$7.27 - \$4.41 \approx \$2.86 million$. Therefore, when $I < \$4.42 million$, there is no value in waiting and the option should be exercised today. On the other hand, when $I \geq \$4.42 million$, the value of the option becomes greater than it's payoff. Hence, the optimal investment strategy would be to keep the option alive and wait (see Figure 7) [28].

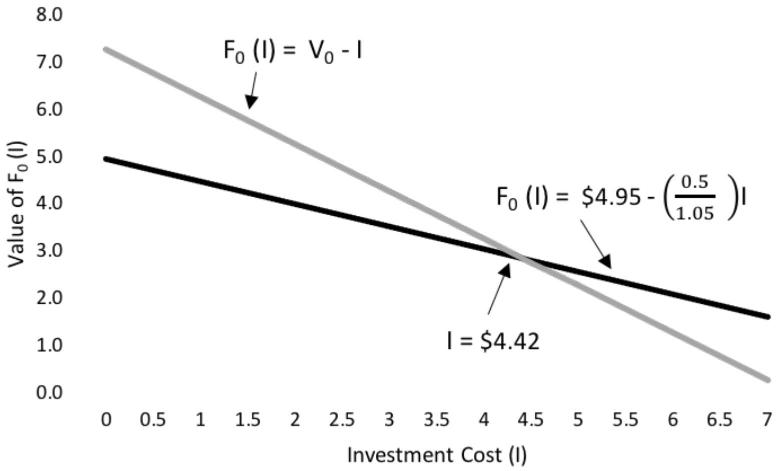


Figure 7: The Option to Invest as a Function of I.

5.2.2 Changing the Cash Flow

Let us consider another scenario where we keep the cost fixed ($I = \$4.70 million$), but we let the initial cash flow vary (C_0). The uncertainty over the cash flow is 50% as before and it will rise/fall with the same probabilities as before of $q = 0.5$ and $(1-q) = 0.5$, respectively. If we invest today (in the 5th year from initially starting the project), the first revenue stream is expected in the next year, before knowing whether the cash flow will rise or fall. In that year, the final product will be completed, uncertainty will be resolved, and the expected cash flow will either rise ($1.5C_0$) or fall ($0.5C_0$). For now, we simplify reality and make the assumption that the cash flow will stay at that level until the patent expires (see Figure 8).

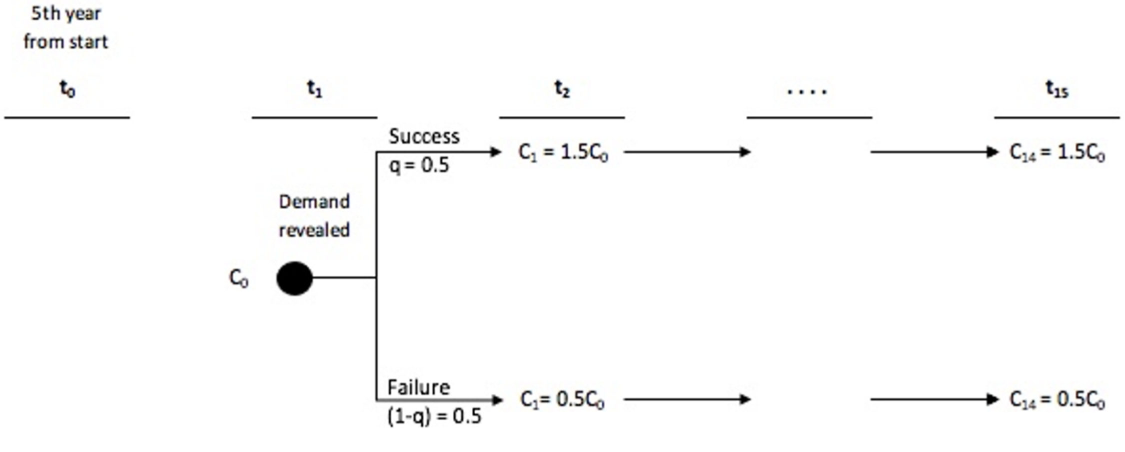


Figure 8: Expected Cash Flow until Patent Expires.

The value of our replicating portfolio today is $\Pi_0 = F_0 - \Delta C_0$. If we wait for one year until the final product is completed we will only invest if the product is successful, market conditions are favorable, and the expected cash flow rises to $C_1 = 1.5C_0$. Now the value of the option in one year (6th year from initial start, t_1), becomes a function of the initial cash flow as follows

$$\begin{aligned} F_1 &= -\$4.70 + \sum_{t=1}^{t=14} \frac{C_1}{(1.05)^t} \\ &= -\$4.70 + \sum_{t=1}^{t=14} \frac{1.5C_0}{(1.05)^t} \\ &\approx -\$4.70 + 14.85C_0 \end{aligned}$$

We will assume that the initial cash flow C_0 is in a range such that if it goes up it will be worth investing but if it goes down the value of the option (F_1) is zero. Therefore, the value of the portfolio in one year will be either $\Pi_1 = F_1 - \Delta 1.5C_0$ or $\Pi_1 = 0 - \Delta 0.5C_0$, and we can find Δ such that it will not matter whether the expected cash flow rises or falls.

$$\begin{aligned} \Pi_1 &= F_1 - \Delta 1.5C_0 = 0 - \Delta 0.5C_0 \\ &\Rightarrow -\$4.70 + 14.85C_0 - \Delta 1.5C_0 = -\Delta 0.5C_0 \\ &\Rightarrow \Delta C_0 = -\$4.70 + 14.85C_0 \\ &\Rightarrow \Delta = 14.85 - \frac{\$4.70}{C_0} \end{aligned}$$

When we have found Δ, we can calculate the value of the portfolio in one year.

$$\begin{aligned} \Pi_1 &= F_1 - \Delta 1.5C_0 \\ \Pi_1 &= -\$4.70 + 14.85C_0 - \left(14.85 - \frac{\$4.70}{C_0}\right)1.5C_0 \\ \Pi_1 &= \$2.35 - 7.42C_0 \end{aligned}$$

The payment required from the short position becomes $0.05\Delta C_0 = 0.05(14.85C_0 - \$4.70)$, and we can calculate the portfolio's return as follows

$$\begin{aligned} Return &= \Pi_1 - \Pi_0 - 0.05\Delta C_0 \\ &= \Pi_1 - (F_0 - \Delta C_0) - 0.05(14.85C_0 - \$4.70) \\ &= (\$2.35 - 7.42C_0) - (F_0 - (14.85C_0 - \$4.70)) - 0.74C_0 + \$0.23 \\ &= \$2.35 - 7.42C_0 - F_0 + 14.85C_0 - \$4.70 - 0.74C_0 + \$0.23 \\ &\approx -\$2.12 + 6.68C_0 - F_0 \end{aligned}$$

For the portfolio to be risk free, it must earn the risk free rate and it's return must equal $0.05\Pi_0$. Hence the value of the investment opportunity today (F_0) becomes

$$\begin{aligned} -\$2.12 + 6.68C_0 - F_0 &= 0.05\Pi_0 \\ -\$2.12 + 6.68C_0 - F_0 &= 0.05(F_0 - \Delta C_0) \\ -\$2.12 + 6.68C_0 - F_0 &= 0.05(F_0 - (14.85C_0 - \$4.7)) \\ -\$2.12 + 6.68C_0 - F_0 &= 0.05F_0 - 0.74C_0 + \$0.23 \\ 1.05F_0 &= -\$2.35 + 7.42C_0 \\ F_0 &= 7.07C_0 - \$2.24 \end{aligned}$$

We have obtained the equation for the option to invest as a function of C_0 under the assumption that we will only invest if the price goes up. When $C_0 = \$0.70 million$, the value of the option is $\$2.71 million$ and we would wait as stated before. There has to be some value of C_0 that is low enough so that we would not invest at all. We can see from the equation for F_0, that when $C_0 \approx \$0.32 million$ (exact value, $C_0 = \$0.316541771 million$) the value of the option is zero. We can also see that the payoff from an immediate investment is worthless as follows

$$
\begin{aligned}
F_0 &= max\Big(V_0 - I, 0\Big) \\
&\Rightarrow max\Big(\sum_{t=1}^{t=15} \frac{\$C_0}{(1.05)^t} - I, 0\Big) = max\Big(10.38C_0 - \$4.70, 0\Big) \\
&\Rightarrow max\Big(10.38(\$0.32) - \$4.70, 0\Big) = max\Big(-\$1.38, 0\Big) \\
&\Rightarrow F_0 = 0
\end{aligned}
$$

Thus, when $C_0 \leq \$0.32 million$ we will never invest. There has to be some value of C_0 that is high enough to justify an immediate investment. Recall that we should invest today if the value of the investment exceeds the sum of the investment cost and the opportunity cost as follows

$$
\begin{aligned}
V_0 &> I + F_0 \\
10.38C_0 &> \$4.70 + \big(7.07C_0 - \$2.24\big) \\
3.31C_0 &< \$2.46 \\
C_0 &> \frac{\$2.46}{3.31} \\
C_0 &> \$0.74 million
\end{aligned}
$$

This value of the expected cash flow is the critical value of our investment opportunity, $C_0^* = \$0.74 million$ (exact value, $C_0^* = \$0.743957632 million$). When C_0 exceeds this value the option to invest is so deep in the money that it's better to invest today and receive next years profit instead of waiting to see if the cash flow rises. If we consider $C_0 = \$0.75, million$ the payoff from an immediate investment becomes $F_0 = max\Big(V_0 - I, 0\Big) = max\Big(10.38(\$0.75) - \$4.7, 0\Big) = \$3.08 million$, while the value of waiting is $F_0 = 7.07(\$0.75) - \$2.24 = \$3.06 million$. Thus, even though we know that the expected cash flow will rise in one year, it's of more value to invest today when $C_0 > \$0.74 million$. On the other hand, if $C_0 \leq \$0.74 million$ the payoff from immediate investment becomes $F_0 = max\Big(10.38(\$0.74) - \$4.7, 0\Big) = \$2.98 million$, less than the value of waiting $F_0 = 7.07(\$0.74) - \$2.24 = \$2.99 million$. Hence, we have found the optimal investment rule for this investment opportunity, illustrated in Table 4.

Table 4: The Optimal Investment Rule ($ in millions)

Cash Flow Range	**Value of the Option to Invest**	**Optimal Investment Rule**
$C_0 > \$0.74$	$F_0 = 10.38C_0 - \$4.70$	An immediate investment.
$\$0.32 < C_0 \leq \$0.74$	$F_0 = 7.07C_0 - \$2.24$	Wait and only invest if the cash flow rises in one year.
$C_0 \leq \$0.32$	$F_0 = 0$	Never invest.

It can bee seen in Figure 9, that the option to invest is a piecewise linear function of the expected cash flow C_0 and that the optimal investment rule depends on C_0. The investment rule changes from waiting to an immediate investment at the critical value $C_0^* = \$0.74 million$ [28].

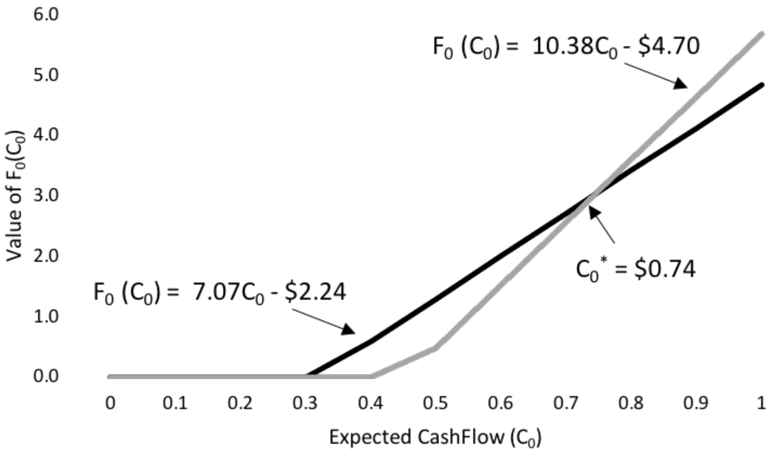


Figure 9: The Option to Invest as a Function of C_0.

5.2.3 Changing the Probabilities

We can also consider how the probabilities of the cash flow going up or down will affect the option to invest. To do this we let C_0 be a random variable and the investment cost be constant ($I = \$4.70 million$). Recall that the size of the up and down movements in the cash flow are calculated as follows

$$u = \frac{C_{up}}{C_0} = \frac{\$1.05}{\$0.70} = 1.5 = 1 + r_u$$
$$d = \frac{C_{down}}{C_0} = \frac{\$0.35}{\$0.70} = 0.5 = 1 + r_d$$

where r_u is the rate of an up movement and r_d is the rate of a down movement. The number of units we need to hold in the short position (Δ), only depends on this uncertainty, not on the probability of this happening (q). Therefore, we have the same value of $\Delta = 14.85 - \frac{\$4.70}{C_0}$ as in the previous section, same value of the option $F_1 = -\$4.7 + 14.85C_0$, and the same value of the portfolio in one year $\Pi_1 = \$2.35 - 7.42C_0$. However, the payment for holding the short position depends on the expected capital gain in one year which depends on the probability of an up or down movement in the cash flow. To calculate the expected payment required for holding the short position, we calculate the expected value of the cash flow in one year $E[C_1]$.

$$E[C_1] = quC_0 + (1-q)dC_0 = q1.5C_0 + (1-q)0.5C_0$$
$$E[C_1] = q1.5C_0 + 0.5C_0 - q0.5C_0 = qC_0 + 0.5C_0$$
$$E[C_1] = (q + 0.5)C_0$$

Knowing the expected value of the cash flow we can calculate the expected rate of capital gain in one year as follows

$$
\begin{aligned}
&E[C_1]-C_0=(q+0.5)C_0-C_0=qC_0+0.5C_0-C_0\\
&E[C_1]-C_0=(q-0.5)C_0\\
&\frac{E[C_1]-C_0}{C_0}=(q-0.5)
\end{aligned}
$$

Now the annual payment required for holding the short position is the risk free rate, minus the expected rate of capital gain given by

$$
\begin{aligned}
\big(r-(q-0.5)\big)\Delta C_0 &= \big(0.05-(q-0.5)\big)\Delta C_0\\
&=(0.55-q)\Delta C_0
\end{aligned}
$$

The risk free return on the portfolio becomes $\Pi_1-\Pi_0-(0.55-q)\Delta C_0=0.05\Pi_0$ and we can obtain the value of the option to invest as follows

$$
\begin{aligned}
\Pi_1-\Pi_0-(0.55-q)\Delta C_0 &= 0.05\Pi_0\\
\Pi_1-(F_0-\Delta C_0)-(0.55-q)\Delta C_0 &= 0.05(F_0-\Delta C_0)\\
(\$2.35-7.42C_0)-\big(F_0-(14.85C_0-\$4.70)\big)...&\\
...-\big((0.55-q)(14.85C_0-\$4.70)\big) &= 0.05\big(F_0-(14.85C_0-\$4.70)\big)\\
\$2.35-7.42C_0-(F_0-14.85C_0+\$4.70)...&\\
...-(8.17C_0-14.85C_0q-\$2.585+\$4.70q) &= 0.05F_0-0.74C_0+\$0.23\\
\$2.35-7.42C_0-F_0+14.85C_0-\$4.70...&\\
...-8.17C_0+14.85C_0q+\$2.585-\$4.70q &= 0.05F_0-0.74C_0+\$0.23\\
14.85C_0q-\$4.70q &= 1.05F_0\\
F_0 &= 14.14C_0q-\left(\frac{\$4.70}{1.05}\right)q
\end{aligned}
$$

Now we have obtained the equation of the option to invest as a function of C_0 and q. The reader can verify that by letting $C_0=\$0.70million$ and $q=0.5$ the value of the option to invest becomes $F_0=\$2.71million$ as before. From the optimal investment rule presented in Table 4, we know that when $C_0\le\$0.32million$, $F_0=0$, even if there is a 100% probability that the price will increase in the upcoming year. As one would expect when $C_0>\$0.32million$ the value of the option increases as q increases, since when q increases, it's more likely that the price is going up in the next period. Now because the option to invest depends on q we need to find when it's better to invest today rather than wait as a function of q. Recall that it's better to invest today if $V_0>I+F_0$. Now $V_0=\frac{C_0}{1.05}+\sum_{t=2}^{t=15}\frac{(q+0.5)C_0}{(1.05)^t}\approx(5.67+9.43q)C_0$ where the investment is made in period 0 and the cash flow is not expected to rise in the first period. Thereafter the expected value of the cash flow equals $(q+0.5)C_0$ as discussed before. Thus, to justify an immediate investment we need

$$
\begin{aligned}
V_0 &> I + F_0 \\
(5.67 + 9.43q)C_0 &> \$4.70 + 14.14C_0 q - \left(\frac{\$4.70}{1.05}\right)q \\
5.67C_0 + 9.43qC_0 &> \$4.70 + 14.14C_0 q - \left(\frac{\$4.70}{1.05}\right)q \\
(5.67 - 4.71q)C_0 &> \$4.70 - \left(\frac{\$4.70}{1.05}\right)q \\
C_0 &< \frac{\$4.70(1 - \frac{q}{1.05})}{5.67 - 4.71q}
\end{aligned}
$$

When $q = 0.5$ the critical value is $C_0^* = \$0.74 million$ as before and if the cash flow exceeds this critical value, one should invest immediately. When we increase the probability of an up movement to e.g. $q = 0.7$, we can see that the critical value decreases to $C_0 = \$0.66 million$. Thus, when q increases the range for C_0 where the optimal investment rule is to wait decreases, and a lower value of C_0 is needed to justify an immediate investment. With an increased probability, it is more likely that the cash flow will rise and the cost of waiting, which is is the revenue foregone in t_1 (see Figure 8), will exceed the value of waiting [28].

5.2.4 Increasing Uncertainty

Until now we have considered a volatility of 50% over the cash flow and the probability of an up/down movement has been $q = 0.5$ and $(1-q) = 0.5$, respectively. Now we will look at how an increase in the cash flow's uncertainty to 75%, will affect the value of the option to invest, while keeping the probabilities the same. In the previous section we changed the probabilities of the cash flow movements which resulted in an altered expected value in t_2. Changes in the cash flow uncertainty do not affect the expected value, thus we have an expected value of $E[C_1] = \frac{1}{2}\big(1.75(\$0.70) + 0.25(\$0.70)\big) = \$0.70 million$ and the payoff from an immediate investment is $F_0 = V_0 - I_0 = \$2.57 million$ (see Figure 6). To obtain the value of the option to wait one year, we go through the same steps as before letting C_0 be a random variable. The process for the cash flow is presented in Figure 10.

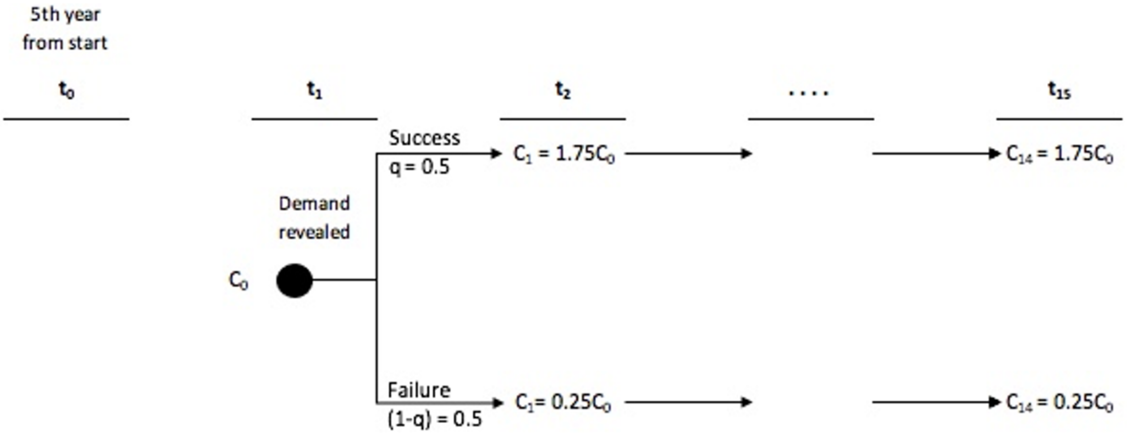


Figure 10: Expected Cash Flow with Uncertainty of 75%.

Now the value of the option to invest in one year, assuming that we will only invest if the cash flow rises, is calculated as follows

$$\begin{aligned} F_1 &= -\$4.7 + \sum_{t=1}^{t=14} \frac{C_1}{(1.05)^t} \\ &= -\$4.7 + \sum_{t=1}^{t=14} \frac{1.75C_0}{(1.05)^t} \\ &\approx 17.32C_0 - \$4.7 \end{aligned}$$

The value of the portfolio in one year will either be $\Pi_1 = F_1 - \Delta 1.75C_0$, or $\Pi_1 = 0 - \Delta 0.25C_0$, and we can obtain Δ as earlier

$$\begin{aligned} \Pi_1 &= F_1 - \Delta 1.75C_0 = 0 - \Delta 0.25C_0 \\ &\Rightarrow -\$4.7 + 17.32C_0 - \Delta 1.75C_0 = -\Delta 0.25C_0 \\ &\Rightarrow \Delta 1.5C_0 = -\$4.7 + 17.32C_0 \\ &\Rightarrow \Delta = 11.55 - \frac{\$4.7}{1.5C_0} \end{aligned}$$

When we have found Δ so that the portfolio will be riskless we can calculate the return on the portfolio $\Pi_1 - \Pi_0 - 0.05\Delta C_0 = 0.05\Pi_0$ and find the value of the option to invest

$$\begin{aligned} 0.05\Pi_0 &= \Pi_1 - \Pi_0 - 0.05\Delta C_0 \\ 0.05(F_0 - \Delta C_0) &= (F_1 - \Delta 1.75C_0) - (F_0 - \Delta C_0) - 0.05\Delta C_0 \\ 0.05F_0 - 0.05\left(11.55 - \frac{\$4.70}{1.5C_0}\right)C_0 &= F_1 - \left(11.55 - \frac{\$4.70}{1.5C_0}\right)1.75C_0 - F_0 ... \\ &\quad ... + \left(11.55 - \frac{\$4.70}{1.5C_0}\right)C_0 - 0.05\left(11.55 - \frac{\$4.70}{1.5C_0}\right)C_0 \\ 0.05F_0 - 0.58C_0 + \$0.16 &= 17.32C_0 - \$4.70 - 20.21C_0 + \$5.48 - F_0 ... \\ &\quad ... + 11.55C_0 - \frac{\$4.70}{1.5} - 0.58C_0 + \$0.16 \\ 1.05F_0 &= 8.66C_0 - \$2.35 \\ F_0 &= 8.25C_0 - \$2.24 \end{aligned}$$

If we let $C_0 = \$0.70 million$ the value of the option to invest becomes $F_0 \approx \$3.54 million$. This is higher than the value calculated under an uncertainty of 50%, or $F_0 = \$2.71 million$. Hence, an increased uncertainty increases the value of the option to wait and we can see that the critical value of C_0 that will justify an immediate investment becomes

$$\begin{aligned} V_0 &> I + F_0 \\ 10.38C_0 &> \$4.70 + (8.25C_0 - \$2.24) \\ 2.13C_0 &< \$2.46 \\ C_0 &> \frac{\$2.46}{2.13} \\ C_0 &> \$1.15 million \end{aligned}$$

The range where the option to wait is valuable increases from the threshold that we obtained before of $C_0^* = \$0.74 million$ to this new critical value of $C_0^* = \$1.15 million$. Therefore, when uncertainty increases it is more likely that we will wait rather than invest today [28].

5.3 The "Bad News Principle"

We can also consider how "bad news" (a downward movement in the cash flow) and "good news" (an upward movement in the cash flow) affect the critical value C_0^*. To do this we let the probability of an up movement q and the size of the up/down movements vary. Now the cash flow in t_1 will either go up by an amount u with a probability of q or down by an amount d with the probability $(1-q)$ (see Figure 11).

$$C_1 = \begin{cases} (1+u)C_0 & \text{with probability q} \\ (1\text{-}d)C_0 & \text{with probability (1-q)} \end{cases}$$

Figure 11: Probability of an Up/Down Movement in the Expected Cash Flow.

If we invest today the NPV of our investment opportunity becomes a function of C_0, q, u and d as follows

$$\begin{aligned}
NPV &= -I + \frac{C_0}{1.05} + q\Big(\sum_{t=2}^{t=15} \frac{(1+u)C_0}{(1.05)^t}\Big) + (1-q)\Big(\sum_{t=2}^{t=15} \frac{(1-d)C_0}{(1.05)^t}\Big) \\
NPV &= -I + C_0\Big(\frac{1}{1.05} + 9.43q(1+u) + 9.43(1-q)(1-d)\Big) \\
NPV &= -I + 9.43C_0\big(0.10 + q + qu + 1 - d - q + qd\big) \\
NPV &= -I + 9.43C_0\big(1.10 + q(u+d) - d\big) \qquad (20)
\end{aligned}$$

However, if we wait the NPV becomes

$$\begin{aligned}
NPV &= q max\Big(V_{1_{up}} - I, 0\Big) + (1-q) max\Big(V_{1_{down}} - I, 0\Big) \\
NPV &= q max\Big(\sum_{t=2}^{t=15} \frac{\$(1+u)C_0}{(1.05)^t} - \frac{I}{1.05}, 0\Big) + (1-q) max\Big(\sum_{t=2}^{t=15} \frac{\$(1-d)C_0}{(1.05)^t} - \frac{I}{1.05}, 0\Big) \\
NPV &= q max\Big(9.43(1+u)C_0 - \frac{I}{1.05}, 0\Big) + (1-q) max\Big(9.43(1-d)C_0 - \frac{I}{1.05}, 0\Big)
\end{aligned}$$

Recall that we will only invest if the $R\&D$ results in a high quality product and market conditions turn out favorable. In this case the expected cash flow rises and therefore, the value of the option to wait simplifies to

$$NPV = q\Big(9.43(1+u)C_0 - \frac{I}{1.05}\Big) \qquad (21)$$

By equating the NPV from an immediate investment in Eq.20, and the value of waiting from Eq.21, we can solve for C_0 and find the critical value of the cash flow that warrants an immediate investment.

$$
\begin{aligned}
-I+9.43C_0\big(1.10+q(u+d)-d\big) &= q\Big(9.43(1+u)C_0-\frac{I}{1.05}\Big)\\
-I+10.37C_0+9.43C_0qu+9.43C_0qd-9.43C_0d &= 9.43C_0q+9.43C_0qu-\frac{qI}{1.05}\\
10.37C_0+9.43C_0qd-9.43C_0d-9.43C_0q &= I-\frac{qI}{1.05}\\
9.43C_0\big(1.10+qd-d-q\big) &= I\Big(\frac{1.05}{1.05}-\frac{q}{1.05}\Big)\\
9.43C_0\big(0.10+(1-q)(1-d)\big) &= I\Big(\frac{1}{1.05}\Big)\big(1.05-q\big)\\
C_0\big(0.10+(1-q)(1-d)\big) &= I\Big(\frac{1}{9.43}\Big)\Big(\frac{1}{1.05}\Big)\big(0.05+1-q\big)\\
C_0^* &= 0.10I\Big(\frac{0.05+(1-q)}{0.10+(1-q)(1-d)}\Big)
\end{aligned}
$$

It can be seen, that the critical value C_0^* does not depend in any way on the size of the upward movement (u). It only depends on the size of the downward movement (d), and the probability of the cash flow decreasing $(1-q)$. Furthermore, the larger the size of the downward movement, the larger the critical value C_0^*, and the range where waiting is optimal increases. When d is large, the greater is the magnitude of possible bad news and the motivation to wait increases [28].

5.4 Cost Uncertainty

Most $R\&D$ projects involve substantial uncertainty over cost. Even though the cost of the investment is known today, it's value might change as time goes by. Material costs, employee salaries, or government regulations are examples of factors that can affect the cost of the investment. To see how uncertainty over investment costs can affect the decision to invest, we will go back to our previous example, with a constant expected cash flow of $\$0.70 million$ and now we assume an uncertainty of 75% over the investment cost. Therefore, the cost of the investment today of $\$4.70 million$ can either rise to $\$8.23 million$ or fall to $\$1.18 million$ in one year. We will let the probability of these movements be $q = 0.5$ and $(1-q) = 0.5$, respectively. If we invest today the NPV of our investment opportunity is the same as before, $F_0 = \$2.57 million$ (see Figure 6). If we wait, we will only invest in one year, if the investment cost falls to $\$1.18 million$. In the case of waiting, the NPV of our investment opportunity becomes

$$
\begin{aligned}
V_0 &= \sum_{t=2}^{t=15}\frac{\$0.70}{(1.05)^t} = \$6.60 million\\
I_0 &= \frac{\$1.18}{1.05} = \$1.12 million
\end{aligned}
$$

$$
\begin{aligned}
NPV &= q max\Big(V_0-I_0,0\Big)\\
&= 0.5 max\Big(\$6.60-\$1.12,0\Big)\\
&= \$2.74 million
\end{aligned}
$$

In this case it's better to wait than to invest right away. On the other hand, if we would have assumed an uncertainty over cost of 50% instead of 75%, the NPV of waiting would have been lower than the one calculated from an immediate investment. Thus the optimal investment rule would have been to invest right away.

It might appear that an increased uncertainty always increases the incentive to wait, or at least, increases the expected rate of return. However, this is not always the case. Some uncertainty over costs such as the amount of time, effort or materials needed to finish a project, can only be resolved by undertaking the project. As time unfolds these costs may be greater or lower than initially estimated and the total cost of the investment will not be known until the project is completed. Sometimes investing provides information that will not be obtained unless the project is undertaken.

Example. Let us assume that if we invest $\$4.70 million$ today, there is a 65% probability that we will finish production and marketing and start selling the product, receiving an expected cash flow of $\$0.70 million$ until the patent expires ($V_0 = \$7.27 million$). However, there is a 35% probability that we will need extra material and effort to finish production, resulting in an overtime pay and an additional cost of $\$7.50 million$. One might calculate the NPV of this scenario as $-\big(\$4.70 + 0.35(\$7.50)\big) + \$7.27 \approx -\$0.06 million$, discouraging an investment. On the other hand, this calculation ignores the value of the information that will be obtained by investing $\$4.70 million$ at the onset, where there is a 65% probability that no additional cost will be needed. As well, it ignores the fact that we can abandon the project if an additional cost of $\$7.50 million$ will be required. The correct value of the project in this example would be $-\$4.70 + 0.65(\$7.27) = \$0.03 million$.

When uncertainty over cost is mostly influenced by market risk that is not controlled by the firm, such as material costs or government regulations, an increased uncertainty creates the incentive to wait. On the other hand, when the risk is firm specific and can be partly or completely resolved by investing, it can have the opposite effect, to accelerate the investment [28] [32].

5.5 Interest Rate Uncertainty

We can also consider how uncertainty over interest rates affects the decision to invest. For a better explanation of this subject we will assume that the interest rate today is 10%. With a probability $q = 0.5$ the interest rate will rise to 15% and with probability $(1-q) = 0.5$, it will fall to 5%. We will let the expected cash flow of $\$0.70 million$ and the investment cost of $\$4.70 million$ stay constant. If we invest today and there is no uncertainty over interest rates the NPV becomes

$$NPV = -\$4.70 + \sum_{t=1}^{t=15} \frac{\$0.70}{(1.1)^t} = \$0.62\$ million$$

The NPV under the assumption of waiting, when there is no uncertainty over the interest rates is

$$NPV = \frac{-\$4.70}{1.1} + \sum_{t=2}^{t=15} \frac{\$0.70}{(1.1)^t} = \$0.42\$ million$$

Clearly, when there is no uncertainty over interest rates, it is better to invest today rather than wait. When the interest rates are uncertain, the NPV of investing today becomes

$$NPV = -\$4.70 + 0.5\sum_{t=1}^{t=15}\frac{\$0.70}{(1.15)^t} + 0.5\sum_{t=1}^{t=15}\frac{\$0.70}{(1.05)^t} = \$0.98 million$$

Hence, the value of investing today increases when interest rates are uncertain from $NPV = \$0.62 million$, to $NPV = \$0.98 million$. Now suppose that we wait for one year. With an uncertainty over interest rates, today's NPV of waiting is either

$$NPV = 0.5\Big(-\frac{\$4.70}{1.15} + \sum_{t=2}^{t=15}\frac{\$0.70}{(1.15)^t}\Big) = -\$0.30 million$$

or

$$NPV = 0.5\Big(-\frac{\$4.70}{1.05} + \sum_{t=2}^{t=15}\frac{\$0.70}{(1.05)^t}\Big) = \$1.06 million$$

We will only invest in one year if the interest rate falls to 5%. When interest rates are uncertain, the NPV of waiting ($\$1.06 million$) is higher than the value of investing today ($\$0.98 million$), thus, it's better to wait.

This simple analysis of interest rate uncertainty shows that mean preserving volatility in interest rates will increase the expected value of a project but it will also increase the incentive to wait. The value of waiting for more information increases when uncertainty over interest rates increases. Therefore, if a government aims at encouraging investment in a society, a policy that leads to stable and predictable interest rates may be more important than the level of interest rates. Low but volatile interest rates could discourage investment, increasing the incentive to wait [28].

6 Continuous Time Valuation

Until now, we have considered our investment opportunity as a discrete time valuation problem. This is a simplification of reality where expected cash flow, investment costs and uncertainty can change continuously through time. In Section 5, we made the assumption that uncertainty would resolve over one period and the cash flow would stay at that level until maturity. In reality the future is always uncertain and the amount of uncertainty will increase as the time horizon is extended. To model our investment opportunity as a continuous time problem we shall introduce some mathematical tools that are gaining increasing use in finance and economics. The two primary tools that we will examine are the stochastic processes and Itô's Lemma. These provide important results for the derivation of the BSM model (see Section 6.5) and the real options theory (see Section 7).

6.1 Stochastic Processes

Stochastic processes describe the probabilistic evolution of any variable whose value changes in an uncertain way over time. Stochastic processes can be classified as either discrete or continuous. A discrete stochastic process is one where the value of a variable can change only at certain fixed points in time for example the random walk (see Section 4). Continuous time stochastic process is one where changes can take place any time. A Markov process is a stochastic process where the past history of the variable is irrelevant and only the PV of the variable can be used to predict the future, denoted as

$$E[V_{t+1} \big| V_t, V_{t-1}, V_{t-2} ... V_0] = E[V_{t+1} \big| V_t]$$

The Markov process indicates the efficient market hypothesis, which states that it is not possible to predict trends in the market through technical analysis of the past. All relevant information is already in the current value of the underlying asset [24] [28].

6.2 The Wiener Process

A Wiener process is a continuous time stochastic process. It is also known as Brownian motion where in physics it has been used to describe the collisions of particles. A variable w_t follows a Wiener process if it satisfies the two following properties:

1. The change in w_t over a small period of time equals $w_{t+\Delta t} - w_t = \Delta w_t = \epsilon_t \sqrt{\Delta t}$. The term ϵ_t represents a random variable from a standard normal distribution with mean 0 and standard deviation 1.
2. The value of Δw_t is independent of $w_j, j \leq t$.

The second property implies that the Wiener process is a Markov process. The past history of the variable w_t, is irrelevant for forecasting its future value w_{t+1}. Furthermore, if $t_1 < t_2 < t_3$ than the change in the Wiener process in two non-overlapping intervals $(w_{t_3} - w_{t_2})$ and $(w_{t_2} - w_{t_1})$ are independent of each other.

A Wiener process at $t = 0$ equals $w_0 = 0$. If we divide a timeline from $t = 0$ to $t = T$ into n time steps, each of length Δt ($n\Delta t = T$), the value of the Wiener process at time T, is the sum of all the independent increments up to time T.

$$w_T = \sum_{i=0}^{n-1} \epsilon_i \sqrt{\Delta t} = \left(\epsilon_0 + \epsilon_1 + \epsilon_2 + ... + \epsilon_{n-1}\right)\sqrt{\Delta t} \tag{22}$$

Because the increments $\epsilon_t \sim N(0,1)$ are independent of each other we can apply the Central Limit Theorem to their sum. Since the expected value of each increment is zero, the expected value of w_T is also zero, $E[w_T] = 0$. Because the variance of each increment is 1, the variance of w_T is the sum of all the increments' variances multiplied with $(\sqrt{\Delta t})^2$ or $Var(w_T) = n\Delta t = T$. Thus we know that w_T is normally distributed with a mean of zero and variance T, $w_T \sim N(0,T)$. This means that if the value of a variable is w_0 at t_0 then at time T it is normally distributed with mean w_0 and standard deviation $\sqrt{T}$. Since $w_T - w_0 \sim N(0, T-0)$ it follows that the change in w_t over any time interval e.g. $w_{t_2} - w_{t_1}$ is also normally distributed with mean zero and variance equal to the length of the time interval $\sim N(0, t_2 - t_1)$. The variance of a Wiener process grows linearly with time or in other words, the variance at time t is smaller than the variance at time $t + \Delta t$.

It is important to realize why ϵ_t is multiplied by $\sqrt{\Delta t}$ instead of Δt. This is because as $\Delta t \to 0$, which is necessary in continuous time analysis when the time steps become infinitely small, the square root of Δt goes at a much slower rate towards zero preventing the process from stopping (e.g. when $\Delta t = 0.01$ then $\sqrt{\Delta t} = 0.1$). Similarly, when Δt is large $\sqrt{\Delta t}$ increases at a much slower rate than Δt. By letting Δt become infinitesimally small we can write the Wiener process in continuous time as

$$dz = \epsilon_t \sqrt{dt} \tag{23}$$

with mean of $E[dz] = 0$ and variance $E[(dz)^2] = dt$. The Wiener process has been used to describe the process of the change in a stock price. It might seem reasonable that stock prices follow a Wiener process since they satisfy the Markov property and have independent increments. However, we know that their price can never be negative. Therefore, it is not reasonable to assume that price changes follow a normal distribution. Nevertheless, we can assume that changes in stock prices are lognormally distributed. This means that the changes in the logarithm of the price is normally distributed. With transformation of this kind we can use the Wiener process to describe the behavior of stock prices as well as other variables that change stochastically and continuously through time [11] [24] [28].

6.2.1 Generalized Wiener Process

The Wiener process can be generalized into other processes. One generalization is the Wiener process with drift, given by

$$dx = \alpha dt + \sigma dz \tag{24}$$

where the drift rate α and the variance rate σ^2 are constants and dz represents the increment of the Wiener process. The change in the variable x over a small time interval is normally distributed with an expected value of $E[dx] = \alpha dt$ and variance $E[\sigma^2(dz)^2] = \sigma^2 dt$. The term $\sigma dz = \sigma \epsilon_t \sqrt{dt}$ is often referred to as added noise or variability of the path followed by x. Sample paths for Eq.24 are shown in Figure 12. The trajectory for x was calculated over 15 years assuming a time step of 1 month ($\Delta t = 1/12$) as follows

$$x_{t+1} = x_t + \alpha \Delta t + \sigma \epsilon_t \sqrt{\Delta t} \tag{25}$$

The paths all start at $x_0 = 0$ and have an annual drift of $\alpha = 0.3$ and standard deviation of $\sigma = 1$. Note that this is a Markov process because the value of x_{t+1} only depends on the value of x_t, not on its past history.

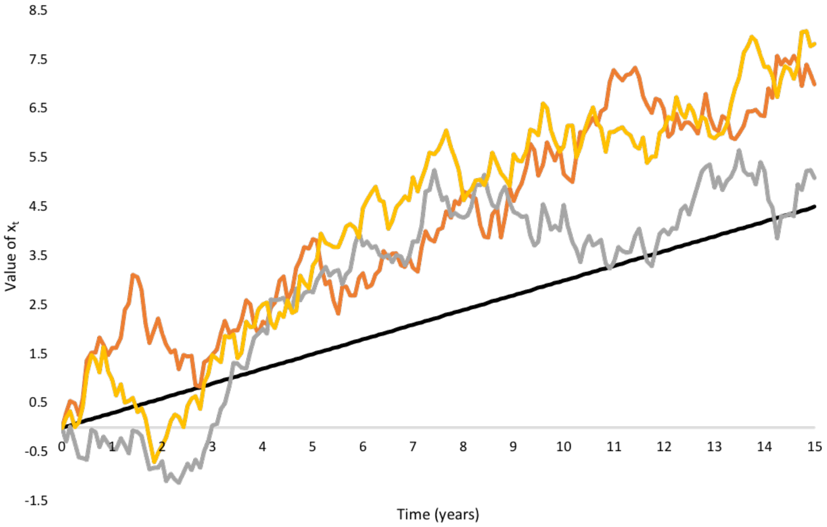


Figure 12: Sample Paths of a Generalized Wiener Process.

The difference between the four paths in Figure 12, is due to the random value of ϵ_t at each time step. When $\epsilon_t = 0$ the process in Eq.24 has no variability and simplifies to $dx = \alpha dt$. This is the equation for a straight line with a slope α, as shown by the black path in Figure 12. The other trajectories all have independent increments of the Wiener process, where ϵ_t is a random variable drawn from a standard normal distribution at each time step, resulting in three different paths. By looking at the whole picture, we can see that the value of α controls the trend of the process. In the short run when t is small the dominant factor of the process is the volatility $\sigma\sqrt{t}$, since $\sqrt{t} >> t$. As time increases the reverse is true, $t >> \sqrt{t}$ and the drift αt becomes a dominant determinant.

The optimal forecast for the same stochastic process as given in Eq.25 with $\alpha = 0.3$ and $\sigma = 1$ is illustrated in Figure 13. The trajectory for x_t was forecasted from the beginning of the 7th year (x_7) until the end of the 15th year. The expected value of x_7, T months into the future is given by

$$x_{7+T} = x_7 + \alpha \frac{T}{12}$$

Because of the Markov property, only the value of x in the beginning of the 7th year (x_7) is needed to forecast its future, the past is irrelevant.

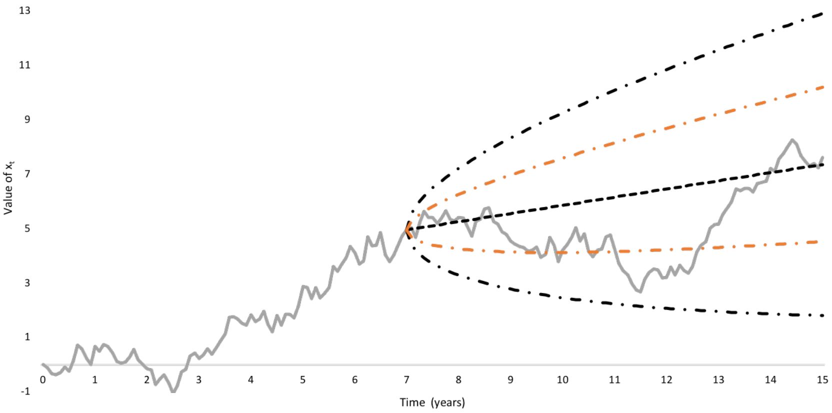


Figure 13: Forecasted Value of x with 95% and 66% Confidence Interval.

The orange dotted line shows the 66% confidence interval of the forecasted path followed by x. This is calculated as the expected value of x_7 plus/minus one standard deviation.

$$x_7 + \alpha\frac{T}{12} \pm \sigma\sqrt{\frac{T}{12}}$$

Similarly the outer black dots display the 95% confidence interval for the forecasted value of x_7 plus/minus 1.96 standard deviation.

$$x_7 + \alpha\frac{T}{12} \pm 1.96\sigma\sqrt{\frac{T}{12}}$$

Because the Wiener process grows linearly with time, the standard deviation increases as the square root of time. This means that as we move further into the future the forecast becomes less accurate than in the beginning of the forecasted time horizon [24] [28].

6.3 Itô Processes

An Itô process is a generalized Wiener process where the drift and variance rate are a function of both the underlying variable and time. The change in the underlying variable in a very short time interval is approximately normally distributed. The expected drift and variance rate of an Itô process are likely to change over time. However, it is assumed that they stay constant over a small time interval $(t + \Delta t)$. An Itô process for the underlying asset x, with a drift of $a(x,t)$ and variance rate of $b^2(x,t)$, can be defined in terms of the Wiener process dz, as follows

$$dx = a(x,t)dt + b(x,t)dz \tag{26}$$

The change in x only depends on its current value, not on it's history. Thus an Itô process is a Markov process [24] [28].

6.3.1 Geometric Brownian Motion (GBM)

A special case of the Itô process presented in Eq.26 is the geometric Brownian motion (GBM) (see Eq.27).

$$dx = \alpha x dt + \sigma x dz \tag{27}$$

The drift (α) and the variance (σ^2) are constants over a small interval of time, x is a random variable, and dz is the increment of the Wiener process. This process leads to the most widely used model of stock price behavior where the random variable is the stock price, μ is the expected return on the stock, and σ is the volatility of the stock.

$$dS = \mu S dt + \sigma S dz \tag{28}$$

This process is convenient in modeling stock prices because the expected percentage return investors require on a stock (μ) is independent of the stock price. Hence μ stays constant over a small time interval but the drift rate changes in proportion to the stock price: μS. The expected increase in the stock price over a small time interval becomes μdt (in a risk neutral world $\mu = r$). Similarly it is reasonable to assume that the variability (σ) in the stock returns over a small time period is the same, independent of the price of the stock. Therefore, the standard deviation of the change should be in proportion to the price of the stock: σS. Figure 14 illustrates a sample path of the process given in Eq.28, assuming that the current value of the stock is $S_0 = 50$, with an expected rate of return $\mu = 12\%$, and volatility $\sigma = 25\%$. The path was generated by assuming monthly time steps over 5 years [24].

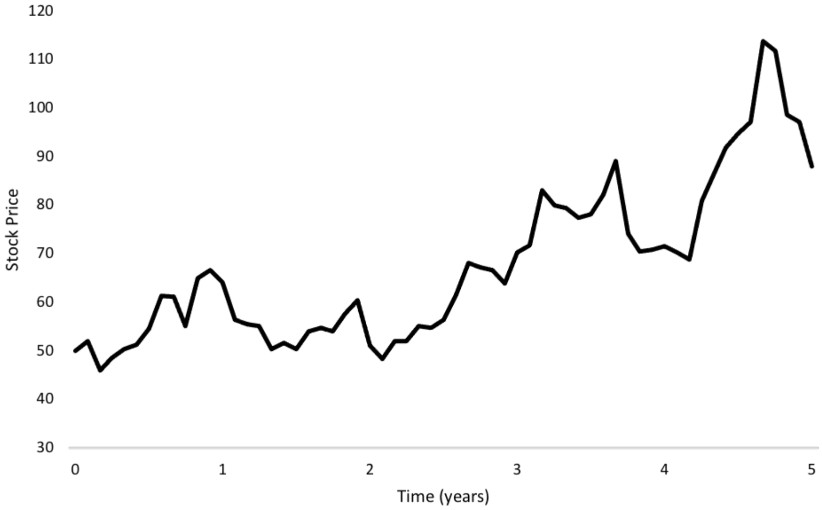


Figure 14: Sample Path of a GBM followed by a Stock Price.

6.4 Itô's Lemma

The price of any derivative can be written as a function of the stochastic variables underlying the derivative and time. The mathematician, K. Itô discovered this phenomena in 1951, known as Itô's Lemma [33]. If a variable x follows an Itô process, and a function

$F(x,t)$ can be differentiated once with respect to t and twice with respect to x, then $F(x,t)$ also follows a stochastic process, given by

$$dF(x,t) = \Big[\frac{\partial F(x,t)}{\partial x}a + \frac{\partial F(x,t)}{\partial t} + \frac{1}{2}\frac{\partial^2 F(x,t)}{\partial x^2}b^2\Big]dt + \frac{\partial F(x,t)}{\partial x}bdz \tag{29}$$

where a is the drift rate and b^2 is the variance rate. The function $F(x,t)$ and the variable x are affected by the same underlying source of uncertainty dz, where $F(x,t)$ has the mean

$$\Big[\frac{\partial F(x,t)}{\partial x}a + \frac{\partial F(x,t)}{\partial t} + \frac{1}{2}\frac{\partial^2 F(x,t)}{\partial x^2}b^2\Big]$$

and variance

$$\Big(\frac{\partial F(x,t)}{\partial x}\Big)^2 b^2$$

For a derivation of Itô's Lemma see Hull (2015) p.341 [24]. It's result is important in finding the value of any derivative. One of the main assumption underlying the BSM model is that stock prices are lognormally distributed. We can use Itô's Lemma to prove this, by letting $F(S,t) = lnS_t$, where S_t follows the GBM illustrated in Eq.28. Then the price of the option $F(S,t)$ also follows a stochastic process given by

$$dF(S,t) = \Big[\frac{\partial F(S,t)}{\partial S}\mu S + \frac{\partial F(S,t)}{\partial t} + \frac{1}{2}\frac{\partial^2 F(S,t)}{\partial S^2}\sigma^2 S^2\Big]dt + \frac{\partial F(S,t)}{\partial S}\sigma S dz \tag{30}$$

By differentiating $F(S,t)$ with respect to S and t we get

$$\frac{\partial F(S,t)}{\partial S} = \frac{1}{S}\frac{\partial F(S,t)}{\partial t} = 0\frac{\partial^2 F(S,t)}{\partial S^2} = -\frac{1}{S^2}$$

Substituting these results into Eq.30 we obtain

$$dF(S,t) = \Big(\mu - \frac{1}{2}\sigma^2\Big)dt + \sigma dz$$

We have shown that $F(S,t) = lnS_t$ follows a generalized Wiener process with a constant drift rate $\left(\mu - \frac{1}{2}\sigma^2\right)$ and variance σ^2. Therefore, the natural logarithm of the change in the stock price from $t = 0$ up to time T, is normally distributed with mean $\left(\mu - \frac{1}{2}\sigma^2\right)T$ and variance $\sigma^2 T$.

$$ln\Big(\frac{S_T}{S_0}\Big) \sim N\Big[\Big(\mu - \frac{1}{2}\sigma^2\Big)T, \sigma^2 T\Big]$$

The solution to the process followed by S is

$$ln(S_T) = ln(S_0) + \Big(\mu - \frac{1}{2}\sigma^2\Big)T + \sigma dz$$

or

$$S_T = S_0 e^{\left(\mu - \frac{1}{2}\sigma^2\right)T + \sigma dz}$$

When there is no volatility this equation simplifies to

$$S_T = S_0 e^{\mu T}$$

which is the expected value of the stock at time T [24].

6.5 The Black-Scholes-Merton (BSM) Model

When first introduced, the aim of the BSM model was to value financial assets e.g. stocks. It builds on the idea that one can construct a replicating portfolio consisting of the underlying asset and the option. For the portfolio to be risk free it must earn the risk free rate. The assumptions underlying the discrete time binomial approach and the BSM model are the same, as outlined in Section 4.1. However, in the binomial approach it is assumed that the underlying asset follows a discrete random walk while the BSM model assumes that it follows a continuous stochastic process (see Section 6.3.1). The BSM model therefore accounts for continuous trading such that the replicating portfolio only maintains risk free over a very short period of time and needs to be rebalanced frequently. It also assumes that there are no dividends during the life of the option but this assumption can be relaxed (see Section 6.6.1.1).

6.5.1 The BSM Differential Equation

Let us consider an option's time to maturity as $(T-t)$, where t is not necessarily $t=0$. In this way, we can obtain the value of a derivative, at any time t. To derive the BSM differential equation we let the function $F(x,t)$ from Eq.29 be the price of the derivative, contingent on the price of the underlying asset which follows the GBM presented in Eq.28. Denoting the underlying asset as V, the discrete version of the process becomes

$$\Delta V = \mu V \Delta t + \sigma V \Delta z \tag{31}$$

The price of the derivative $F(V,t)$ in discrete time is

$$\Delta F(V,t) = \Big[\frac{\partial F(V,t)}{\partial V}\mu V + \frac{\partial F(V,t)}{\partial t} + \frac{1}{2}\frac{\partial^2 F(V,t)}{\partial V^2}\sigma^2 V^2\Big]\Delta t + \frac{\partial F(V,t)}{\partial V}\sigma V \Delta z \tag{32}$$

where ΔV is the change in the value of the underlying asset and $\Delta F(V,t)$ is the change in the price of the option, over a small time interval Δt. Now we will consider a replicating portfolio, with a long position in the option and a short position in $\Delta = \frac{\partial F(V,t)}{\partial V}$ units of the underlying asset. The value of our portfolio is given by

$$\Pi = F(V,t) - \frac{\partial F(V,t)}{\partial V}V \tag{33}$$

The change in the portfolio over a small period Δt can be denoted as

$$\Delta\Pi = \Delta F(V,t) - \frac{\partial F(V,t)}{\partial V}\Delta V \tag{34}$$

By substituting Eq.31 and Eq.32 into Eq.34, the stochastic component of the equations falls out $\left(\frac{\partial F(V,t)}{\partial V}\sigma V \Delta z\right)$ and we obtain

$$\Delta\Pi = \left[\frac{\partial F(V,t)}{\partial t} + \frac{1}{2}\frac{\partial^2 F(V,t)}{\partial V^2}\sigma^2 V^2\right]\Delta t \tag{35}$$

This means that the change in the portfolio must be risk free over a small period Δt. In absence of arbitrage opportunities its return must equal the risk free rate as follows

$$\Delta\Pi = r\Pi\Delta t. \tag{36}$$

By substituting Eq.33 and Eq.35 into Eq.36, we get

$$\left[\frac{\partial F(V,t)}{\partial t} + \frac{1}{2}\frac{\partial^2 F(V,t)}{\partial V^2}\sigma^2 V^2\right]\Delta t = r\left(F(V,t) - \frac{\partial F(V,t)}{\partial V}V\right)\Delta t$$

And by rearranging terms this equation simplifies to the *BSM* differential equation, given by

$$\frac{\partial F(V,t)}{\partial V} rV + \frac{\partial F(V,t)}{\partial t} + \frac{1}{2}\frac{\partial^2 F(V,t)}{\partial V^2}\sigma^2 V^2 = rF(V,t) \tag{37}$$

The price of any financial derivative needs to satisfy this equation. Note, even though the expected return from the underlying asset is μ, as illustrated in Eq.31, the price of the derivative does not depend on μ, only on the risk free rate r. Hence, the price of any financial derivative is risk neutral. This is not necessarily the case for real options (see Section 7).

6.5.2 The BSM Pricing Formula

The *BSM* differential equation has different solutions for the various derivatives that it satisfies. It's solution depends on the boundary conditions used, that define the values of the derivative, for possible values of the underlying asset and time. In this section we will introduce its well known solution for pricing a European call option. In the following section we will consider its application in patent valuation. The boundary conditions for a European call option at $t = T$, on the underlying asset V is $F = max(V - I, 0)$. It's solution is given by

$$c = VN(d_1) - Ie^{-r(T-t)}N(d_2) \tag{38}$$

where V is the current price/value of the underlying asset, I is the strike price of exercising the option, $T - t$ is the time to maturity, and r is the risk free rate. The variables d_1 and d_2 are calculated as follows

$$d_1 = \frac{ln\left(\frac{V}{I}\right) + (r + \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{(T-t)}} \tag{39}$$

$$d_2 = \frac{ln\left(\frac{V}{I}\right) + (r - \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{(T-t)}} = d_1 - \sigma\sqrt{(T-t)} \tag{40}$$

The cumulative probability distribution function $N(x)$, tells us the probability that a variable with a normal distribution is less than x, $P\left(X \leq x\right)$. When $V > I$, $N(d_1)$ and $N(d_2)$ tend to 1. Hence, when t approaches T, Eq.38 tends to $V - I$. On the other hand, when $I > V$, $N(d_1)$ and $N(d_2)$ tend to 0. In this case when $t \to T$, Eq.38 tends to $max\left(V - I, 0\right)$.

By differentiating the price of a call option in Eq.38 twice with respect to V and once with respect to t, we can identify that it satisfies the BSM differential equation. The first derivative of c with respect to V becomes

$$\frac{\partial c}{\partial V} = N(d_1) + VN'(d_1)\frac{\partial d_1}{\partial V} - Ie^{-r(T-t)}N'(d_2)\frac{\partial d_2}{\partial V}$$

Since $N(x)$ is the cumulative probability distribution function, $N'(x)$ is the probability density function for a standard normal distribution, given by

$$N'(x) = \frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}$$

Because $d_1 = d_2 + \sigma\sqrt{(T-t)}$, it follows that $N'(d_1)$ is

$$\begin{aligned} N'(d_1) &= N'\left(d_2 + \sigma\sqrt{(T-t)}\right) \\ &= \frac{1}{\sqrt{2\pi}}e^{-\frac{\left(d_2+\sigma\sqrt{(T-t)}\right)^2}{2}} = \frac{1}{\sqrt{2\pi}}e^{-\frac{\left(d_2^2+2d_2\sigma\sqrt{(T-t)}+\sigma^2(T-t)\right)}{2}} \\ &= N'(d_2)e^{\left(-d_2\sigma\sqrt{(T-t)}-\frac{1}{2}\sigma^2(T-t)\right)} \end{aligned}$$

Now since $d_2 = \frac{ln\left(\frac{V}{I}\right)+(r-\frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{(T-t)}}$, we have

$$\begin{aligned} N'(d_1) &= N'(d_2)e^{\left(-\left(ln\left(\frac{V}{I}\right)+(r-\frac{1}{2}\sigma^2)(T-t)\right)-\frac{1}{2}\sigma^2(T-t)\right)} \\ &= N'(d_2)e^{\left(ln\left(\frac{I}{V}\right)-r(T-t)\right)} \\ &= N'(d_2)\frac{Ie^{-r(T-t)}}{V} \end{aligned}$$

We can write $d_1 = \frac{lnV - lnI + (r+\frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{(T-t)}}$ and $d_2 = \frac{lnV - lnI + (r-\frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{(T-t)}}$. Thus, $\frac{\partial d_1}{\partial V} = \frac{\partial d_2}{\partial V} = \frac{1}{V\sigma\sqrt{T-t}}$ and $\frac{\partial c}{\partial V}$ simplifies to

$$\frac{\partial c}{\partial V} = N(d_1)$$

Now we have shown that delta, the change in the price of the call option with respect to the change in the price/value of the underlying asset is equal to $N(d_1)$. Next we'll need to find the second derivative of c with respect to V as follows

$$\begin{aligned}\frac{\partial^2 c}{\partial V^2} &= N'(d_1)\frac{\partial d_1}{\partial V}\\ &= N'(d_1)\frac{1}{V\sigma\sqrt{T-t}}\end{aligned}$$

And by differentiating c with respect to t we get

$$\frac{\partial c}{\partial t} = VN'(d_1)\frac{\partial d_1}{\partial t} - rIe^{-r(T-t)}N(d_2) - Ie^{-r(T-t)}N'(d_2)\frac{\partial d_2}{\partial t}$$

We know that $VN'(d_1) = N'(d_2)Ie^{-r(T-t)}$ and therefore

$$\frac{\partial c}{\partial t} = VN'(d_1)\big(\frac{\partial d_1}{\partial t} - \frac{\partial d_2}{\partial t}\big) - rIe^{-r(T-t)}N(d_2)$$

And because $d_1 - d_2 = \sigma\sqrt{(T-t)}$ we have $\frac{\partial}{\partial t}\big(\sigma\sqrt{(T-t)}\big) = \frac{-\sigma}{2\sqrt{(T-t)}}$ and the above equation simplifies to

$$\frac{\partial c}{\partial t} = \frac{-VN'(d_1)\sigma}{2\sqrt{(T-t)}} - rIe^{-r(T-t)}N(d_2)$$

Summarizing the above result we find

$$\begin{aligned}\frac{\partial c}{\partial V}rV + \frac{\partial c}{\partial t} + \frac{1}{2}\frac{\partial^2 c}{\partial V^2}\sigma^2V^2 &= N(d_1)rV + \frac{-VN'(d_1)\sigma}{2\sqrt{(T-t)}} - rIe^{-r(T-t)}N(d_2)...\\ &...+\frac{1}{2}N'(d_1)\frac{1}{V\sigma\sqrt{T-t}}\sigma^2V^2\\ &= N(d_1)rV - rIe^{-r(T-t)}N(d_2)\\ &= r\big(VN(d_1) - Ie^{-r(T-t)}N(d_2)\big)\\ &= rc\end{aligned}$$

We have shown that the closed form solution for the price of a European call option indeed satisfies the BSM differential equation [24].

6.6 Patent as a Call Option

A patent has the characteristics of a call option. It gives its holder the right but not the obligation to buy the underlying asset (invest in the patented project) for a certain strike price (the cost of the investment). In Section 5 we considered the option to invest or wait for more information to be obtained. We saw that we could value our investment opportunity in the same way as a financial call option.

Damodaran (2002) [8], also considered the value of a patent as the option to wait. They argued that the monetary value of the patent should equal the price of the call option on the PV of future cash flow. The exercise price was the direct investment cost and the option's maturity was the expiration date of the patent. The value of the patent was calculated with the closed form solution for the price of a European call option (see Eq.38), assuming the underlying asset followed a GBM (see Eq.28). A dividend yield (δ) was included in the analysis. In real option valuation the dividend yield represents the cost of delaying the project's commercialization. In this section we will employ this approach, discuss its features, and its application in real option valuation.

6.6.1 The Dividend Yield

To understand the importance of the dividend yield (δ) in real option valuation we shall begin by considering a financial call option. The expected return on a dividend paying stock is the sum of the expected dividends and the growth in the stock, $\mu = \delta + \alpha$. Both the dividend yield and the growth rate are calculated as a percentage amount of the stock price. When $\delta > 0$ and there is a significant rise in the stock price, a holder of an American call option could find it optimal to exercise right before an ex-dividend date, to receive the expected dividend payment. On the ex-dividend date the price of the stock will decline by the amount of the dividend payment. When $\delta = 0$ the return from the stock equals its growth rate $\mu = \alpha$. The entire return is captured in the price movements and the holder of an American call option, would not generally exercise prior to maturity [28].

Dixit and Pindyck (1994) [28] argued that there is always some dividend yield $\delta > 0$, in the case of investment opportunities. They denoted this yield as the difference between the expected return (μ) and the expected growth rate in the underlying asset (α). By assuming $\mu > \alpha$, the dividend yield could be treated as the cost of waiting, instead of keeping the option to invest alive. They claimed, when $\delta = 0$ there is no difference between the expected return rate and the expected rate of capital gain. No matter how high the NPV would be, there would be no cost of waiting and one would never invest. Moreover, when δ is large, the cost of waiting is large and the value of the option becomes small, accelerating investment (see Section 7.1.2).

Just as the underlying stock price will decrease after a dividend payment has been made, the value of the expected cash flow from a real asset will decrease as time to maturity decreases. This is due to the revenues that are foregone in the waiting period. Generally, the patented product will generate excess returns only while it's patented. When the patent expires competitors will start producing similar or the same products and take a part of, or the whole market share. Thus a delay of commercializing the product while it is patented will cost the firm a part of those patent protected returns, and every year that goes by without having the patented product commercialized, comes with a cost. When the cost of delaying is estimated from this point of view, δ can be calculated as the percentage change in the PV of cash flow, over a particular period $(\frac{PV_{NextPeriod} - PV_{Today}}{PV_{Today}})$. However, if the cash flow is evenly distributed between years, the cost of waiting can be estimated as a fraction of the time that is left until the option expires. For example, if there are twenty years until a patent expires, a one year delay of commercializing the product could cost $\delta = 1/20$. Similarly, if there were fifteen years until expiration, the cost could equal $\delta = 1/15$. When there is less time until the patent expires, the cost of waiting increases. Hence, the probability of delay is greater in the early stages of the patent's life [8].

6.6.1.1 Adjusting BSM for the Dividend Yield

When there are dividends the expected rate of change in the underlying asset is less than it would be in the absence of dividends or $\alpha = \mu - \delta$. In a risk neutral world we have $\alpha = r - \delta$ and the GBM in Eq.28 can be adjusted as follows

$$dx = (r - \delta)xdt + \sigma xdz \tag{41}$$

Including the dividend yield in the analysis in Section 6.5.1, the differential equation becomes

$$\frac{\partial F(V,t)}{\partial V}(r-\delta)V + \frac{\partial F(V,t)}{\partial t} + \frac{1}{2}\frac{\partial^2 F(V,t)}{\partial V^2}\sigma^2 V^2 = rF(V,t) \tag{42}$$

When the expected growth rate in the underlying asset is $(r-\delta)$, its future value at time T becomes $V_t e^{(r-\delta)(T-t)}$. To provide a way to adjust the BSM option pricing formula for dividends, we introduce a simple rule. If the underlying asset with dividends grows from V_0 today to V_T at time T, then in the absence of dividends it will grow from V_0 to $V_T e^{\delta T}$. Alternatively, if the value of the asset today, in the absence of dividends is $V_0 e^{-\delta T}$, it will grow to V_T at time T. Hence, when valuing a European option that expires at time T on a dividend paying asset, we can reduce the current value V_t to $V_t e^{-\delta(T-t)}$, and value the option as there were no dividends (see Eq.43) [24].

$$c = V_t e^{-\delta(T-t)} N(d_1) - I e^{-r(T-t)} N(d_2) \tag{43}$$

Since $ln\left(\frac{V_t e^{-\delta(T-t)}}{I}\right) = ln\left(\frac{V_t}{I}\right) - \delta(T-t)$, d_1 and d_2 become

$$d_1 = \frac{ln\left(\frac{V_t}{I}\right) + (r - \delta + \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{(T-t)}} \tag{44}$$

$$d_2 = \frac{ln\left(\frac{V_t}{I}\right) + (r - \delta - \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{(T-t)}} = d_1 - \sigma\sqrt{(T-t)} \tag{45}$$

Example. We will proceed with the example from Section 5. The investment strategy was to only invest in the favorable state where the value of the project was $V = \sum_{t=1}^{t=14} \frac{\$1.05}{(1.05)^t} = \$10.39 million$. The risk free rate was estimated 5%. The cost of production and commercialization was $I = \$4.70 million$ and we will assume an annual volatility of 50%. Since the cash flow is evenly distributed between years, the cost of waiting is calculated as a fraction of the number of years that are left until the patent expires $\left(\delta = \frac{1}{(T-t)} = \frac{1}{(20-7)} = \frac{1}{13}\right)$. From Figure 6, it can be seen that the payoff from an immediate investment was $\$2.57 million$ and the value of waiting was $\$2.71 million$.

Now the value of our investment opportunity, by using the BSM option pricing formula for a call option, can be calculated with d_1, d_2, $N(d_1)$ and $N(d_2)$ as follows

$$d_1 = \frac{ln(\frac{\$10.39}{\$4.70}) + (0.05 - \frac{1}{13} + \frac{1}{2}0.5^2)(20-7)}{0.5\sqrt{(20-7)}} = 1.147$$
$$d_2 = \frac{ln(\frac{\$10.39}{\$4.70}) + (0.05 - \frac{1}{13} - \frac{1}{2}0.5^2)(20-7)}{0.5\sqrt{(20-7)}} = -0.655$$

$$N(d_1) = normcdf(-\infty, 1.1475, 0, 1) = 0.874$$
$$N(d_2) = normcdf(-\infty, -0.655, 0, 1) = 0.256$$

The value of the call option becomes

$$c = \$10.39e^{-\frac{1}{13}(13)}0.874 - \$4.70e^{-0.05(13)}0.256 = \$2.71$$

We can see that the BSM option pricing formula generates the same results as obtained in Section 5. It is no coincidence because the two models build on the same replicating portfolio strategy.

Damodaran (2002) [8], takes the position that the value of a patent should equal the price of the call option (c). It may be an unreasonable estimation and the value of the patent should rather be the amount of freedom that is acquired by holding an exclusive right to a product. This freedom is simply the option premium calculated as the difference between an immediate investment and the value of waiting, resulting in a monetary value of

$$\begin{aligned}\text{Value of Patent} &= c - NPV \\ &= \$2.71 - \$2.57 \\ &= \$0.14 million\end{aligned}$$

This is the cost we would be willing to pay for receiving an investment opportunity that is flexible rather than one that only allows for an immediate investment (see Section 5). As the patent's life shortens, the cost of delay will increase and the expected value of the call option declines. To illustrate this point the price of the call option was calculated, letting time vary, while other variables were held constant (see Figure 15). Here the optimal investment strategy would be to keep the option alive until there are 12 years to patent expiration. At that time the option's payoff (NPV) will exceed the value of waiting.

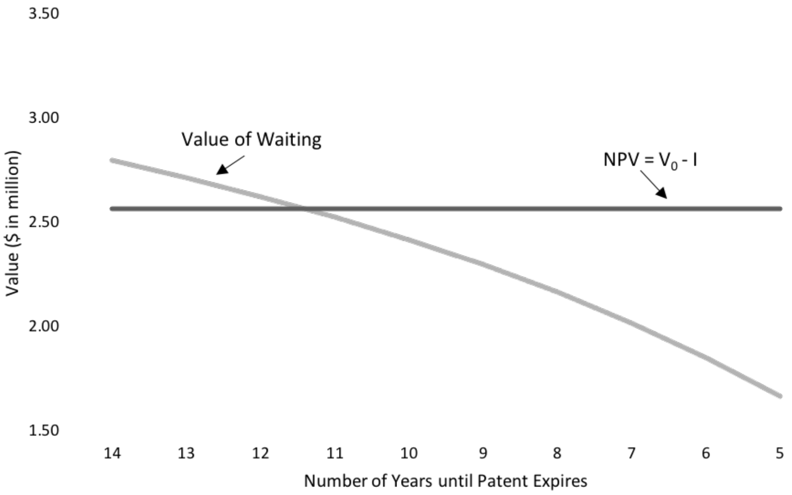


Figure 15: The Value of Waiting vs. NPV.

It can be seen that the value of the option to wait/delay commercialization declines as time to expiration shortens. If there is external competition, then there will be some other firm or individual planning on entering the same market. The competing product might be able to fulfill the same need and demand as our patented product. Hence, the option's time to maturity would be the time until the competitor launches it's product. The greater the number of competing products on the way to the market, the less likely it is that the option to wait will be valuable [8] [28].

In our previous examples we have assumed that the option to invest or wait expires when the patent expires. This is a reasonable estimation if it is expected, due to competition, that there will be no cash flow generated after the patent expires. However, it is possible that the firm has some competitive advantage after the patent expires. Many companies are able to form a strong brand name while their product is patent protected and keep generating excess returns after the patent expires. In this case the life of the option needs to be defined as the expected period over which this competitive advantage can be sustained. An extended lifetime, will result in an increased value of the patent, increasing the incentive to wait [8].

6.6.2 The BSM Model in ROV

The BSM model was introduced in order to value financial assets. Although it is convenient in use, it may not be applicable in valuing options on real assets. For example, it assumes a constant deterministic strike price. In ROV the costs associated with materials, production, or labor etc. are liable to vary through time and often there is significant uncertainty in investment costs. It also assumes that the volatility in the underlying asset is constant but in real projects this parameter is likely to change over time. The BSM model includes an assumption about the probability distribution of the underlying asset. Furthermore, it makes the assumption that the underlying asset can be traded regularly and assumes that investors can adjust their investment portfolios continuously. It values an option at its maturity which may not be convenient in the case of real assets where investments prior to maturity are probable due to external competition or other unexpected events. When the assumptions underlying the BSM model do not hold, the result will become less reliable. However, the model can yield qualitative insights [3] [8] [11] [23].

7 Real Options Theory

In this section we will review the mathematical theory underlying the valuation of real options. The content is mainly derived from the first educational book that deals exclusively with real options, by Dixit and Pindyck (1994) [28]. We begin by considering an investment opportunity with an infinite time horizon, starting with a deterministic case where there is no uncertainty. Then the stochastic case will be solved with both DP and CCA in order to compare those two approaches. We will examine how the problem changes when the time horizon is finite, and finally, introduce equivalent risk neutral valuation.

7.1 Infinite Time Horizon

The model derived in this section was first developed by Mcdonald and Siegel (1986) [34], and later presented by Dixit and Pindyck (1994) [28]. The aim was to find at what time it would be optimal to invest in an irreversible investment opportunity, with a PV of V. At an initial time t the value of the project is known but its future is uncertain. Therefore, it is assumed that V evolves according to the following GBM.

$$dV = \alpha V dt + \sigma V dz$$

At an unknown future time T an investment will be made. The value of the investment opportunity is calculated as the expected payoff discounted to the present by the discount rate ρ, as follows

$$F(V) = maxE\left[(V_T - I)e^{-\rho T}\right] \tag{46}$$

We need to assume that $\rho > \alpha$ such that $\delta = \rho - \alpha > 0$. This is similar to a perpetual call option on a dividend paying stock. We need $\delta > 0$, else one would never exercise the option and waiting would always be the optimal investment strategy [28].

7.1.1 The Deterministic Case

When there is no uncertainty ($\sigma = 0$), the process followed by V simplifies to

$$dV = \alpha V dt$$

Since V follows a lognormal distribution, its expected value at some random future time t becomes

$$ln(V_t) = ln(V_0) + \alpha t$$
$$V_t = V_0 e^{\alpha t}$$

The value of the investment opportunity is

$$F(V) = \left(V_0 e^{\alpha t} - I\right)e^{-\rho t} \tag{47}$$

If we let $\alpha \leq 0$, we can see that V either stays constant or decreases with time. The optimal investment strategy would be to invest immediately or never invest, $F(V_t) = max\big(V_t - I, 0\big)$. Let us consider the scenario where $0 \leq \alpha \leq \rho$. Now $Ve^{\alpha t}$ will grow with time. If the current value of the project is less than the investment cost ($V_0 < I$), there might be some time t, where the opposite would hold. To find the optimal time to invest we differentiate $F(V)$ with respect to t as follows

$$\begin{aligned} \frac{F(V)}{dt} &= \frac{\partial}{\partial t}\big(V_0 e^{-(\rho-\alpha)t} - I e^{-\rho t}\big) = 0 \\ &\Rightarrow -V_0(\rho-\alpha)e^{-(\rho-\alpha)t} + \rho I e^{-\rho t} = 0 \end{aligned}$$

By solving for t we obtain

$$\begin{aligned} -ln\big(V_0(\rho-\alpha)\big) + (\rho-\alpha)t + ln(\rho I) - \rho t = 0 \\ \Rightarrow ln\Big[\frac{\rho I}{V_0(\rho-\alpha)}\Big] = \alpha t \\ \Rightarrow t = \frac{1}{\alpha} ln\Big[\frac{\rho I}{V_0(\rho-\alpha)}\Big] \end{aligned}$$

The optimal time to invest becomes

$$t^* = max\left[\frac{1}{\alpha} ln\Big[\frac{\rho I}{V_0(\rho-\alpha)}\Big], 0\right] \tag{48}$$

If I isn't too much smaller than V_0, we will have $t^* > 0$, indicating that it is optimal to wait. If $t^* = 0$ we should invest immediately. To find for what value of V^* an immediate investment would be preferred, we let $t^* = 0$ as follows

$$\begin{aligned} t^* &= \frac{1}{\alpha}\Big(ln[\rho I] - ln\big[V_0(\rho-\alpha)\big]\Big) = 0 \\ &\Rightarrow ln[\rho I] = ln[V_0(\rho-\alpha)] \\ &\Rightarrow V_0^* = \frac{\rho}{\rho-\alpha} I \end{aligned} \tag{49}$$

When $\alpha = 0$, we have $V_0^* = I$ and it becomes optimal to invest if $V_0 > V_0^*$, else we will never invest. In the case where $\alpha > 0$, we will invest when the value of the investment exceeds the investment cost by an amount $\frac{\rho}{\rho-\alpha}$, but if $V_0 < V_0^*$ we should wait (see Figure 16).

To derive the optimal investment rule when $t^* > 0$, we simply substitute the formula for the optimal timing (Eq.48), into the equation for the value of our investment opportunity (Eq.47) as follows

$$
\begin{aligned}
F(V) &= V_0 e^{-(\rho-\alpha)t} - I e^{-\rho t} \\
&= V_0 e^{-\rho\left[\frac{1}{\alpha} ln\left[\frac{\rho I}{V_0(\rho-\alpha)}\right]\right]} e^{\alpha\left[\frac{1}{\alpha} ln\left[\frac{\rho I}{V_0(\rho-\alpha)}\right]\right]} - I e^{-\rho\left[\frac{1}{\alpha} ln\left[\frac{\rho I}{V_0(\rho-\alpha)}\right]\right]} \\
&= V_0 e^{ln\left[\frac{\rho I}{V_0(\rho-\alpha)}\right]^{(-\rho/\alpha)}} e^{ln\left[\frac{\rho I}{V_0(\rho-\alpha)}\right]} - I e^{ln\left[\frac{\rho I}{V_0(\rho-\alpha)}\right]^{(-\rho/\alpha)}} \\
&= V_0 \left[\frac{\rho I}{V_0(\rho-\alpha)}\right]^{(-\rho/\alpha)} \left[\frac{\rho I}{V_0(\rho-\alpha)}\right] - I \left[\frac{\rho I}{V_0(\rho-\alpha)}\right]^{(-\rho/\alpha)} \\
&= \left[\frac{\rho I}{V_0(\rho-\alpha)}\right]^{(-\rho/\alpha)} \left(\frac{\rho I}{\rho-\alpha} - I\right) \\
&= \left[\frac{V_0(\rho-\alpha)}{\rho I}\right]^{(\rho/\alpha)} \left(\frac{\alpha I}{(\rho-\alpha)}\right)
\end{aligned}
$$

Figure 16 shows the value of our investment opportunity as a function of V w.r.t three different growth rates (α), assuming an investment cost of $I = \$4.70 million$ and a discount rate of $\rho = 0.15$. The critical point $V^* = \frac{\rho}{\rho-\alpha} I$, is exactly where the value of waiting and the value of an immediate investment intercept. When $\alpha = 0$, the optimal investment strategy is to invest if $V > I$, else we will never invest. For $\alpha = 0.02$ we can see that waiting is optimal until $V \approx \$5.40 million$ and when $\alpha = 0.05$ we should wait until $V \approx \$7.10 million$. Hence, an increased growth in V increases the incentive to wait and the value of our investment opportunity [28].

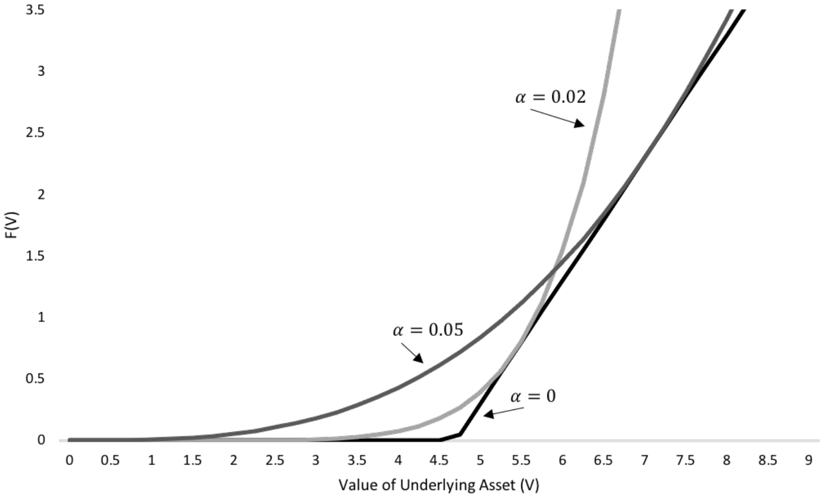


Figure 16: The Value of the Investment Opportunity w.r.t. Different Growth Rates.

7.1.2 The Stochastic Case - Solution by DP

DP is a mathematical optimization method that was first developed by Richard Bellman in the 1950s [35]. It's essence is to solve a problem by breaking it into its subproblems. In the case of an investment decision we can break a sequence of many future decisions into two parts, an immediate decision and the expected value of continuation.

In continuous time analysis, each time period becomes infinitely small ($\Delta t \to 0$), and the Bellman equation is defined as

$$\rho F(x,t) = \max\left[\pi(x,t) + \frac{1}{dt}E\left[dF(x,t)\right]\right] \tag{50}$$

where ρ is the discount factor and $\rho F(x,t)$ is the return required from the asset. The immediate payoff from the asset is $\pi(x,t)$ and $\frac{1}{dt}E\left[dF(x,t)\right]$ is the limit of $\frac{1}{\Delta t}E\left[F(x+\Delta x, t+\Delta t)\right]$, denoting the expected rate of capital gain over a small period of time (dt).

Let us assume that the underlying asset of our investment opportunity (V) follows the GBM, given by

$$dV = \alpha V dt + \sigma V dz$$

When the underlying asset follows a stochastic process, we can not determine the optimal investment time t^*. However, we can find the optimal investment strategy dependent on the critical value V^*, such that it is optimal to invest when $V > V^*$. Since the investment opportunity does not generate any cash flow until investment has been made, the only return from holding it is the rise in capital appreciation. Thus, the Bellman equation (see Eq.50), simplifies to

$$\rho F(V) = \frac{1}{dt}E[dF(V)] \tag{51}$$

Expanding the right hand side of this equation using Itô's Lemma, we get

$$dF(V) = \left[\frac{\partial F(V)}{\partial V}\alpha V + \frac{\partial F(V)}{\partial t} + \frac{1}{2}\frac{\partial^2 F(V)}{\partial V^2}\sigma^2 V^2\right]dt + \frac{\partial F(V)}{\partial V}\sigma V dz$$

Because we are considering the investment opportunity as infinite, it is not a function of time and $\frac{\partial F(V)}{\partial t} = 0$. Since $E[dz] = 0$ (see Section 6.2), the expected value of the change in $F(V)$ becomes

$$E\left[dF(V)\right] = \left[\frac{\partial F(V)}{\partial V}\alpha V + \frac{1}{2}\frac{\partial^2 F(V)}{\partial V^2}\sigma^2 V^2\right]dt$$

By substituting this result into the Bellman equation (see Eq.51) we get the following differential equation

$$\frac{\partial F(V)}{\partial V}\alpha V + \frac{1}{2}\frac{\partial^2 F(V)}{\partial V^2}\sigma^2 V^2 - \rho F(V) = 0$$

In order to compare this solution to the CCA approach (see Section 7.1.3), we let $\alpha = \rho - \delta$. As explained earlier we need $\rho > \alpha$ and $\delta = \rho - \alpha > 0$, to make sure there will be an optimal solution. Now the Bellman equation becomes

$$\frac{\partial F(V)}{\partial V}(\rho - \delta)V + \frac{1}{2}\frac{\partial^2 F(V)}{\partial V^2}\sigma^2 V^2 - \rho F(V) = 0 \tag{52}$$

$F(V)$ must satisfy this equation subject to the following boundary conditions

- $F(0) = 0$
  This is a boundary condition due to the stochastic process followed by V. If it goes to zero, it will stay at zero. Hence V $= 0$ *and the investment opportunity will be of no value.*

- $F(V^*) = V^* - I$
  This is the value matching condition. At the critical value V^* *investment is optimal and we will receive the payoff* $V^* - I$*, upon investing.*

- $\frac{\partial F(V^*)}{\partial V} = 1$
  This is the smooth pasting condition that determines a unique stopping point. If $F(V^*)$ *is not continuous and smooth at the critical exercise point* V^* *than it might be better to exercise at another point.*

By rearranging terms in the second condition we can also say that the critical value (V^*) needs to equal or exceed the sum of the opportunity cost of waiting and the direct investment cost or $V^* = F(V^*) + I$. This verifies that the NPV rule is flawed by indicating that one should invest when $V \geq I$.

To find the value of our investment opportunity we need to solve the second order homogeneous differential equation (see Eq.52) subject to the boundary conditions. To do this it is assumed that the solution takes the form

$$F(V) = AV^{\beta} \tag{53}$$

where A is constant. Substituting this solution into Eq.52, we get the following quadratic function (denoted as Q)

$$Q = \frac{1}{2}\sigma^2\beta(\beta - 1) + (\rho - \delta)\beta - \rho = 0 \tag{54}$$

The possible values for the roots (β) are

$$\beta_1 = \frac{1}{2} - \frac{\rho - \delta}{\sigma^2} + \sqrt{\left(\frac{\rho - \delta}{\sigma^2} - \frac{1}{2}\right)^2 + \frac{2\rho}{\sigma^2}} > 1$$

$$\beta_2 = \frac{1}{2} - \frac{\rho - \delta}{\sigma^2} - \sqrt{\left(\frac{\rho - \delta}{\sigma^2} - \frac{1}{2}\right)^2 + \frac{2\rho}{\sigma^2}} < 0$$

The quadratic function in Eq.54 is illustrated in Figure 17. The two roots can be seen where the function crosses the horizontal axis, one to the right of one ($\beta_1 > 1$) and one to the negative side ($\beta_2 < 0$). When $\beta \to \pm\infty$, $Q \to +\infty$. When $\beta = 0$ we have $Q = -\rho$ and when $\beta = 1$ the function equals $Q = -\delta$. Recall that we are assuming $\rho > \delta > 0$, otherwise waiting would always be optimal and one would never invest. Thus we are concentrating on the positive root (β_1) which we will denote as β.

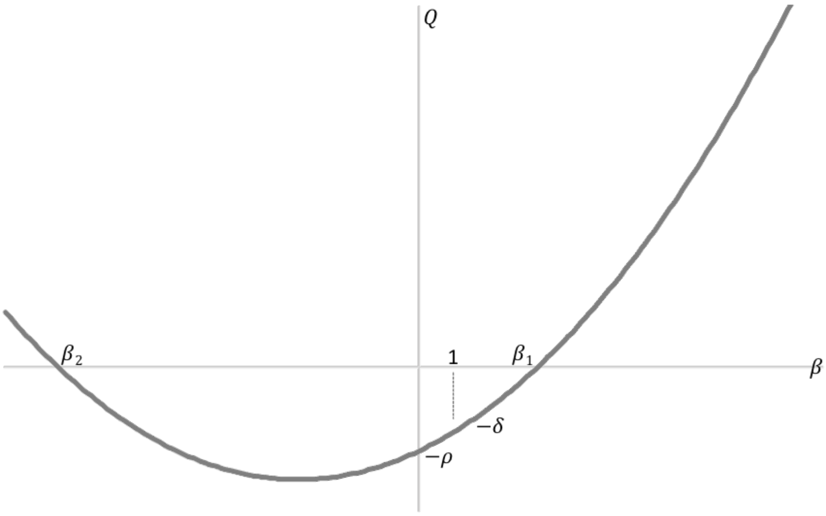


Figure 17: The Quadratic Function

Because Eq.52 is linear in the dependent variable $F(V)$ and its derivatives, the general solution can be written as a linear combination of two independent solutions $F(V) = A_1V^{\beta_1} + A_2V^{\beta_2}$. However, the first boundary condition needs to hold $F(0) = 0$. Because $\beta_2 < 0$ we need $A_2 = 0$ and the solution simplifies to Eq.53, where $\beta = \beta_1 > 1$. To see if this solution satisfies the boundary conditions outlined above, we substitute and find

$$\begin{aligned} F(0) &= A0^{\beta} = 0 \\ F(V^*) &= AV^{*\beta} = V^* - I \Rightarrow I = V^*(1 - AV^{*\beta-1}) \\ \frac{\partial F(V)}{\partial V^*} &= A\beta V^{*\beta-1} = 1 \Rightarrow V^{*\beta-1} = \frac{1}{A\beta} \end{aligned}$$

Now we can solve for the critical value V^*, observing

$$\begin{aligned} I &= V^*\left(1 - A\frac{1}{A\beta}\right) = V^*\left(1 - \frac{1}{\beta}\right) \\ \Rightarrow V^* &= \frac{I}{1-\frac{1}{\beta}}\left(\frac{\beta}{\beta}\right) \\ \Rightarrow V^* &= \frac{\beta}{\beta-1}I \end{aligned} \tag{55}$$

Because we have $\beta > 1$, the factor $\frac{\beta}{\beta-1}$ will always be larger than one (see Eq.55). Therefore, the critical value V^* at which it is optimal to invest will always exceed the investment cost I by an amount determined by this factor. When $\beta \to 1$, the factor becomes larger and the gap between V^* and I increases. This indicates that the traditional NPV rule, encouraging an investment when $V \geq I$, is incorrect.

To solve for the constant A we use the second boundary condition $F(V^*)$ as follows [36]

$$F(V^*) = AV^{*\beta} = V^* - I$$

$$\Rightarrow A = \frac{V^* - I}{V^{*\beta}}$$

By substituting V^* (see Eq.55) into this result for A, we get

$$\begin{aligned}
A &= \frac{\frac{\beta}{\beta-1}I - I}{\left(\frac{\beta}{\beta-1}I\right)^{\beta}} = \left(I\left[\left(\frac{\beta}{\beta-1}\right)-1\right]\right)\left(\frac{\beta}{\beta-1}I\right)^{-\beta} = I^{1-\beta}\left[\left(\frac{\beta}{\beta-1}\right)^{1-\beta} - \left(\frac{\beta}{\beta-1}\right)^{-\beta}\right] \\
&= \frac{1}{I^{\beta-1}}\left[\left(\frac{\beta-1}{\beta}\right)^{\beta-1} - \left(\frac{\beta-1}{\beta}\right)^{\beta}\right] = \frac{1}{I^{\beta-1}}(\beta-1)^{\beta-1}\left[\beta^{1-\beta} - \frac{\beta-1}{\beta^{\beta}}\right] \\
&= \frac{1}{I^{\beta-1}}(\beta-1)^{\beta-1}\left[\beta^{1-\beta} - \left(\beta^{1-\beta} - \beta^{-\beta}\right)\right] = \frac{1}{I^{\beta-1}}(\beta-1)^{\beta-1}\left(\beta^{-\beta}\right) \\
\Rightarrow A &= \frac{(\beta-1)^{\beta-1}}{I^{\beta-1}(\beta)^{\beta}} \qquad (56)
\end{aligned}$$

Since β is a function of σ, ρ and δ we can examine how changes in these parameters will affect the factor $\frac{\beta}{\beta-1}$ as well as the optimal investment rule. Figure 18 shows how change in σ affects the value of β, keeping ρ and δ constant.

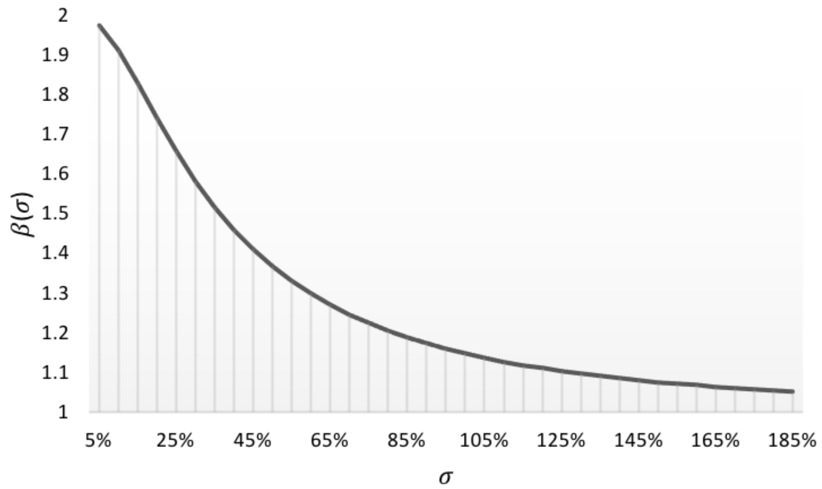


Figure 18: Changes in β w.r.t. σ ($\rho = 0.2$ and $\delta = 0.1$).

It can be seen that as σ increases, β decreases. When β decreases the factor $\frac{\beta}{\beta-1}$ increases. Hence, the required difference increases between V^* at which it is optimal to invest and the investment cost I, increasing the value of the option to invest. In context to this analysis, Chang (2005) [2] examined how change in σ would affect the constant A, keeping other parameters constant (see Figure 19).

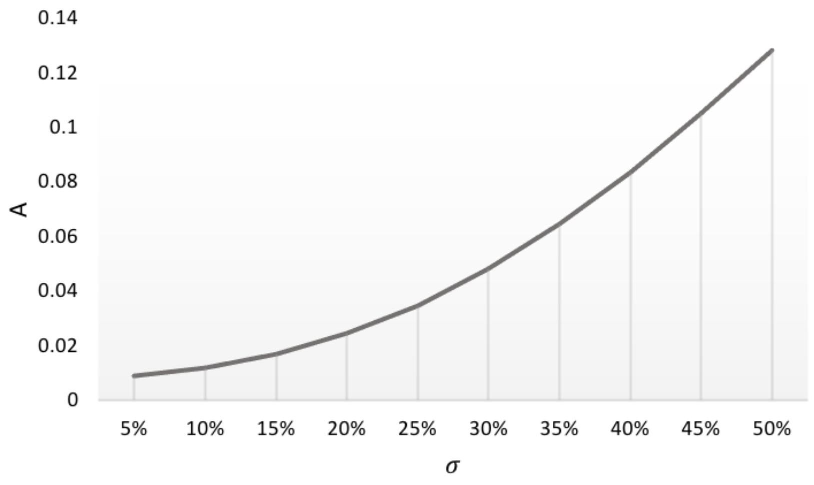


Figure 19: Changes in A w.r.t. σ ($I = 30$, $\rho = 0.2$ and $\delta = 0.1$).

Apparently, an increase in σ has the opposite effect on A than it has on β. Since the value of the option is $F(V) = AV^{\beta}$, it's value is ambiguous under the trade off between A and β when volatility increases. Interestingly, when the option is deep in the money, it's value decreases as σ increases. This is because a decrease in β overrides the increase in A when volatility is low. Hence an option with low volatility is more valuable than one with high volatility (see Figure 20). This is in contrast with finite horizon financial options, where an increase in volatility, always increases the value of the option (see Table 2).

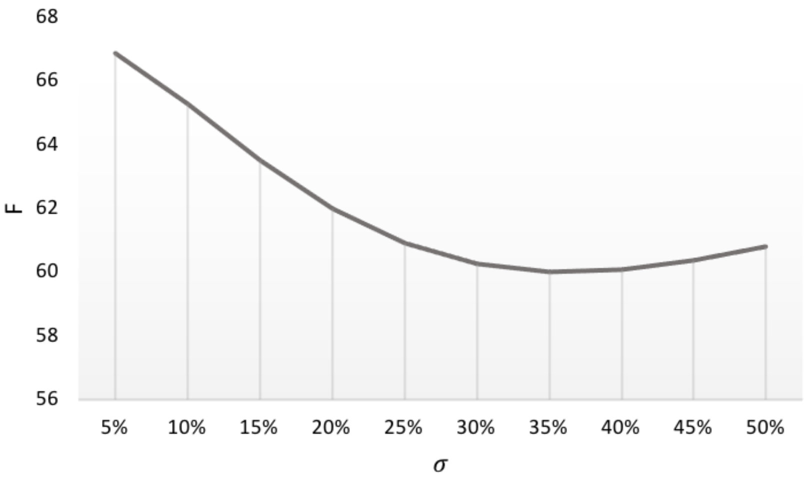


Figure 20: Option Value w.r.t. σ ($V = 90$, $I = 30$, $\rho = 0.2$ and $\delta = 0.1$).

However, when the option is not deep in the money, out of money, or at the money, an increase in σ will increase its value (see Figure 21).

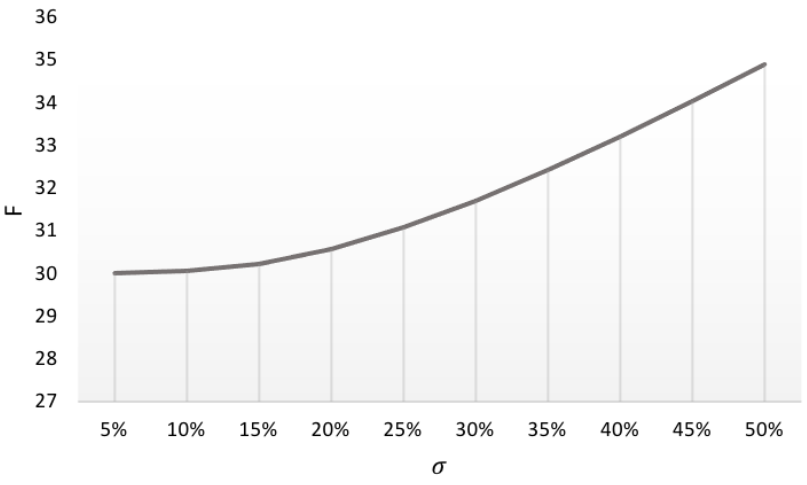


Figure 21: Option Value w.r.t. σ ($V^* = 60$, $I = 30$, $\rho = 0.2$ and $\delta = 0.1$).

Let us revisit Figure 18, considering the extreme case where $\sigma \to \infty$. Under this scenario it can be seen that $\beta \to 1$ and thus, $V^* \to \infty$. Here it becomes unlikely that the value of the underlying asset reaches the critical value V^* and waiting will always be optimal. When $\sigma \to 0$ and $\alpha > 0$ the quadratic function simplifies to

$$\Rightarrow \frac{1}{2}\sigma^2\beta(\beta-1) + (\rho-\delta)\beta - \rho = 0$$
$$\Rightarrow \beta = \frac{\rho}{\rho-\delta}$$

By substituting this value of β into Eq.55 for V^*, we obtain

$$V^* = \left[\frac{\frac{\rho}{\rho-\delta}}{\frac{\rho}{\rho-\delta}-1}\right]I = \left[\frac{\frac{\rho}{\rho-\delta}}{\frac{\rho-(\rho-\delta)}{\rho-\delta}}\right]I = \left(\frac{\rho}{\rho-\delta}\right)\left(\frac{\rho-\delta}{\delta}\right)I = \frac{\rho}{\delta}I$$
$$\Rightarrow V^* = \frac{\rho}{\rho-\alpha}I$$

This is the same result as derived under the deterministic case (see Section 7.1.1). It shows that an increase in $\alpha > 0$ will increase the critical value V^*, increase the incentive to wait, and the value of our investment opportunity (see Figure 16). Figure 22 shows how change in δ affects the value of β, holding σ and ρ constant.

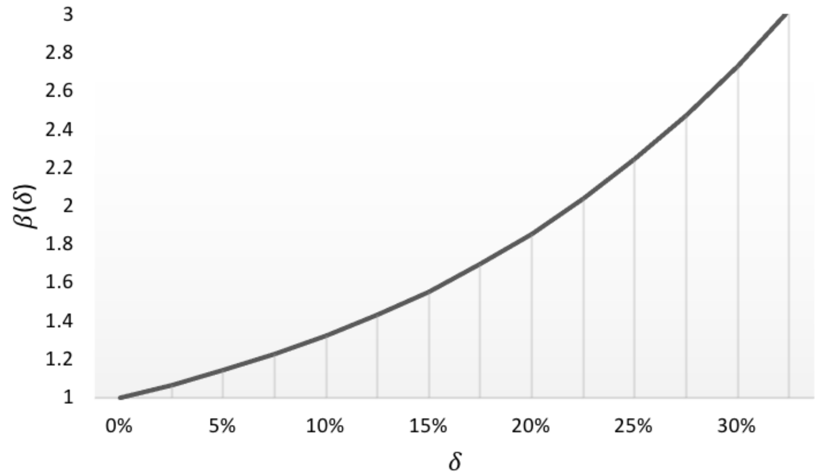


Figure 22: Changes in β w.r.t. δ ($\sigma = 0.3$ and $\rho = 0.35$).

When δ increases it can be seen that β increases. As a consequence the factor $\frac{\beta}{\beta-1}$ decreases as well as the critical value V^*. Hence higher cost of waiting (δ) accelerates investment. In the extreme case where $\delta \to \infty$ the value of the option goes to zero. Figure 23 displays how change in the discount factor ρ affects the value of β, keeping other parameters constant.

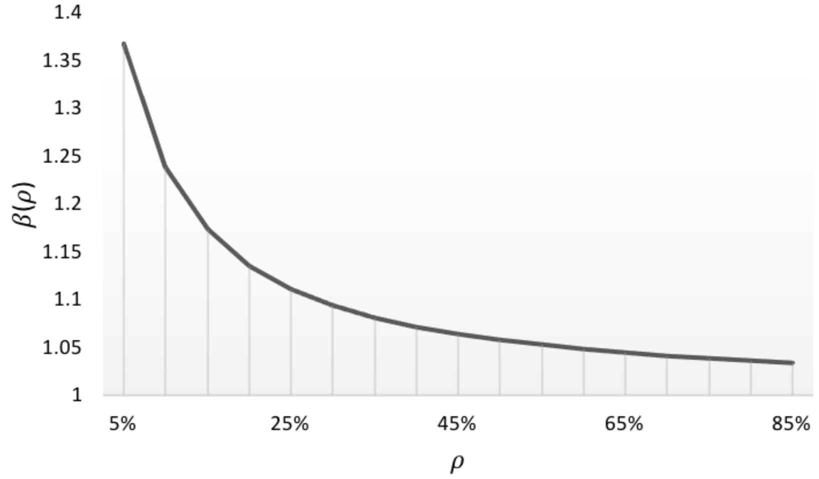


Figure 23: Changes in β w.r.t. ρ ($\sigma = 0.3$ and $\delta = 0.03$).

When ρ increases we can see that β decreases. Thus, the factor $\frac{\beta}{\beta-1}$ increases, increasing the incentive to wait [28].

7.1.3 The Stochastic Case - Solution by CCA

The main assumption underlying the CCA approach is that the stochastic asset underlying the investment opportunity (V) can be replicated with another traded asset, often called the spanning asset. To be able to construct a replicating portfolio the price of the spanning asset needs to be perfectly correlated with the price of the underlying asset. In the absence of arbitrage opportunities the value of the replicating portfolio must equal the value of the investment opportunity. This is the same methodology as explained in Section 5.1 and under the BSM model (see Section 6.5.1). If the underlying asset were a commodity that is traded in the market, it would be convenient to use the CCA approach. It is seldom that a spanning asset exists for $R\&D$ projects such as young patented assets. Nevertheless, we will assume that an asset exists ($\bar{v}$) that is perfectly correlated with the stochastic variable underlying our investment opportunity (V). Because these two assets are perfectly correlated their correlation with the market is the same ($\rho_{\bar{v}m} = \rho_{Vm}$), thus they have the same β. Now we assume that the spanning asset evolves according to the following stochastic process

$$d\bar{v} = \mu\bar{v}dt + \sigma\bar{v}dz$$

where μ is the expected return that reflects the non-diversifiable risk, obtained from the $CAPM$ model (see Section 2.1.3). We assume that the expected percentage change in V is less then the expected rate of return an investor would require from holding the spanning asset $\bar{v}$. Thus $\alpha < \mu$ and $\delta = \mu - \alpha$, represents the cost of waiting. To construct a replicating portfolio we take a long position in our investment opportunity $F(V)$, and short $\Delta = \frac{\partial F(V)}{\partial V}$ units of the underlying asset (this is the same as shorting the spanning asset because the two assets are perfectly correlated). Hence the value of our portfolio is $\Pi = F(V) - \Delta V$. No rational investor would hold the long position unless receiving at least the risk adjusted expected return (μ). Now $\mu = \alpha + \delta$ and since the short position includes $\Delta = \frac{\partial F(V)}{\partial V}$ of the underlying asset, it will require a payment of $\delta V \Delta = \delta V \frac{\partial F(V)}{\partial V}$. The composition of the portfolio may change over time as V changes. However, over a very short period of time dt we hold Δ fixed. Thus the return from the portfolio over the time interval dt is

$$Return = dF(V) - \frac{\partial F(V)}{\partial V}dV - \delta V \frac{\partial F(V)}{\partial V}dt$$

Using Itô's Lemma and the properties of the Wiener process as earlier, we get

$$dF(V) = \frac{\partial F(V)}{\partial V}dV + \frac{1}{2}\frac{\partial^2 F(V)}{\partial V^2}dV^2$$

The change in $F(V)$ is not a function of time since we are assuming infinite time horizon. From the stochastic process followed by V we know that the variance is $dV^2 = \sigma^2 V^2 dt$. Substituting for $dF(V)$ the return becomes

$$Return = \frac{1}{2}\frac{\partial^2 F(V)}{\partial V^2}\sigma^2 V^2 dt - \delta V \frac{\partial F(V)}{\partial V}dt$$

This return is risk free. To avoid arbitrage opportunities it should equal $r\Pi dt$ as follows

$$\frac{1}{2}\frac{\partial^2 F(V)}{\partial V^2}\sigma^2 V^2 dt - \delta V \frac{\partial F(V)}{\partial V}dt = r\left(F(V) - \frac{\partial F(V)}{\partial V}V\right)dt$$

Rearranging terms and dividing through by dt we obtain the following differential equation

$$\frac{\partial F(V)}{\partial V}(r-\delta)V + \frac{1}{2}\frac{\partial^2 F(V)}{\partial V^2}\sigma^2 V^2 - rF(V) = 0 \tag{57}$$

It can be seen that this equation is closely related to the differential equation derived under the *DP* approach (see Eq.52). The difference is that the risk free rate r has replaced the discount factor ρ. The value of our investment opportunity $F(V)$ needs to satisfy Eq.57 subject to the same boundary conditions as outlined earlier (see Section 7.1.2). The solution takes the same form as before

$$F(V) = AV^{\beta}$$

where V^* is given by Eq.55, the constant A is given by Eq.56, and the root β is defined as

$$\beta = \frac{1}{2} - \frac{r-\delta}{\sigma^2} + \sqrt{\left(\frac{r-\delta}{\sigma^2} - \frac{1}{2}\right)^2 + \frac{2r}{\sigma^2}}$$

7.1.4 Comparing DP and CCA

We have seen that *DP* and *CCA*, satisfy very similar differential equations. Their difference lies in the estimation of the expected rate of return. The *DP* approach requires an estimation of a discount rate (ρ) that reflects the decision maker's subjective judgement of risk. Under the *CCA* approach on the other hand, this rate can be estimated according to the capital market equilibrium (*CAPM*).

Although the *CCA* approach offers a better way of estimating the expected rate of return, it does require that the investment opportunity can be replicated with another traded asset. When it is assumed that the underlying asset follows a stochastic process, the stochastic component of the spanning asset has to be perfectly correlated to the stochastic component of the underlying asset. This can be quite demanding, especially in the case of a new unique invention [28].

Under the assumption of risk neutrality ($\rho = r$), the *CCA* solution is equivalent to the *DP* solution. In this case the change in σ, δ, or r (holding other parameters fixed), will have the same effect on β and the critical value V^* (see Section 7.1.2). However, when investors are risk averse and the expected return under the *CCA* approach is calculated according to the *CAPM* model (see Section 2.1.3), we have the following relationship

$$\delta = \mu - \alpha = \left(r + \frac{\rho_{\bar{v}m}}{\sigma_m}\sigma(r_m - r)\right) - \alpha$$

If we let δ vary (keeping α constant), an increase in the risk free rate r as well as an increase in σ will have the opposite effect on the critical value V^*. In this case, an increase in these parameters is likely to be accompanied with an increase in μ, which implies an increase in δ. This lowers the value of the investment opportunity and accelerates investment [28] [34]. This effect is prevalent even when the option is out of the money [2].

7.2 Finite Time Horizon

Many real life investment opportunities do not come with an infinitely lived option to invest. An example of this is a patent that has a known expiration date. When the time horizon is finite, the investment opportunity becomes a function of time. We can go through the same steps as before and construct a risk free portfolio. Assuming a spanning asset exists, we will obtain the same risk free return as earlier.

$$Return = dF(V,t) - \frac{\partial F(V,t)}{\partial V}dV - \delta V\frac{\partial F(V,t)}{\partial V}dt$$

However, when we use Itô's Lemma to expand $F(V,t)$, the time variable does not vanish as in the previous section and we obtain

$$Return = \frac{1}{2}\frac{\partial^2 F(V,t)}{\partial V^2}\sigma^2 V^2 dt + \frac{\partial F(V,t)}{\partial t}dt - \delta V\frac{\partial F(V,t)}{\partial V}dt$$

To avoid arbitrage opportunities this return should equal the risk free rate ($r\Pi dt$), and we get

$$\frac{1}{2}\frac{\partial^2 F(V,t)}{\partial V^2}\sigma^2 V^2 dt + \frac{\partial F(V,t)}{\partial t}dt - \delta V\frac{\partial F(V,t)}{\partial V}dt = r\Big(F(V,t) - \frac{\partial F(V,t)}{\partial V}V\Big)dt$$

Rearranging terms and dividing through by dt we obtain the following differential equation

$$\frac{\partial F(V,t)}{\partial V}(r-\delta)V + \frac{\partial F(V,t)}{\partial t} + \frac{1}{2}\frac{\partial^2 F(V,t)}{\partial V^2}\sigma^2 V^2 - rF(V,t) = 0 \tag{58}$$

This is the same equation as we saw earlier after adjusting the BSM model for dividends (see Eq.42). In very few cases the partial differential equation leads to a closed form solution such as the BSM equation for the European call option (see Section 6.5.2). However, a reasonable set of boundary conditions for Eq.58 could be

- $F(0,t) = 0$
- $F(V^*,t) = V^* - I$
- $\frac{\partial F(V^*,t)}{\partial V} = 1$
- $F(V,T) = max\big(V - I, 0\big)$

The first three conditions are similar to the ones used in the case of infinite time horizon (see Section 7.1.2). We need to define a boundary condition for the expiration of the finite time horizon. The fourth condition is an example of such boundary where we will exercise at time T if the stochastic variable V is greater than I. In contrast to the infinite time horizon in the previous section, Eq.58 subject to these boundary conditions can not be solved analytically [37].

Real option solution methods can be divided into two groups, analytical and numerical methods. The former are the DP and CCA which can sometimes yield a closed form solution as illustrated. When the time horizon is finite, or the number of variables increases, or the variables are inconstant, some problems can not be solved analytically. In that case there is a need for numerical solution methods that can approximate the underlying stochastic variables [28] [34]. The most common numerical methods are the binomial tree approach (see Section 4.1), Monte Carlo simulation (see Section 8.1), and finite difference methods.

In the beginning of the real options analysis era, most problems were solved as infinite since it offers a convenient, one dimensional solution [37]. It has been argued that it is possible to account for the omission of finite time horizon in a model by letting the value of the underlying asset follow a jump process. In such a process, the underlying variable jumps to zero at some unknown date, and stays there. It is also possible to let the underlying asset have an average downward drift (δ) that is expected to decline at a rapid rate, such that the omission of finite time horizon may be insignificant [34]. However, it is questionable to value a finite investment opportunity as an infinitely lived one. This is because an infinitely lived patent will always be worth more than a finitely lived one. For example, Schwartz (2004) [16] found that by extending the duration of a patent by 10%, the value of the underlying project increased by 35%.

7.3 Equivalent Risk Neutral Valuation

In this section we will look at an example of an equivalent risk neutral valuation that further reflects the relationship between DP and CCA. Let us consider a project with profit flow $\pi(x,t)$. The underlying asset x follows the GBM, given by

$$dx = \alpha x dt + \sigma x dz$$

The value of the firm at a current time t is $F(x,t)$ and at a future time $t+dt$ the value of the asset changes to $x+dx$. We will assume that the project will end at a finite time T, with a terminal payoff of $\Omega(x_T,T)$. With dynamic programming we specify a discount rate ρ and the current value of the firm $F(x,t)$ is the expected PV, based on information as of time t, given by

$$F(x,t) = E_t\Big[\int_t^T e^{-\rho(\tau-t)}\pi(x_\tau,\tau)d\tau + e^{-\rho(T-t)}\Omega(x_T,T)\Big] \tag{59}$$

After a small time period dt the value of the underlying asset has changed to $F(x+dx,t+dt)$. Since dx is a random increment from the stochastic process we need to take the expected value discounted back to the present, over the small time period dt, as follows

$$F(x,t) = \pi(x,t)dt + e^{-\rho dt}E_t\big[F(x+dx,t+dt)\big] \tag{60}$$

This equation is one kind of the Bellman equation since the time period from t to T is broken into two components, the current short interval dt $\big(\pi(x,t)dt\big)$ and the continuation value $\big(e^{-\rho dt}E_t\big[F(x+dx,t+dt)\big]\big)$. To find a solution we expand the right hand side of the equation using Itô's Lemma (ignoring dt^2 and higher terms).

$$
\begin{aligned}
&\pi(x,t)dt + e^{-\rho dt}E_t\big[F(x+dx,t+dt)\big] \\
&= \pi(x,t)dt + (1-\rho dt)\big[F(x,t) + \frac{\partial}{\partial t}F(x,t)dt \ldots \\
&\ldots + \frac{\partial}{\partial x}F(x,t)\alpha x dt + \frac{1}{2}\frac{\partial^2}{\partial x^2}F(x,t)\sigma^2x^2dt\big] \\
&= F(x,t) + \big[\pi(x,t) - \rho F(x,t) + \frac{\partial}{\partial t}F(x,t) + \frac{\partial}{\partial x}F(x,t)\alpha x \ldots \\
&\ldots + \frac{1}{2}\frac{\partial^2}{\partial x^2}F(x,t)\sigma^2x^2\big]dt
\end{aligned}
$$

By substituting this result into Eq.60 we see that $F(x,t)$ satisfies the following differential equation

$$\pi(x,t) + \frac{\partial F(x,t)}{\partial x}\alpha x + \frac{\partial F(x,t)}{\partial t} + \frac{1}{2}\frac{\partial^2 F(x,t)}{\partial x^2}\sigma^2x^2 - \rho F(x,t) = 0 \tag{61}$$

subject to the following boundary condition

$$F(x,T) = \Omega(x,T) \qquad \text{for all } x \tag{62}$$

We have shown that Eq.59 is the solution to the differential equation in Eq.61. This is a special case of a general result known as the Feynman-Kae formula [28]. Now if we consider the differential equation for CCA, subject to the boundary condition (see Eq.62), we have

$$\pi(x,t) + \frac{\partial F(x,t)}{\partial x}(r-\delta)x + \frac{\partial F(x,t)}{\partial t} + \frac{1}{2}\frac{\partial^2 F(x,t)}{\partial x^2}\sigma^2x^2 - rF(x,t) = 0 \tag{63}$$

We do not have the solution to this differential equation beforehand. Although, we can see that this equation is quite similar to Eq.61, here r replaces ρ and $(r-\delta)$ replaces α. Since $\mu = \delta + \alpha$, and by assuming that the required return is $\mu = r$, we can write $r = \delta + \alpha$ or $\alpha = r - \delta$. Adjusting the stochastic process for the underlying variable x, we obtain the following equivalent process

$$dx' = (r-\delta)x'dt + \sigma x'dz$$

The artificial variable x' starts at the same initial value as x, but then it follows a different stochastic process dx'. Therefore, we can write the solution to Eq.63 as follows [28]

$$F(x,t) = E'_t\Big[\int_t^T e^{-r(\tau-t)}\pi(x'_\tau,\tau)d\tau + e^{-r(T-t)}\Omega(x'_T,T)\Big]$$

8 The Model

The proposed model of this study is built on the model originally presented by Schwartz (2004) [16], later developed by Ernst, Legler, and Lichtenthaler (2010) [9] and simplified by Hernández, Güemes, and Ponce (2018) [15]. These studies aimed at determining the value of a patent in the pharmaceutical and chemical industries. A simulation approach was implemented and the decision process was viewed from the perspective of abandonment. Even though abandonment is discussed extensively in the literature, most often it is the option to default which is under consideration. That said, abandonment does describe the decision to default, in that the project will be stopped when the decision is taken.

When constructing a model in order to value a patent it makes a difference which industry is under consideration. Patented projects can be seen in industries such as for example pharmaceuticals, telecommunications, computer and technologies, semiconductors, food and beverages, automobiles, and biotechnologies. The $R\&D$ process, competition interactions and market conditions can vary between industries. Most research on patent valuation, using real options, is to be found in the pharmaceutical industry, which has become a research-oriented sector. In that particular industry the product goes through many stages in $R\&D$ until it is accepted by the FDA. This process is expensive, time consuming and only one out of 10,000 discovered compounds become a prescription drug. Hence the probability of failure is high and the option to abandon has been found valuable. This industry offers a long history of returns in the market, a great variety of possible spanning assets, and a wide experience in $R\&D$. Therefore, it may be straight forward to estimate parameters in the purpose of a valuation. In contrast, a pioneer venture may have no available data of historical returns and no comparable asset in the market. The management may need to prove demand for their new invention and enhance the company's market position if that market or a spin off product should develop in the upcoming years. Thus, at the outset the cash flow is likely to be dominated by expenses, and the most valuable option might be a growth option on possible future growth opportunities [11].

No project has the same characteristics. The more options that are allowed in a model the wider the range of managerial flexibility - adding value to the patent. Precautions need to be taken when estimating which option or combination of options is the most appropriate. Estimations like these depend on the nature of the project to be valued. The aim of this thesis is to provide a general model that can be applied to obtain the monetary value of a granted patent. To do this we examined what patented projects have in common. The first commonality is that their future is uncertain. That means that there is a need to account for uncertainty in both expected cash flow and investment costs. Second, a new invention usually has to go through some $R\&D$ before the product is fully developed and marketable. At this stage the time to completion and expected development costs are both uncertain. If the product is successfully developed and market conditions are favorable, production and marketing will take place. Finally, they all have a finite horizon protection from competition (usually 20 years). When the patent expires the market share will decline or even vanish but this effect will be different between various projects and industries.

The model developed aims to value a patented project as a sequential choice over continuation, expansion, or abandonment. It is reasonable to assume that if the project is marginal, the management may have the opportunity to expand its scope. However, if the option to expand does not make the project favorable the decision to default will be taken. This situation can arise if costs turn out to be higher than expected and/or cash flows turn out to be lower than anticipated. The concentration will be on the uncertainty over expected future cash flow, uncertainty in development costs, time to completion of the development phase, and uncertainty in the estimated costs of commercialization. The values we use when implementing our model are hypothetical and not industry specific. That said, in reality, market conditions need to be evaluated within the parameters of the type of industry and the nature of the project.

The simulation approach used to solve the real options problem will be introduced and the variables used in the model will be examined. We will explain the algorithm and present the results from the simulation. To better realize how different parameters affect the value of the patent a sensitivity analysis will be performed.

8.1 Solution Method

There are several variables that need to be evaluated when selecting the most appropriate solution method. The evaluator needs to distinguish between European and American options, how many state variables are to be taken into account, and whether they are path dependent. For example, it is convenient to use simulation when European options are to be valued because the simulation approach is forward looking - meaning that it works from the beginning to the end of the life of the option. American options on the other hand do have features of dynamic programming. They are valued by working from the end of the life of the option to the time being observed. In that case the binomial tree approach is the most suitable numerical solution method since it allows for comparing the value of an early exercise to the one at maturity (see Section 4.1). Research has been conducted in order to find an approach that can approximate the value of American options by simulation, using simple least squares, see Longstaff and Schwartz (2001) [38]. Although the binomial tree approach has been widely used, it is not convenient when there are many state variables or where there are path dependencies [24]. The model presented includes a European sequential option where the state variables are path dependent. Thus the simulation approach was found to be the most appropriate solution method for this study.

8.1.1 Monte Carlo Simulation

The simulation approach is a useful tool for solving real options valuation problems. It is computationally efficient, can be easily implemented, intuitive, transparent, and flexible [39]. This approach is well suited for valuing path dependent options since it simulates new values depending upon the values from the state before. The method becomes relatively more efficient as the number of underlying variables increases or where there are many stochastic variables.

It is believed that the Monte Carlo simulation approach satisfies the requirements and needs of the proposed model. In each simulation run the expected value of the investment opportunity is calculated where the underlying stochastic variables are subject to random movements. At every decision node the expected payoff (F_k) is calculated. The process is performed 10,000 times. The simulation constructs a probability distribution of possible outcomes where the final expected value is the mean of all the possible option values ($\bar{F}$).

$$\bar{F} = \frac{\sum_{k=1}^{n=1e4} F_k}{n}$$

As the number of simulations increases the more accurate the result will be. When ($n \to \infty$), then $\bar{F} \to E[F_k]$, with probability 1 [15] [39] [40].

8.2 Model Derivation

In the following subsections we will explain the variables used in the model and their characteristics. Most $R\&D$ projects involve considerable uncertainty over costs and future cash flow. In the model, both of these will be taken into account. It is assumed that investment costs are irreversible and an amount invested is a sunk cost. To maintain computational efficiency the model is built in a discrete time manner.

8.2.1 Cost To Completion

As earlier stated (see Section 5.4), there are two types of uncertainties over costs. One is related to the overall economy (non-diversifiable) such as government regulations and material costs. The other is firm specific such as the amount of time and effort needed to complete a project. The former is often called input cost uncertainty, and the latter, technical uncertainty. Input cost uncertainty will change whether or not the firm is investing. This kind of uncertainty increases the incentive to wait for more information to be obtained. The opposite holds for technical uncertainty which can only be resolved by investing, and thus, encourages investment [32]. For $R\&D$ projects, technical uncertainty is far more important [28]. This is because it is not known at the outset how much time, effort, or cost is needed to complete the project. Thus to resolve uncertainty, there is a need to invest.

In the beginning of the analysis it is estimated that the total cost needed to finish the development phase is $K_0 = \$7.50 million$. The variable for the cost to completion is supposed to follow a stochastic process until the $R\&D$ has been completed as follows

$$dK = -Idt + \sigma_K \sqrt{IK} dz$$

where I is the expected investment cost (see Section 8.2.4), σ_K is the uncertainty over the cost to completion, and dz is the increment of the Wiener process (see Section 6.2). The uncertainty over cost to completion (σ_K) should reflect the variability of the estimated and realised development costs. This estimation can be obtained from the firm's earlier experience in $R\&D$, or from similar projects/firms in the same industry. Usually, costs regarding $R\&D$ projects are highly uncertain. Therefore, it is reasonable to estimate a variability of 50%. The first term $-Idt$ illustrates when investment takes place the remaining cost to completion will decrease by the amount of the investment. The second term $\sigma_K \sqrt{IK} dz$ reflects the technical uncertainty which will only be resolved by investing. The discrete approximation to this process can be written as

$$K_{t+1} = K_t - I_t \Delta t + \sigma_K \sqrt{I_t K_t} \sqrt{\Delta t}(\epsilon)$$

where ϵ is a random number drawn from a standard normal distribution. The remaining cost to completion in the next period (K_{t+1}) is the current cost to completion (K_t) minus the investment that is made (I_t), subject to uncertainty that will cause the remaining cost to increase or decrease in the upcoming period.

8.2.2 Development Time

This variable represents the time expected to complete the development phase (D_T). To estimate this variable, development time for other similar $R\&D$ projects can be used as a proxy. In the beginning of the analysis it is estimated that development will take 3 years. Because the cost to completion (K_t) is a random variable, the development phase may take shorter or longer time than expected.

8.2.3 Time Until Patent Expires

It is estimated that there are 20 years until the patent expires. To reflect actual circumstances, the project will be evaluated quarterly ($N = 4$). Therefore, the time steps in the simulations are $\Delta t = \frac{1}{N} = 0.25$. To include the current time ($t = 0$) in the analysis the number of evaluations needs to be $T_E = 20 + \frac{1}{N}$, resulting in $NT_E = (4)(20 + \frac{1}{4}) = 81$ periods.

8.2.4 Investment Costs

We will allow for two types of investment costs in this model. The first one is the periodic investment in $R\&D$ (I_t). It is estimated that the company has a certain budget per year to invest in development. This cost will reduce the remaining cost to completion (K_t) each period, but the total investment needed to complete $R\&D$ is not known, because of the uncertainty concerning K_t. At an initial time t_0, this cost is estimated as the cost to completion, divided by the number of years development is expected to take.

$$I_0 = \frac{K_0}{D_T}$$

The annual rate of investment therefore becomes $I_0 = \frac{\$7.50}{3} = \$2.50 million$. With a time step of $\Delta t = 0.25$ the periodic rate of investment is $I_t = \$0.63 million$ and will be the same until development is completed. However, in the last period it may be lower since it only needs to be large enough to finish the last payment of development (K_t).

The other investment cost is the expected cost of commercialization (I_{m_t}). At the outset it is estimated that the periodic cost of commercialization is $\$1.73 million$. However, only if the development phase is successful does the management have the choice to invest in production and marketing. While $R\&D$ is taking place, the final product is unknown, and thus, there is no general direction to which the expected marketing costs will develop. Therefore, it is assumed that I_m follows the stochastic process given by

$$dI_m = I_m dt + \sigma_{I_m} I_m dz$$

The initial estimation of this cost may rise or fall by dI_m each period until development has been completed. Hence, the expected costs of commercialization may turn out to be higher or lower than anticipated. The discrete approximation of this process can be written as

$$I_{m_{t+1}} = I_{m_t}\Delta t + \sigma_{I_m} I_{m_t}\sqrt{\Delta t}(\epsilon)$$

It is assumed that the uncertainty over costs of commercialization is the same as for the cost to completion (σ_{I_m}). It is estimated that the commercialization will take two periods after $R\&D$ has been completed ($Y_D = 2$).

8.2.5 Expected Cash Flow

The future benefits from the project are uncertain and need to be estimated by management to the best of its knowledge. The expected future cash flow is calculated as the expected annual net revenue (after subtracting the expected operational expenses and taxes). The expected annual cash flow is estimated at the beginning of the analysis $C_0 = \$2.70 million$ per year. Then it is supposed to follow the GBM given by

$$dC = \alpha C dt + \sigma_C C dz$$

where α is the drift in the cash flow and σ_C is the volatility over expected cash flow. The cash flow is expected to grow at the rate of inflation which is assumed to be 2.5% per year. The return on successful $R\&D$ projects can be used to estimate the cash flow volatility but we will assume $\sigma_C = 30\%$. We know from our previous discussion that the expected return over future prices is the sum of the expected growth rate in the cash flow plus the dividend yield ($\mu = \alpha + \delta$). By assuming risk neutrality ($\mu = r$), we obtain $\alpha = r - \delta$ where r is the risk free rate and δ represents the cost of waiting. Hence, the risk equivalent process followed by the cash flow becomes

$$dC = (r - \delta) C dt + \sigma_C C dz$$

The solution to this process (see Section 6.4) can be written in a discrete time manner as follows

$$C_{t+1} = C_t e^{\left((r-\delta)-\frac{1}{2}\sigma_C^2\right)\Delta t + \sigma_C\sqrt{\Delta t}(\epsilon)} \tag{64}$$

The trajectory of the cash flow starts in C_0. Because no cash flow is received until $R\&D$ has been completed, the path followed by the cash flow until that time is the anticipated cash flow (C_{e_t}) and is not taken into the PV calculations. Once the development is completed, and if conditions turn out favorable, the firm will exercise the option to invest in commercialization. When commercialization has been completed, the initial value of the realized cash flow (C_t), is the last value of the trajectory followed by the anticipated cash flow (C_{e_t}). Thus, both C_{e_t} and C_t follow the same stochastic process (see Eq.64), but they do have different initial values. The initial value of the path followed by the anticipated cash flow (C_{e_t}) is $C_0 = \$2.70 million$ while the initial value of the realized cash flow (C_t) is the last value of the path followed by C_{e_t}. A reasonable estimation for the risk free rate is 5% (see Section 2). Under the risk neutral valuation we have $\delta = r - \alpha = 2.5\%$. It is assumed that the risk free rate and the cost of waiting will stay constant over the life of the project.

8.2.6 Market Share

It is well known that after the patent expires a part of, or the whole market share, will be lost as a consequence of competition. For instance, Fujimoto et.al (2019) [41] found that branded therapeutic markets experienced a decrease in the quantity of sales in the first three years after patent expiration by 49%, 65%, and 67% respectively. There is no common methodology of valuation when it comes to the expiration of the patent. It has been assumed that the terminal value can be estimated as the PV of the cash flow in the last period, multiplied with some constant M. This is similar to assuming that a terminal value of a firm is a multiple of earnings over a given period of time [9] [16]. The PV of cash flow in future periods after patent expiration has also been estimated with the formula for growing perpetuity, multiplied with a possible market share (m) [15]. Alternative assumptions can also be applied. For example, the life of the patent can be extended by some number of years where the market share is suppose to decline by a certain percentage each period after patent expiration.

In this thesis we will consider another alternative. If C_{T_E} is the cash flow in the last period of patent protection, then the expected value of cash flow after the patent expires is $E[C_t] = C_{T_E} e^{\alpha t}$. By letting C_{T_E} be equal to the initial level of cash flow, and discounting back at the risk free rate, we obtain the expected PV of the cash flow.

$$E[PV_{C_t}] = \int_0^{\infty} C_{T_E} e^{\alpha t} e^{-rt} dt = \frac{C_{T_E}}{(r-\alpha)} = \frac{C_{T_E}}{\delta}$$

This value is then multiplied with a relevant market share (m) and discounted back to the present time of evaluation (i).

$$PV_{C_{exp}}(i) = \frac{C_{T_E}}{\delta} e^{-r\Delta t(NT_E - i)}(m)$$

It is estimated that the market share will be m = 50%, after the patent expires.

8.2.7 Patent Value

The value of the patent (V$_{Patent}$) is measured as the opportunity cost we would be willing to pay for receiving an investment opportunity that is flexible rather than one that only allows for a now or never investment. This value is found by looking at the difference between the value of the project with flexibility and without flexibility.

The project is evaluated at every point in time (i) over the life of the patent (see Section 8.3). The PV of future expected cash flow is calculated as follows

$$PV_C(i) = \sum_{t=D_T+Y_D}^{NT_E} C_t e^{-r\Delta t(t-i)} + PV_{C_{exp}}(i)$$

where D_T is the time when development is completed and Y_D is the number of periods that apply to commercialization. The cash flow (C_t) will be received when $R\&D$ and commercialization has been completed, and until the patent expires (NT_E). The time steps in the simulations are Δt. The value of the project at patent expiration, at every point in time (i), is $PV_{C_{exp}}(i)$. The PV of investment costs is calculated as follows

$$PV_I(i) = \sum_{t=i}^{D_T} I_t e^{-r\Delta t(t-i)} + \sum_{t=D_T+1}^{D_T+Y_D} I_{m_t} e^{-r\Delta t(t-i)}$$

where I_t is the periodic investment in $R\&D$ and I_{m_t} is periodic investment in commercialization. The value of the project without flexibility, at every time step i, is calculated as follows

$$V(i) = PV_C(i) - PV_I(i)$$

If the project is unfavorable ($V(i) < 0$), the option to expand is introduced. This option is calculated as follows

$$V_{expand}(i) = max\Big(\big(PV_C(i)x - I_x - PV_C(i)\big), 0\Big)$$

where x is the percentage of expansion and I_x is the cost of expansion. If the expansion option is of no value, its payoff is zero, and the option to default is introduced. The option to default is a call option on the PV of future cash flow (PV_C) with a strike price equal to the PV of investment costs (PV_I). Therefore, the payoff from the option to default, at every time step i, becomes

$$V_{flex}(i) = max\big(PV_C(i) - PV_I(i), 0\big) = max\big(V(i), 0\big)$$

According to the equations above, we can write a more precise illustration of this payoff, given by

$$V_{flex}(i) = max\Big(\big(\sum_{t=D_T+Y_D}^{NT_E} C_t e^{-r\Delta t(t-i)} + PV_{C_{exp}}(i)\big) - \big(\sum_{t=i}^{D_T} I_t e^{-r\Delta t(t-i)} + \sum_{t=D_T+1}^{D_T+Y_D} I_{m_t} e^{-r\Delta t(t-i)}\big), 0\Big)$$

where i is the time of evaluation, Δt is the time step in the simulations, C_t is the expected cash flow at time t, T_E is the time of patent expiration, D_T is the time to complete development, Y_D is the number of periods in commercialization, PV_{exp} is the PV of cash flow at expiration, I_t is the investment cost at time t, and I_{m_t} is the expected cost of commercialization. If the expansion option is valuable ($V_{expand}(i) > 0$), we evaluate if its value is large enough to make the project favorable.

$$V_{flex}(i) = max\big(V(i) + V_{expand}(i), 0\big)$$

If the expansion makes the project favorable we will continue and move on to the next evaluation point i. However, if the project does not become favorable with the option to expand, we will default with a payoff of zero ($V_{flex}(i) = 0$). The option to invest becomes worthless and we move on to the next simulation. If we exercise the option to continue at all times, the value of the project becomes the one calculated at $t = 0$ in matrix $i = NT_E$, given by

$$V_{final} = \Big(\sum_{t=D_T+Y_D}^{NT_E} C_t e^{-r\Delta t(t-1)} + \frac{C_{T_E}}{\delta} e^{-r\Delta t(NT_E-1)}\Big) - \Big(\sum_{t=1}^{D_T} I_t e^{-r\Delta t(t-1)} + \sum_{t=D_T+1}^{D_T+Y_D} I_{m_t} e^{-r\Delta t(t-1)}\Big)$$

The PV of the option to expand - if there were any expansions, is added to the final value of the project as follows

$$V_{flex} = V_{final} + \sum_{i=1}^{NT_E} V_{expand}(i) e^{-r\Delta t(i-1)}$$

The final value of the project with flexibility in every simulation k, is denoted as $V_{k_{flex}}$, and without flexibility as V_k. If we default and the respective simulation stops before we get to the last period of evaluation ($i = NT_E$) we have

$$V_{k_{flex}} = 0 \quad \text{and} \quad V_k = V(i)$$

However, if the option to continue is exercised at all times (i), we have

$$V_{k_{flex}} = V_{flex} \quad \text{and} \quad V_k = V_{final}$$

After 10,000 simulations (k) the value of flexibility is obtained by subtracting the mean of the simulated values with no flexibility ($\bar{V}$) from the mean of values with flexibility ($\bar{V}_{flex}$), as follows

$$\bar{V}_{flex} = \frac{\sum_{k=1}^{n=1e4} V_{k_{flex}}}{n} \quad \text{and} \quad \bar{V} = \frac{\sum_{k=1}^{n=1e4} V_k}{n}$$

$$\text{Flexibility} = \bar{V}_{flex} - \bar{V}$$

The value of the patent is then calculated by subtracting the median of all values without flexibility from the median of all values with flexibility. It is believed that this is a reasonable estimation of value of the patent because the median of the distribution is the value that separates the higher half of the data sample from the lower half.

$$V_{Patent} = \text{median}(V_{k_{flex}}) - \text{median}(V_k)$$

8.2.8 Parameters Used in Simulations

For actual valuation processes, all parameters used in the simulations need to be estimated with care. Estimations like these can be hard to acquire. However, it can be helpful to look at the history of the company's success of coming up with new products, consider similar projects/firms in the same industry, or go to industry averages. As stated earlier, the objective of this study is to develop a general model, thus the values used in implementing the model are not industry specific. The hypothetical parameters used in the simulations can be seen in Table 5.

Table 5: Parameters Used in Simulations ($ in millions).

Variable	**Value**
Cost to Completion (K_0)	\$7.50
Cost Uncertainty (σ_K)	50%
Development Time (D_T)	3 years
Periodical Investment Costs (I_t)	\$0.63
Costs of Commercialization per Period (I_{m_t})	\$1.73
Uncertainty in Costs of Commercialization (σ_{I_m})	50%
Number of Periods in Commercialization (Y_D)	2
Expected Annual Cash Flow (C_0)	\$2.70
Cash Flow Uncertainty (σ_C)	30%
Annual Drift in Expected Cash Flow (α)	2.5%
Percentage of Expansion (x)	45%
Cost of Expansion (I_x)	\$2.00
Time Until Patent Expires (T_E)	20 years
The Cost of Waiting (δ)	2.5%
Risk Free Interest Rate (r)	5%
Market Share After Patent Expiration (m)	50%
Times Steps per Year (Δ_t)	0.25
Number of Simulations (k)	10,000

8.3 The Algorithm

The simulation was implemented in the programming platform MATLAB R2019a [42]. The number of simulations was set to 10,000. In every simulation (k) we go through i time steps, from the initial time of analysis up to NT_E. The number of decision points is $i = [1:NT_E] = [1:81]$. At every decision point i, we create a matrix $Q_{k,i}$ with $j = [1:NT_E]$ rows and 5 columns. The first column applies to the realized cash flow (C_t), the second to the periodic investment (I_t), the third to the remaining cost to completion (K_t), the fourth to the costs of commercialization (I_{m_t}), and the fifth to the anticipated cash flow (C_{e_t}). In the beginning of the analysis (t_0), we are positioned in the first matrix $Q_{k,i=1}$. The first line ($j = 1$) in the matrix displays the initial values for all of the variables $C_{t_{j,1}} = 0$, $I_{t_{j,2}} = I_0$, $K_{t_{j,3}} = K_0$, $I_{m_{t_{j,4}}} = I_{m_0}$ and $C_{e_{t_{j,5}}} = C_0$. Their trajectories are then calculated according to the processes outlined in Section 8.2. No cash flow will be received while the $R\&D$ is taking place. However, the expected cash flow (C_{e_t}) will follow a stochastic process until the actual cash flow (C_t) is received. The last value of the trajectory followed by the expected cash flow is the initial value of the actual cash flow. At t_0 the program evaluates if the future cash flow, received after development and commercialization has been completed, exceeds the cost of the investment. If not the expansion option is introduced. If the expansion does not make the project favorable we will default in matrix $Q_{k,i=1}$ and move on to the next simulation. However, if we exercise the option to continue, we will pay the first investment cost (I_0) in the respective simulation. Then we move to the next period in the matrix ($Q_{k,i=2}$). Here we start in the second line ($j = 2$). This is because the costs that were paid in matrix $i = 1$ and line $j = 1$ are sunk costs and should not be taken into the analysis. We calculate the trajectories for all the random variables again ($C_{t_{j,1}}$, $I_{t_{j,2}}$ $K_{t_{j,3}}$, $I_{m_{t_{j,4}}}$ and $C_{e_{t_{j,5}}}$). As before, if investment costs exceed the future expected cash flow at this point in time, the expansion option is evaluated. If that option does not make the project favorable the option to default will be exercised and we move on to the next simulation. However, if we exercise the option to continue we move on to matrix $Q_{k,i=3}$. Here we start in period three ($i = 3$) and line three ($j = 3$). The values for the first two periods will be the same as earlier but the trajectories for the random variables from then on are calculated again. This process continues until we reach the last period ($i = NT_E$), or the option to default has been exercised. Abandonment may arise if

1. The PV of expected cash flow is lower than anticipated.
2. The cost to completion is higher than anticipated.
3. The $R\&D$ takes more time than expected.
4. The option to expand does not make the project favorable.

The option to default will add value to the project, because, even though the costs that have already been paid in previous periods are sunk costs, the option to halt the project prevents further losses when conditions are unfavorable. If the product is successfully developed and commercialized the project will not be abandoned/expanded from there. The algorithm can be seen in Appendix A.

8.4 Results

An example of the trajectories followed by the expected cost to completion and the expected cash flow are illustrated in Figure 24.

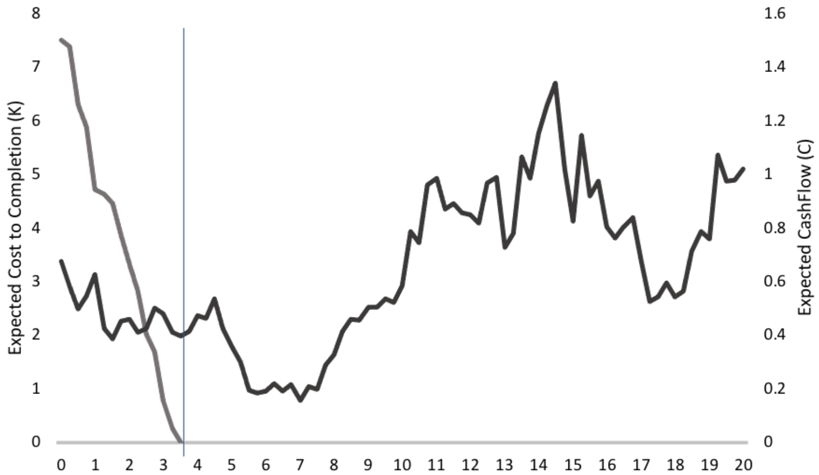


Figure 24: Example of the Trajectories followed by the Cost to Completion (K_t) and the Expected/Actual Cash Flow (C_{e_t}/C_t).

This figure demonstrates only one trajectory out of the 10,000 that were sampled. The blue line represents the point in time that development was completed and the cost to completion reaches zero. In the beginning of the analysis the development phase was expected to take 3 years, but in this case it took a little longer. The total investment needed to complete the $R\&D$ phase was $\$8.38million$, which is above the expected cost to completion of $\$7.50million$. Although the development took more time than expected and the costs needed to complete the development phase were higher than anticipated, the project was not abandoned. The path followed by the cash flow to the left of the blue line represents the anticipated cash flow (C_{e_t}) and the path to the right represents the realized cash flow (C_t). In the beginning of the analysis it was estimated that the annual cash flow would be $\$2.70million$. Because the project was evaluated quarterly in the simulations, the expected cash flow was $\$0.68million$ in the first quarter. In this example, the path followed by the cash flow rose over the life of the patent.

According to the parameters given in Table 5, the results from five different simulation runs are given in Table 6.

Table 6: Results from Five Different Simulation Runs ($ in millions).

Value of The Project with Flexibility	$23.48	$23.97	$23.08	$22.92	$24.50
Portion of Paths Abandoned	33.74%	34.07%	33.75%	34.61%	33.32%
Portion of Paths Expanded	16.24%	15.39%	16.29%	15.69%	16.2%
Value of The Project without Flex.	$21.33	$21.89	$20.98	$20.81	$22.40
Value of Flexibility	$2.14	$2.08	$2.08	$2.11	$2.10
Value of Patent	$0.53	$0.53	$0.53	$0.46	$0.47

It can be seen that the mean value of the project with flexibility for the given parameters was $\$23.59million$ and $\$21.48million$ without flexibility. The project was halted in 33.96% of the simulated paths and expanded 15.96% of the time. The value of flexibility could be found by subtracting the mean value of the project without flexibility from the mean value with flexibility. The value of the patent was found by looking at the difference between the median values of the distribution of the project with and without flexibility. The median of the distribution is the value that separates the higher half of the data sample from the lower half. In all of the simulations the median was slightly to the left of the mean which indicates that a larger number of the samples were lower than the mean. The option to default was most often exercised in the first quarter or in 6.8% of the simulated paths. It was last exercised after 4.75 years and only once over the 10,000 runs (see Figure 25).

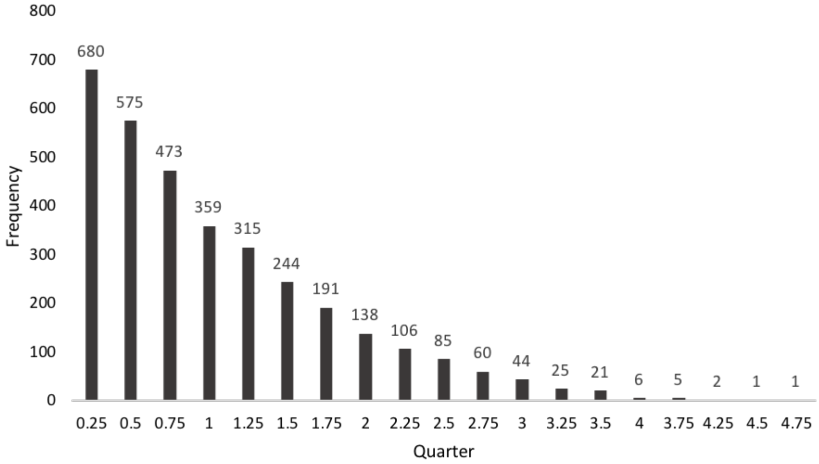


Figure 25: The Frequency of Defaulting per Quarter.

The same applies to the option to expand which was most often exercised in the first quarter or in 3.44% of the simulations and last exercised in the 5th year (see Figure 26).

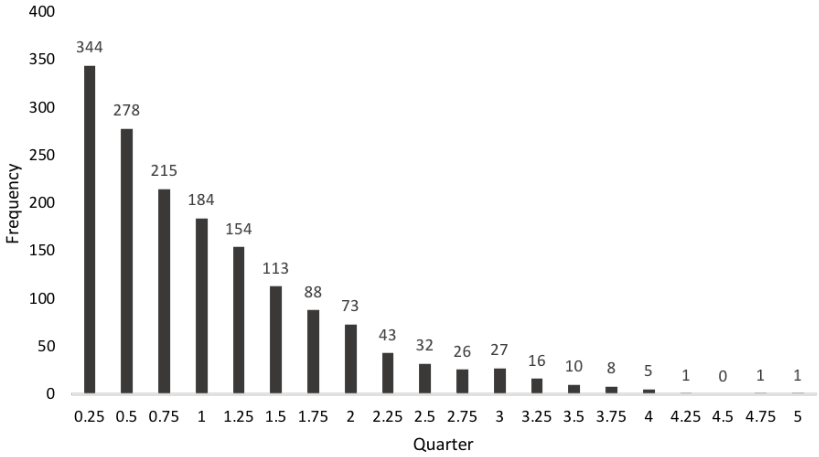


Figure 26: The Frequency of Expansion per Quarter.

If we did not allow for the option to expand, the proportion of paths abandoned would have risen to approximately 45%. The mean of the simulated costs to completion was $\$7.65 million$, slightly higher than the estimated $\$7.50 million$. Because we could default if conditions were unfavorable, the mean cost that was actually spent in development over the simulated paths was $\$5.64 million$, which represents that the project was halted before $R\&D$ was completed.

8.5 Sensitivity Analysis

Sensitivity analysis with respect to volatilities in cash flow and costs is illustrated in Table 7. It can be seen how adjusting these parameters affects the probability of abandonment, expansion, the value of flexibility, and the value of the patent.

Table 7: Sensitivity Analysis w.r.t. Volatilities in Cash Flow and Costs.

Cash Flow Uncertainty	Percent Abandoned	Percent Expanded	Value of Flexibility	Value of Patent
20%	4.61%	6.28%	\$0.61	\$0.32
25%	15.66%	11.63%	\$1.23	\$0.43
30%	33.74%	16.24%	\$2.14	\$0.53
35%	54.62%	18.62%	\$3.07	\$1.84
40%	72.31%	17.07%	\$3.90	\$3.12

Cost Uncertainty	Percent Abandoned	Percent Expanded	Value of Flexibility	Value of Patent
40%	32.95%	15.26%	\$1.66	\$0.43
45%	33.52%	15.28%	\$1.87	\$0.48
50%	33.74%	16.24%	\$2.14	\$0.53
55%	35.1%	16.51%	\$2.39	\$0.62
60%	35.89%	15.34%	\$2.64	\$0.63

The table shows that an increase in cash flow uncertainty increases the probability of abandonment and the value of the patent. This is in harmony with the "Bad News Principle" (see Section 5.3) and previous discussion of an increased uncertainty over cash flow (see Section 5.2.4). When cash flow uncertainty increases, the probability of bad news increases, which increases the probability of abandonment. Increase in uncertainty also increases the probability of possible good news, increasing the value of our investment opportunity. In the context of the real options theory, an increase in cash flow uncertainty increases the opportunity cost (the flexibility), the value of the investment opportunity, and the incentive to wait. This effect is not prevalent when uncertainty in costs are increased. It can be seen that uncertainty over costs does not have as decisive an effect on the probability of abandonment and the value of the patent. This is because of the technical uncertainty that was included in the process followed by the cost to completion. This kind of uncertainty can only be resolved by investing, which accelerates investment. However, this effect levels off because the technical uncertainty was not included in the process followed by the costs of commercialization. As uncertainties over costs and cash flow increase, the probability of expansion also increases. However, in the last line of the two panels in Table 7, it can be seen that the probability of expansion decreases when uncertainty becomes significantly high. This is because the value of the expansion option, when the project is marginal, is not high enough to prevent the project from defaulting.

Sensitivity analysis with respect to the risk free rate and drift in expected cash flow is illustrated in Table 8.

Table 8: Sensitivity Analysis w.r.t. The Risk Free Rate and Drift in Cash Flow ($ in millions).

Risk Free Interest Rate	Percent Abandoned	Percent Expanded	Value of Flexibility	Value of Patent
3%	9.69%	5.8%	$0.76	$0.20
4%	24.54%	12.07%	$1.62	$0.38
5%	33.74%	16.24%	$2.14	$0.53
6%	42.19%	18.02%	$2.43	$0.57
7%	49.2%	19.77%	$2.68	$1.19

Cash Flow Drift	Percent Abandoned	Percent Expanded	Value of Flexibility	Value of Patent
0.5%	52.75%	21.11%	$3.03	$1.64
1.5%	43.17%	17.99%	$2.54	$0.60
2.5%	33.74%	16.24%	$2.14	$0.53
3.5%	23.05%	11.88%	$1.55	$0.37
4.5%	7.92%	4.9%	$0.60	$0.17

These result are in line with the discussion in Section 5.5, where an increase in the risk free rate increases the value of flexibility. It can be seen that by increasing the drift the value of flexibility decreases. An increase in the drift is likely to be due to higher expected inflation. This would also affect the risk free rate and thus, these two effect would mitigate one another.

Sensitivity analysis with respect to change in the expected cost to completion, the expected cash flow, the time to completion of the development phase, and the market share after the patent expires is illustrated in Table 9.

Table 9: Sensitivity Analysis w.r.t. Costs, Cash Flow, Time to Completion, and Market Share ($ in millions).

Expected Cost To Completion	Percent Abandoned	Percent Expanded	Value of Flexibility	Value of Patent
$5.5	26.51%	11.16%	$1.51	$0.28
$6.5	31.01%	13.64%	$1.85	$0.37
$7.5	33.74%	16.24%	$2.14	$0.53
$8.5	37.69%	18.84%	$2.47	$0.70
$9.5	42.15%	20.56%	$2.89	$0.72

Expected Cash Flow	Percent Abandoned	Percent Expanded	Value of Flexibility	Value of Patent
$2.30	43.73%	18.27%	$2.53	$0.76
$2.50	38.81%	16.83%	$2.33	$0.57
$2.70	33.74%	16.24%	$2.14	$0.53
$2.90	29.2%	15.13%	$1.88	$0.46
$3.10	26.49%	13.58%	$1.76	$0.28

Time To Completion	Percent Abandoned	Percent Expanded	Value of Flexibility	Value of Patent
1 year	6.83%	5.23%	$0.41	$0.11
2 years	18.87%	11.97%	$1.14	$0.34
3 years	33.74%	16.24%	$2.14	$0.53
4 years	49.05%	19.44%	$3.05	$1.01
5 years	62.15%	19.68%	$3.97	$2.04

Market Share After Patent Expiration	Percent Abandoned	Percent Expanded	Value of Flexibility	Value of Patent
0%	33.84%	16.01%	$2.08	$0.41
50%	33.74%	16.24%	$2.14	$0.53
100%	33.4%	16.11%	$2.04	$0.56

These variables all have predictable effects on the probability of abandonment, expansion, and the value of the patent. When the expected costs to completion increase, the probability of the options being exercised increases as well as the value of flexibility and the patent. When the expected cash flow increases, the probability of the options being exercised decreases as well as the value of flexibility and the patent. As the expected time to completion increases, the total cost to completion is divided over a longer period, lowering the periodical investment. This increases the probability of the options being exercised and increases the opportunity cost of the investment opportunity. Interestingly, changing the market share after patent expiration has almost no effect on the value of the patent.

9 Conclusion

Flexibility can be an important component in the value of investment opportunities, especially when uncertainty is high, investments are irreversible and when we have the ability to control the timing of our investment. The standard valuation approaches (DCF), indicating that one should invest immediately when the future benefits from the project exceed the investment costs ($V \geq I$), tend to undervalue uncertain investment opportunities. The standard NPV approach fails to take into account the future opportunities and the flexibility to respond to unexpected events as time goes by. To capture the value of flexibility, contingent valuation approaches have been applied (DTA). These methods have their limitations when uncertainty in both investment costs and expected cash flow is significant. The main drawback of these approaches is the estimation of a single discount factor over the life of the project. This assumes that the same source of uncertainty is resolved in every period and that risk increases at a constant rate through time. It can be difficult to estimate the correct discount factor, especially when no comparable asset exists, such as in the case of a new unique invention.

The option pricing framework is a powerful tool to capture the value of flexibility, and invalidates the use of a single discount rate. We have seen that an investment opportunity is analogous to a financial call option on a common stock. By assuming risk neutrality, the option pricing framework yields the same results as the contingent NPV approach. We also saw that the binomial option pricing approach and the BSM model yield the same answer. This is no coincidence, since these two methods build on the same replicating portfolio strategy. We found that valuation approaches used for pricing financial options can easily be applied to capture the value of flexibility involved in an investment opportunity. However, there is a vast difference between real options and financial options. The main difference is that financial assets are most often traded, while very few real assets are traded. Therefore, the parameters used for financial option valuation can be obtained from the market and it is reasonable to assume that investors can adjust their investment portfolios continuously. This is rarely the case for real assets, and the parameters for the valuation process will need to be estimated. Another important difference is that the strike price in financial option valuation is deterministic and constant over the life of the option. This is rare in the case of investment opportunities where uncertainty over costs can be significant.

Most patent protected projects have in common that the underlying asset will need to go through $R\&D$ before the product can be commercialized. In the beginning of this process the time to completion and development costs are uncertain. Therefore, the duration of future expected cash flow is unknown. This makes analysis of $R\&D$ investment projects a difficult investment problem. The model developed in this thesis is aimed at catching these sources of uncertainty. It was assumed that the uncertainty in the underlying stochastic variables was normally distributed. However, in reality we can not assume a normal distribution, because in the real world volatility is not constant. The results from the model were in line with expectations of the option pricing theory. We saw that an increase in uncertainty increases the value of our investment opportunity and the value of the patent. Interestingly, changing the market share after patent expiration had negligible effect.

The main disadvantage of the simulation approach is that it does not allow for defining the optimal stopping rule. Because this approach is forward looking it is useful when estimating option values at maturity (European options). In order to compare early exercise decisions to the ones at maturity (American options), the binomial tree approach would be a better alternative. However, an American option will always be more valuable than a European option. That said, the binomial tree approach is not as suitable when the number of stochastic variables increases or when their value depends on a previous state. Therefore, there is no one right way to go. The solution method and the options that are taken into account in the valuation process will always depend on the nature of the project and the objective of the one performing the evaluation.

Throughout this thesis we have seen that the option pricing framework and real options are essential to capture the value of uncertain investment opportunities. It has been argued that patents should be valued as options because they provide its holder the exclusive right to invest at a cost (strike price) in the investment opportunity (underlying asset) at a certain time in the future. However, the asset underlying real options comes with a much higher price than financial assets underlying financial options (e.g. stocks). This is because the asset underlying real options is the PV of future expected cash flow. Therefore, pricing a patent as an option will result in a very high valuation. The approach proposed in this thesis, to let the value of the patent correspond to the opportunity cost inherent in an investment opportunity, seems like a more reasonable valuation.

The truth is that all valuation models are a simplification of reality. This is because they all build on assumptions. The results from a valuation model can never be better or more reliable than the assumptions underlying the model. There is a tempting and fatal imagination in mathematics. Albert Einstein warned against it, he said: "Leave elegance to the tailor". This means, we can not believe in something because it is a beautiful formula. Even though a valuation model yields a promising result, we can never be sure what the future brings. In the end it is the power of the human mind to observe and manage changes as time goes by, that will lead to accomplishment.

A Algorithm

```
clear all; close all;

%%Parameters used in simulations
N=4; %Number of evaluations per year
delta_t = 1/N; %Time steps in simulation
D_T = 3; %Expected development time estimated at t=0
Y_D = 2; %Number of periods that apply to commercialisation
K_0 = 7.5; %Total development cost estimated at t=0
I_t = (K_0/D_T); %Maximum investment per period
I_m = 1.73*N; %Cost of commercialization per period
I_x = 2; %Cost of expansion
x = 1.45; %Percent expanded
rp = 0; %Risk premium
r = 0.05; %Risk free rate
mu = r + rp; %Expected rate of return
alpha = 0.025; %Growth rate
d = r+rp-alpha; %Cost of waiting
C_0 = 2.7; %Initial expected cash flow
T=20+1/N; %Number of years until patent expires
sigma_I = 0.5; %Uncertainty over investment costs
sigma_c = 0.3; %Uncertainty over the expected cash flow
Nsim = 10000; %Number of simulations
m = 1; %Market share after patent expires

%Create vectors for the data from each simulation
D_t = zeros(Nsim,1); %The number of years development takes place
NPVsim = zeros(Nsim,2); %NPV without flex col1 and with flex col2
patent = zeros(Nsim,1); %Value of the patent
I_total = zeros(Nsim,1); %Total investment spent in development
year = zeros(1,Nsim); %The year we default if we default
SunkCost = zeros(Nsim,1); %Costs that have been paid if we default
Ktotal = zeros(Nsim,1); %The total expected cost to completion
NPVfinal =zeros(Nsim,1); %Final value with flex.
NPVnoflexFinal = zeros(Nsim,1); %Final value without flex.

%Create vectors for PV calculations at each time in each simulation
PV_C = zeros(Nsim,N*T); %PV of cash flow that has been received
PV_I = zeros(Nsim,N*T); %PV of periodic investment
PV_Im = zeros(Nsim,N*T); %PV of commercialization costs
PV_exp = zeros(Nsim,N*T); %PV of cash flow at expiration
V_expand = zeros(Nsim,N*T); %The value of the option to expand
NPV = zeros(Nsim,N*T); %Value of project w. flex. at time step i
NPVnoflex = zeros(Nsim,N*T); %Value of project without flex.
Value = zeros(Nsim,N*T); %Value with expansion option


```

```
%Create vectors for the final values in every simulation
PVC=zeros(Nsim,1); %PV of actual cash flow
PVI = zeros(Nsim,1); %PV of periodic investment
PVcom = zeros(Nsim,1); %PV of commercialization costs
PVexp = zeros(Nsim,1); %PV of cash flow at expiration
PV_expand = zeros(Nsim,1); %PV of expansion options
CostExpand = zeros(Nsim,1); %Total cost spent in expanding

%Count the number of times we exercise the option to expand
NoExpand = zeros(Nsim,N*T); numexpand = 0;
%Count the number of times we will default
counter = 0;

%Start simulations
for k = 1:Nsim

    %Define the starting points in each simulation
    i = 1; %Counts the matrices, i=(1:NT]
    j = 1; %Counts the periods in matrix i, j=[1:NT]

    %Create matrix with NT lines and 5 columns Ct,It,Kt,Im & Ce
    Q{k,i} = zeros(N*T,5);
    Q{k,i}(j,1) = 0; %Realized cash flow
    Q{k,i}(j,2) = I_t*delta_t; %Maximum investment per period
    Q{k,i}(j,3) = K_0; %Expected cost to completion
    Q{k,i}(j,4) = I_m*delta_t; %Cost of commercialization
    Q{k,i}(j,5) = C_0*delta_t; %Anticipated cash flow

    while i <= N*T
    j = 1;

      %Fill in the values that have been achieved or paid
      while j < i
      Q{k,i}(j,1) = Q{k,i-1}(j,1);
      Q{k,i}(j,2) = Q{k,i-1}(j,2);
      Q{k,i}(j,3) = Q{k,i-1}(j,3);
      Q{k,i}(j,4) = Q{k,i-1}(j,4);
      Q{k,i}(j,5) = Q{k,i-1}(j,5);

      j = j+1;
      end

      if j ~= 1
        j = j-1;
      end



```

```
  %Calculate the trajectories until Kt = 0
  while Q{k,i}(j,3) > 0 && (Q{k,i}(j,3) ~= Q{k,i}(j,2))
  j = j+1;

   %The expected cash flow while R&D takes place (Ce)
   Q{k,i}(j,5) = Q{k,i}(j-1,5) * exp(((r-d)-((sigma_c^2)/2))
     *...
       delta_t+(sigma_c*sqrt(delta_t)*randn));
   %Periodic investment in development (It)
   Q{k,i}(j,2) = Q{k,i}(j-1,2);
   %Calculate change in the costs to completion (Kt)
   Q{k,i}(j,3) = Q{k,i}(j-1,3)-Q{k,i}(j-1,2)+(sigma_I*...
       (sqrt(Q{k,i}(j-1,3)*Q{k,i}(j-1,2)))*sqrt(delta_t)*randn)
          ;
   %Calculate change in the marketing costs (Im)
   Q{k,i}(j,4) = Q{k,i}(j-1,4)+...
       (sigma_I*Q{k,i}(j-1,4)*sqrt(delta_t)*randn);

       %If Im < 0 we let Im = 0
       if Q{k,i}(j,4) < 0
          Q{k,i}(j,4) = 0;
       end

       %If Kt < It, we only need to pay It = Kt
       if  Q{k,i}(j,3) <= Q{k,i}(j,2)
        Q{k,i}(j,2) = Q{k,i}(j,3);
       end

       %If Kt < 0, we set Kt = 0
       if Q{k,i}(j,3) < 0
        Q{k,i}(j,3) = 0;
        %We pay the remaining costs from the previous period
        Q{k,i}(j,2) = Q{k,i}(j-1,3)-Q{k,i}(j-1,2);
       end

   %Number of years it takes to complete R&D in simulation k
   D_t(k,1) = i;
   %Number of years it takes to complete R&D in matrix i
   D_j = j;
  end

%When we have calculated the trajectories for I,K,Im & Ce
%we calculate the last values for the paths followed by Im & Ce
%Their paths will last for two more periods until the realized
%cash flow is introduced.

 for j = D_j+1 : D_j+Y_D
%Calculate expected cost of commercialization (Im)
```

```
Q{k,i}(j,4) = Q{k,i}(j-1,4)+(sigma_I*Q{k,i}(j-1,4)...
    *sqrt(delta_t)*randn);
%Calculate the anticipated cash flow (Ce)
Q{k,i}(j,5) = Q{k,i}(j-1,5) * exp(((r - d)-((sigma_c^2)/2))...
   *delta_t+(sigma_c*sqrt(delta_t)*randn));
end

%Calculate the trajectory for realized cash flow until maturity.
%The first value is the last value from Ce.
j=j+1;
 while j <= (N*T)
     if  Q{k,i}(j-1,1) == 0
        Q{k,i}(j,1) = Q{k,i}(j-1,5) * exp(((r - d) -...
            ((sigma_c^2)/2))*delta_t+(sigma_c*sqrt(delta_t)*randn));
     else
        Q{k,i}(j,1) = Q{k,i}(j-1,1) * exp(((r - d) -...
            ((sigma_c^2)/2))*delta_t+(sigma_c*sqrt(delta_t)*randn));
     end
   j = j+1;
 end

%Calculate PV of investment costs at time step i
PV_Itemp = 0;
if i < D_j
 for j = i:D_j
   PV_Itemp = Q{k,i}(j,2)*exp(-r*delta_t*(j-i));
   PV_I(k,i) = PV_I(k,i) + PV_Itemp;
   j = j+1;
 end
end

%Calculate PV of marketing costs at time step i
PV_Imtemp =0;
if i <= D_j
  for j = D_j+1:D_j+Y_D
    PV_Imtemp = Q{k,i}(j,4)*exp(-r*delta_t*(j-i));
    PV_Im(k,i) = PV_Im(k,i) + PV_Imtemp;
  end
elseif D_j < i && i <= D_j+Y_D
  for j = i:D_j+Y_D
    PV_Imtemp = Q{k,i}(j,4)*exp(-r*delta_t*(j-i));
    PV_Im(k,i) = PV_Im(k,i) + PV_Imtemp;
  end
end

```

```

  %Calculate PV of realized cash flow (Ct) at time step i
  PV_Ctemp = 0;
  j=i;
  while j < (N*T)
      PV_Ctemp = Q{k,i}(j,1)*exp(-r*delta_t*(j-i));
      PV_C(k,i) = PV_C(k,i) + PV_Ctemp;
      j=j+1;
  end

 %Calculate PV of cash flow at expiration
 j = (N*T);
 PV_exp(k,i) = (Q{k,i}(j,1)/d)*exp(-r*delta_t*(j-i));

 %Calculate the value of the project with the option to default
 NPV(k,i) = max((PV_C(k,i)+PV_exp(k,i)) - (PV_I(k,i)+PV_Im(k,i)),0);

 %Calculate the value of the project without flexibility
 NPVnoflex(k,i) = (PV_C(k,i)+PV_exp(k,i)) - (PV_I(k,i)+PV_Im(k,i));

 %If NPV(k,i) = 0, we introduce the option to expand
 Value(k,i) = (PV_C(k,i)+PV_exp(k,i)) - (PV_I(k,i)+PV_Im(k,i));
 if Value(k,i) < 0
   %Calculate the value of the expansion option
   V_expand(k,i) = max(((PV_C(k,i)+PV_exp(k,i))*x - I_x -...
       (PV_C(k,i)+PV_exp(k,i))), 0 );
   if V_expand(k,i) > 0
       Value(k,i) = Value(k,i) + V_expand(k,i);
       if Value(k,i) > 0
            %Count the number of expansions
            NoExpand(k,i) = NoExpand(k,i)+1;
            numexpand = numexpand + 1;
            %Cost of expanding
            CostExpand(k,1) = CostExpand(k,1) + I_x;
            %If it was profitable to expand we will not default
            NPV(k,i) = Value(k,i);
       end
   end
 end

%If we could not survive by expanding, we will default.
if NPV(k,i) == 0
   counter = counter +1;
   NPVfinal(k,1) = 0; %Value with the flex. to default
   NPVnoflexFinal(k,1) = NPVnoflex(k,i); %Value without flex.
   year(1,k) = i; %The year the decision is taken to default
```

```

       SunkCostTemp = 0;%Calculate sunk costs
       for j = 1:i
           SunkCostTemp = Q{k,i}(j,2)*exp(-r*delta_t*(j-1));
           SunkCost(k,1) = SunkCost(k,1) + SunkCostTemp;
       end

       i = 1000; %Makes sure that we go to next simulation

    elseif i == (N*T)

    %If we do not default after evaluating the project at
    %every time step i, the value of the project is
    %calculated from t = 0 in matrix NT

    %PV of investment costs
    PVI_temp = 0;
    for j = 1:D_j
      PVI_temp = Q{k,i}(j,2)*exp(-r*delta_t*(j-1));
      PVI(k,1) = PVI(k,1) + PVI_temp;
    end

    %PV of commercialization costs
    PV_IcomTemp = 0;
    for j = D_j+1:D_j+Y_D
     PV_IcomTemp = Q{k,i}(j,4)*exp(-r*delta_t*(j-1));
     PVcom(k,1) = PVcom(k,1) + PV_IcomTemp;
    end

    %PV of realized cash flow
    PVC_temp = 0;
    for u = (D_j+Y_D+1):(N*T)
      PVC_temp = Q{k,i}(u,1)*exp(-r*delta_t*(u-1));
      PVC(k,1) = PVC(k,1) + PVC_temp;
    end

    %PV of the option to expand
     PV_expTemp = 0;
     for ii = 1:N*T
         PV_expTemp = V_expand(k,ii)*exp(-r*delta_t*(ii-1));
         PV_expand(k,1) = PV_expand(k,1) + PV_expTemp;
     end

     %Terminal value of the cash flow
     PVexp(k,1) = (Q{k,i}(N*T,1)/d)*exp(-r*delta_t*(N*T-1))*m;




    %Value of the poject with flexibility
    NPVfinal(k,1) = ((PVC(k,1) + PVexp(k,1)) - (PVI(k,1)+PVcom(k,1)))...
        + PV_expand(k,1);

    %Value of the project without flexibility, calculated in i=1
    NPVnoflexFinal(k,1) = ((PVC(k,1) + PVexp(k,1))...
        - (PVI(k,1)+PVcom(k,1)));
   end

   %If we do not default we go to the next period of evaluation
    i=i+1;
end

%Calculate the costs spent in development
if SunkCost(k,1) > 0
    I_total(k,1) = SunkCost(k,1);
elseif SunkCost(k,1) == 0
    I_total(k,1) = sum(Q{k,D_t(k,1)}(:,2));
end

%Calculate the total cost to completion
Ktotal(k,1) = sum(Q{k,D_t(k,1)}(:,2));

%Write the results for the value of the project
NPVsim(k,1) = NPVnoflexFinal(k,1); %Without flexibility
NPVsim(k,2) = NPVfinal(k,1); %With flexibility
end

%Results from simulations for the project with flexibility
NPV_flex = mean(NPVsim(:,2)); %Value of project
NPV_flMedian = median(NPVsim(:,2)); %Median
NPV_flex_st = std(NPVsim(:,2)); %Standard deviation

%95% Confidence level
NPV_flex_Up = NPV_flex + 1.96*NPV_flex_st/sqrt(k);
NPV_flex_Down = NPV_flex - 1.96*NPV_flex_st/sqrt(k);

%Results from simulations for the project without flexibility
NPV_mean = mean(NPVsim(:,1)); %Value of project
NPV_Median = median(NPVsim(:,1)); %Median
NPV_stdv = std(NPVsim(:,1)); %Standard deviation

%95% Confidence level
NPV_Conf_Up = NPV_mean + 1.96*NPV_stdv/sqrt(k);
NPV_Conf_Down = NPV_mean - 1.96*NPV_stdv/sqrt(k);

```

```
%The value of flexibility
Flex_Value = NPV_flex - NPV_mean;

%The value of the patent
Patent_Value = NPV_flMedian - NPV_Median;

%Porportion of paths abandoned
Abandoned = num2str((counter/k)*100, '%g%%');

%Porportion of paths the option to expand was exercised
Expansion = num2str((numexpand/k)*100, '%g%%');

%Mean of the simulated costs to completion
K_mean = mean(Ktotal(:,1));

%Mean of the costs that were spent in development
I_mean = mean(I_total(k,1));
```

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