On Coupling of Discrete Random Walks on the Line

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30 ECTS thesis submitted in partial fulfillment of a
Magister Scientiarum degree in Mathematics

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Abstract

The main focus of this thesis is on coupling of random walks on the line with discrete step lengths and initial positions. After introducing the relevant coupling concepts, an Ornstein-type approach is used to establish necessary and sufficient conditions for the existence of a successful exact coupling and a successful shift-coupling. This yields results on asymptotic loss of memory. These findings are then applied to establish successful exact coupling of continuous-time regenerative processes with discrete regeneration times. After preliminaries on extension techniques, we finally construct a successful exact coupling of random walks with non-discrete (possibly singular) step lengths that are periodic in a certain sense.
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Swing, pretty girl, swing.
It ain’t real anyway.

David Lynch, Imaginary Girl
Notation

• All random walks in this paper are on the line, $\mathbb{R}$. We shall simply call them random walks.

• If $f$ is a measurable function from the measurable space $(E, \mathcal{E})$ into the measurable space $(G, \mathcal{G})$, we say that $f$ is $\mathcal{E}/\mathcal{G}$-measurable.

• If $A$ is a countable set, the $\sigma$-algebra of all its subsets will be denoted by $2^A$.

• If $A$ is a topological space, we will denote the Borel $\sigma$-algebra on $A$ by $\mathcal{B}(A)$.

• $\mathbb{R}^\infty$ will denote the set $\{(x_0, x_1, x_2, \ldots) : x_i \in \mathbb{R}, i \geq 0\}$.

• If $X$ is a random element with distribution $\mu$, we will write $X \sim \mu$. If two random elements $X$ and $Y$ have the same distribution, we write $X \overset{D}{=} Y$ and we say that $Y$ is a copy of $X$.

• If $P$ is a probability measure, we say that something holds $P$-a.s. ($P$-almost surely) if it holds on a set of $P$-measure 1. We will simply say that something holds almost surely if it is clear which probability measure is being referred to.

• Similarly, if $(\Omega, \mathcal{F}, P)$ is a probability space, we say that something holds for $P$-a.e. ($P$-almost every) $\omega$ in $A \in \mathcal{F}$, if it holds on a subset of $A$ having mass $P(A)$.

• If $\mu$ is a probability measure and $A$ is a set such that $\mu(A) = 1$, we will write $\mu \in A$. 
1 Introduction

A random walk on the line is a sequence $S = (S_n)_{n \geq 0}$ of random variables where $S_n = S_0 + X_1 + X_2 + \ldots + X_n$ for some independent random variables $S_0, X_1, X_2, \ldots$. The walk, $S$, is a random element in $(\mathbb{R}^\infty, \mathcal{B}(\mathbb{R}^\infty))$. We call $S_0$ the initial position of $S$ and $X_1, X_2, \ldots$ are called the step-lengths of $S$. If $S_0 \sim \lambda$ and $X_1, X_2, \ldots$ are i.i.d. having distribution $\mu$ we say that $S$ is a $(\lambda, \mu)$ random walk. If $S_0$ is concentrated on a constant $s$ a.s., we say that $S$ is an $(s, \mu)$ random walk. Most of the random walks we will discuss have discrete step-length and initial distributions (a.s. take their values in some countable set). We say these are discrete.

The main question we will deal with is the following: If $S$ is a $(\lambda, \mu)$ random walk and $S'$ is a $(\lambda', \mu)$ random walk, does there exist a successful exact coupling of $S$ and $S'$? That is, do there exist random elements $(\hat{S}, \hat{S}', T)$, defined on the same probability space $(\Omega, \mathcal{F}, P)$, such that $T$ is a $P$-a.s. finite random time on $\mathbb{N}$ (coupling time), $\hat{S} \overset{D}{=} S$, $\hat{S}' \overset{D}{=} S'$, and $\theta_T \hat{S} = \theta_T \hat{S}'$, where

$$\theta_n : \mathbb{R}^\infty \rightarrow \mathbb{R}^\infty, \quad \theta_n((x_0, x_1, \ldots)) = (x_n, x_{n+1}, \ldots),$$

with the convention that $\theta_T \hat{S} = \theta_T \hat{S}'$ on the null-set $\{T = \infty\}$? In other words, $\hat{S}$ is a $(\lambda, \mu)$ random walk, $\hat{S}'$ is a $(\lambda', \mu)$ random walk and $\hat{S}$ and $\hat{S}'$ are exactly the same from the time $T$ onwards.

Note that since

$$h : (\mathbb{N}, \mathbb{R}^\infty) \rightarrow \mathbb{R}^\infty, \quad h(n, s) = \theta_n(s)$$

is $2^\mathbb{N} \otimes (\mathcal{B}(\mathbb{R}))^\infty / (\mathcal{B}(\mathbb{R}))^\infty$-measurable and

$$(\theta_T \hat{S})(\omega) = \theta_{T(\omega)} \hat{S}(\omega)$$

is a composition of $\omega \mapsto (T(\omega), \hat{S}(\omega))$ and $h$, $\theta_T S$ is a random element, that is, $\theta_T S$ is $\mathcal{F} / (\mathcal{B}(\mathbb{R}))^\infty$-measurable.

A successful shift-coupling of $S$ and $S'$ are random elements $(\hat{S}, \hat{S}', T, T')$ such that $\hat{S} \overset{D}{=} S$, $\hat{S}' \overset{D}{=} S'$ and $\theta_T \hat{S} = \theta_{T'} \hat{S}'$, where $T$ and $T'$ are $P$-a.s. finite random times. That is, $\hat{S}$ and $\hat{S}'$ do visit the same real number (though it may happen at different times) and they are exactly the same from those times onward.

A successful coupling allows us to establish, via the coupling event inequality, asymptotic loss of memory of the random walks; two random walks with different initial distributions but the same step-length distribution will behave in the same way in the end. This method dates back to a 1938 paper by Doeblin [1] where his classical coupling established an irreducible, positive recurrent Markov chain’s convergence.
to equilibrium. In the random walk case, there is no equilibrium possible since the recurrent random walks on $\mathbb{R}$ are null-recurrent.

In 1969, Ornstein [2] constructed a successful exact coupling of (possibly transient) $(\lambda, \mu)$ and $(\lambda', \mu)$ random walks $S$ and $S'$, respectively, on a lattice $d\mathbb{Z}$, $d > 0$, when the set $\mathbb{A} = \{a \in \mathbb{R} : P(X_1 = a) > 0\}$ is strongly aperiodic: A set $\mathbb{A} \subseteq d\mathbb{Z}$, $d > 0$, where $d$ is such that $\mathbb{A}$ is in no strict sub-lattice of $d\mathbb{Z}$ ($\mathbb{A}$ is aperiodic in $d\mathbb{Z}$), is called strongly aperiodic if there is an $a \in \mathbb{A}$ such that $\mathbb{A} - a$ is not contained in any strict sub-lattice of $d\mathbb{Z}$.

Section 2 contains preliminaries on coupling. In Section 3, we use a similar trick as Ornstein to couple discrete random walks that need not be in a lattice. We will also see that Ornstein’s condition of strong aperiodicity of the step-lengths is a special case of a more general theorem about discrete random walks. A theorem that gives a necessary and sufficient condition on $\mathbb{A} = \{a \in \mathbb{R} : P(X_1 = a) > 0\}$ for a successful exact coupling of random walks in the additive subgroup generated by $\mathbb{A}$ to always exist. In Section 4, we will show that a successful shift-coupling of discrete random walks on this group always exists. From our results on discrete random walks, we can deduce results for random walks whose step-length distributions (that need not be identical) have a discrete component. In Section 5, we apply these shift-coupling results to establish results on exact coupling for certain continuous-time regenerative processes. After preliminaries on extension techniques in Section 6, we finally, in Section 7, construct a successful exact coupling for random walks with singular-continuous step-length components that are periodic in a certain sense. This is modeled after Thorisson’s [3] treatment of random walks with spread-out step-lengths (these are step-lengths such that the sum of several of them has a non-singular distribution). As an example we will look at a random walk with a step-length distribution on the “double” Cantor-set, $C \cup (C - 1)$.

## 2 Coupling Concepts

This section is based on Chapter 1, Sections 4 and 5, of [3].

**Definition 2.1.** A coupling of a collection of random elements $Y_i$, $i \in \mathbb{I}$, is a family of random elements $(\hat{Y}_i)_{i \in \mathbb{I}}$, all defined on some probability space $(\Omega, \mathcal{F}, P)$, such that $\hat{Y}_i \overset{D}{=} Y_i$ for all $i \in \mathbb{I}$.

**Definition 2.2.** Let $(\hat{Y}_i)_{i \in \mathbb{I}}$ be a coupling of $Y_i$, $i \in \mathbb{I}$. An event $C$ is called a coupling
event of the coupling \((\hat{Y}_i)_{i \in \mathbb{I}}\) if

\[
\hat{Y}_i = \hat{Y}_j \text{ on } C \quad \text{for all } i, j \in \mathbb{I}.
\]

Another concept important to us is the following.

**Definition 2.3.** For a bounded, signed measure \(\nu\) in some measurable space \((E, \mathcal{E})\), we define its **total variation norm** as

\[
\|\nu\| := \sup_{A \in \mathcal{E}} \nu(A) - \inf_{A \in \mathcal{E}} \nu(A).
\]

It is easy to see that \(|| \cdot ||\) defines a norm on the vector space of all bounded, signed measures on \((E, \mathcal{E})\). We obtain the following proposition on the total variation norm of the difference of two probability measures.

**Proposition 2.4.** Let \(P\) and \(Q\) be probability measures on some measurable space \((E, \mathcal{E})\), then we have

\[
\|P - Q\| = 2 \sup_{A \in \mathcal{E}} (P(A) - Q(A)) = 2 \sup_{A \in \mathcal{E}} |P(A) - Q(A)|.
\]

**Proof.** Since \(A \in \mathcal{E}\) if and only if \(A^c \in \mathcal{E}\) and since for any set \(B \subseteq \mathbb{R}\), \(\inf(-B) = -\sup(B)\), we get

\[
\|P - Q\| = \sup_{A \in \mathcal{E}} (P(A) - Q(A)) - \inf_{A \in \mathcal{E}} (P(A) - Q(A))
\]

\[
= \sup_{A \in \mathcal{E}} (P(A) - Q(A)) + \sup_{A \in \mathcal{E}} (Q(A) - P(A))
\]

\[
= \sup_{A \in \mathcal{E}} (P(A) - Q(A)) + \sup_{A^c \in \mathcal{E}} (Q(A^c) - P(A^c))
\]

\[
= \sup_{A \in \mathcal{E}} (P(A) - Q(A)) + \sup_{A^c \in \mathcal{E}} (1 - Q(A) - (1 - P(A)))
\]

\[
= \sup_{A \in \mathcal{E}} (P(A) - Q(A)) + \sup_{A^c \in \mathcal{E}} (P(A) - Q(A))
\]

\[
= 2 \sup_{A \in \mathcal{E}} (P(A) - Q(A)).
\]

And since \(||P - Q|| = ||Q - P||\) (\(|| \cdot ||\) being a norm) we obtain

\[
\|P - Q\| = 2 \sup_{A \in \mathcal{E}} (P(A) - Q(A)) = 2 \sup_{A \in \mathcal{E}} (Q(A) - P(A)) = 2 \sup_{A \in \mathcal{E}} |P(A) - Q(A)|.
\]

Thus, convergence of probability measures in the total variation norm is stronger than weak convergence. Next, we establish the coupling event inequality of a coupling of two random elements.
**Theorem 2.5.** Let $(\hat{Y}, \hat{Y}')$ be a coupling of two random elements, $Y$ and $Y'$ in $(E, \mathcal{E})$, with coupling event $C$. Then

$$\|P(Y \in \cdot) - P(Y' \in \cdot)\| \leq 2P(C^c) \quad \text{(Coupling Event Inequality)}.$$ 

**Proof.** Since $\hat{Y} = \hat{Y}'$ on $C$ we get, for $A \in \mathcal{E}$,

$$P(\hat{Y} \in A, C) = P(\hat{Y}' \in A, C),$$
and thus

$$P(Y \in A) - P(Y' \in A) = P(\hat{Y} \in A) - P(\hat{Y}' \in A)$$

$$= P(\hat{Y} \in A, C^c) - P(\hat{Y} \in A, C^c)$$

$$\leq P(C^c).$$

Thus

$$\|P(Y \in \cdot) - P(Y' \in \cdot)\| = 2 \sup_{A \in \mathcal{E}} (P(Y \in A) - P(Y' \in A))$$

$$\leq 2P(C^c).$$

Therefore, the greater the total variation difference of $P(Y \in \cdot)$ and $P(Y' \in \cdot)$ is, the smaller the probability of a coupling event will be. This means we can not find a coupling $(\hat{Y}, \hat{Y}')$, of $Y$ and $Y'$, with a coupling event of high probability if the original $Y$ and $Y'$ do not have similar distributions, that is if $\|P(Y \in \cdot) - P(Y' \in \cdot)\|$ is large.

Having defined coupling and the important concepts related to it, we turn to random walks.

## 3 Exact Coupling – Fixed Initial Positions

In this section we shall use an Ornstein-type approach to couple discrete $(s, \mu)$ and $(s', \mu)$ random walks. Ornstein’s method of finding a successful exact coupling of two differently started random walks, $S$ and $S'$, on a lattice was, in short, as follows. Let $S$ and $S'$ be independent and construct a third random walk $S''$, such that the difference walk $S - S''$ is sure to hit zero at an a.s. finite random time $T$. Then define $S'''$ to be the process following $S''$ until time the $T$ and then switching to $S$. The walk $S'''$ turns out to be a copy of $S'$ and hence $(S, S''', T)$ is a successful exact coupling of $S$ and $S'$.

We need one preliminary result on simple symmetric random walks on a lattice $d\mathbb{Z}$, $d > 0$, that is, a random walk with initial distribution on the lattice, with i.i.d. step-length distribution $\nu$, such that $\nu\{-d, 0, d\} = 1$ and $\nu(d) = \nu(-d) > 0$. The result is
Lemma 3.1. A simple, symmetric random walk $R$ on the lattice $d\mathbb{Z}, d > 0$, hits zero with probability one.

Proof. We prove the assertion for $d = 1$, the proof for general $d > 0$ is analogous. Let $\nu$ be the step-length distribution of $R$. Put $q_i = P(R$ never hits $0 \mid R_0 = i), i \in \mathbb{Z}$. Then $q_0 = 0$ and $q_i = q_{-i}$. For $i \geq 1$, we get $q_i = \nu(-1)q_{i-1} + \nu(1)q_{i+1} + \nu(0)q_i$. Thus, $q_{i+1} - q_i = q_i - q_{i-1} = \ldots = q_1 - q_0 = q_1$ and we get

$$q_i = \sum_{k=1}^{i}(q_k - q_{k-1}) = \sum_{k=1}^{i}q_1 = iq_1.$$

Therefore, $q_1 = 0$ and it follows that $q_i = 0$ for all $i \in \mathbb{Z}$. \hfill $\square$

The first theorem gives a necessary and sufficient condition for the existence of a successful exact coupling when $\lambda$ and $\lambda'$ are concentrated on some constants $s$ and $s'$, respectively. But first we set up some terminology and establish a lemma.

Define, for a distribution $\mu$ on $\mathbb{R}$, the set

$$\mathcal{A}_\mu := \{a \in \mathbb{R} : \mu(a) > 0\}.$$

Let $B \subseteq \mathbb{R}$. Define $a(B)$ to be the smallest additive group containing $B$, that is

$$a(B) = \{n_1b_1 + \cdots + n_kb_k : k \in \mathbb{N}, n_1, \ldots, n_k \in \mathbb{Z} \text{ and } b_1, \ldots, b_k \in B\}.$$

Let $B - B$ be the set $\{b_i - b_j : b_i, b_j \in B\}$. Note that $a(B - B) \subseteq a(B)$ for any $B \subseteq \mathbb{R}$ and if $B$ is countable, then $a(B)$ is also countable. Further, if $B$ is non-lattice, that is, $B \not\subseteq d\mathbb{Z}$ for all $d > 0$, then $a(B)$ is dense in $\mathbb{R}$ (see [3], page 61, for a proof).

Lemma 3.2. Let $B \subseteq \mathbb{R}$ be countable. Then $B \subseteq a(B - B)$ if and only if $a(B) = a(B - B)$.

Proof. If $a(B) = a(B - B)$, then obviously $B \subseteq a(B - B)$. If on the other hand $B \subseteq a(B - B)$, then, since $a(B)$ is the smallest additive group containing $B$, it must be contained in $a(B - B)$. But since $a(B - B) \subseteq a(B)$, we have $a(B) = a(B - B)$. \hfill $\square$

We will adopt the following convention throughout the paper: If $X$ is a discrete random element, defined on some probability space $(\Omega, \mathcal{F}, P)$, taking values in some countable set $A$ a.s., we shall assume that $X(\omega) \in A$ for all $\omega \in \Omega$. This is no restriction since we can define $X$ on any null-set as we wish without changing its distribution.
Therefore, in the following, when \(X_1, X_2, \ldots\) are the step-lengths of some discrete \((\lambda, \mu)\) random walk, we will assume that for each \(i \geq 1\) and \(\omega \in \Omega\), \(X_i(\omega) \in \mathcal{A}_\mu\).

We come now to our key result.

**Theorem 3.3.** Let \(S\) and \(S'\) be discrete \((s, \mu)\) and \((s', \mu)\) random walks, respectively. There exists a successful exact coupling, \((\hat{S}, \hat{S}', T)\), of \(S\) and \(S'\) if and only if \(s - s' \in a(\mathcal{A}_\mu - \mathcal{A}_\mu)\).

**Proof.** Let \((\hat{S}, \hat{S}', T)\) be a successful exact coupling of \(S\) and \(S'\), defined on some probability space \((\Omega, \mathcal{F}, P)\), and let \(X_1, X_2, \ldots\) be the step-lengths of \(\hat{S}\) and \(X'_{1}, X'_{2}, \ldots\) the step-lengths of \(\hat{S}'\). Then there is an \(\omega \in \Omega\) such that \(T(\omega) < \infty\). Since \(\hat{S}_T = \hat{S}'_T\), we get

\[
s + \hat{X}_1(\omega) + \hat{X}_2(\omega) + \ldots + \hat{X}_{T(\omega)}(\omega) = s' + \hat{X}'_1(\omega) + \hat{X}'_2(\omega) + \ldots + \hat{X}'_{T(\omega)}(\omega)
\]

which is equivalent to

\[
s - s' = (\hat{X}_1(\omega) - \hat{X}'_1(\omega)) + (\hat{X}_2(\omega) - \hat{X}'_2(\omega)) + \ldots + (\hat{X}_{T(\omega)}(\omega) - \hat{X}'_{T(\omega)}(\omega)).
\]

The right hand side is in \(a(\mathcal{A}_\mu - \mathcal{A}_\mu)\) and the only if part follows.

Now assume \(d := s - s' \in a(\mathcal{A}_\mu - \mathcal{A}_\mu)\). It is no restriction to assume that \(S\) and \(S'\) are independent, defined on some probability space \((\Omega, \mathcal{F}, P)\). Call the step-lengths of \(S\) and \(S'\), \(X_1, X_2, \ldots\) and \(X'_1, X'_2, \ldots\), respectively. We break the rest of the proof into four steps.

**Step 1:** There is an \(r \in \mathbb{N}\) such that

\[
P(X_1 + X_2 + \ldots + X_r - (X'_1 + X'_2 + \ldots + X'_r) = d) > 0.
\]

Since \(d \in a(\mathcal{A}_\mu - \mathcal{A}_\mu)\), we can write, for some \(r \geq 0\) and \(a_1, a_2, \ldots, a_r, a'_1, a'_2, \ldots, a'_r \in \mathcal{A}_\mu\),

\[
d = (a_1-a'_1) + (a_2-a'_2) + \ldots + (a_r-a'_r).
\]

Then

\[
P(X_1 + X_2 + \ldots + X_r - (X'_1 + X'_2 + \ldots + X'_r) = d) \\
\geq P(X_1 = a_1, X_2 = a_2, \ldots, X_r = a_r, X'_1 = a'_1, X'_2 = a'_2, \ldots, X'_r = a'_r).
\]

Since all the \(X_i, X'_i\) are independent and \(\mu(a) > 0\) for all \(a \in \mathcal{A}_\mu\), this is strictly greater than zero.
Step 2: Construction of a third random walk $S''$.

For $k \geq 1$, let

$$Y_k := (X_{(k-1)r+1}, X_{(k-1)r+2}, \ldots, X_{kr}),$$

and

$$Y'_k := (X'_{(k-1)r+1}, X'_{(k-1)r+2}, \ldots, X'_{kr}).$$

Obviously, $(Y_1, Y'_1), (Y_2, Y'_2), \ldots$ is an i.i.d. sequence of random elements in $\mathbb{R}^{2r}$, and since $Y_k$ and $Y'_k$ are independent and have the same distribution, we also have $(Y_k, Y'_k) \overset{D}{=} (Y'_k, Y_k)$.

Define, for $k \geq 1$,

$$L_k := X_{(k-1)r+1} + X_{(k-1)r+2} + \ldots + X_{kr}$$

and

$$L'_k := X'_{(k-1)r+1} + X'_{(k-1)r+2} + \ldots + X'_{kr}.$$ 

Notice that $(L_k, L'_k)$ is a measurable function of $(Y_k, Y'_k)$ and since $(Y_k, Y'_k) \overset{D}{=} (Y'_k, Y_k)$, we get

$$P(Y_k \in \cdot, |L_k - L'_k| = d) = P(Y'_k \in \cdot, |L'_k - L_k| = d). \quad (*)$$

We now construct the step-lengths of a third random walk. Notice that by Step 1,

$$P(|L_k - L'_k| = d) = P(L_k - L'_k = d) + P(L'_k - L_k = d) > 0.$$ 

Let

$$(X''_{(k-1)r+1}, \ldots, X''_{kr}) := \begin{cases} Y_k & \text{if } |L_k - L'_k| \neq d, \\ Y'_k & \text{if } |L_k - L'_k| = d. \end{cases}$$

Since $Y''_k := (X''_{(k-1)r+1}, \ldots, X''_{kr})$ is the same measurable function of $(Y_k, Y'_k)$ for each $k \geq 1$, the $Y''_k$ are i.i.d. Further, for $C \in \mathcal{B}(\mathbb{R}^r)$ and $k \geq 1$, we get

$$P(Y''_k \in C) = P(Y''_k \in C, |L_k - L'_k| \neq d) + P(Y''_k \in C, |L_k - L'_k| = d)$$

$$= P(Y_k \in C, |L_k - L'_k| \neq d) + P(Y'_k \in C, |L_k - L'_k| = d)$$

$$= P(Y_k \in C, |L_k - L'_k| \neq d) + P(Y_k \in C, |L_k - L'_k| = d)$$

$$= P(Y_k \in C),$$

where the third equality follows by $(*)$. Thus, $X''_1, X''_2, \ldots$ are i.i.d. with distribution $\mu$ and if we define a random walk $S''$ with initial position $s'$ and step-lengths $X''_1, X''_2, \ldots$, it is a $(s', \mu)$ random walk. We also have

$$L''_k := X''_{(k-1)r+1} + X''_{(k-1)r+2} + \ldots + X''_{kr} = \begin{cases} L_k & \text{if } |L_k - L'_k| \neq d, \\ L'_k & \text{if } |L_k - L'_k| = d. \end{cases}$$
Step 3: $S$ and $S''$ meet.

Let $R = (R_k)_{k \geq 0}$ be the difference walk of $S$ and $S''$, with step-lengths $(X_k - X_k'')_{k \geq 1}$, beginning in $s - s' = d$. Now, let us observe $R$ only at times $0, r, 2r, 3r, \ldots$. The step-lengths of $(R_{kr})_{k \geq 0}$ are $L_k - L_k''$ and, remembering the definition of $L_k''$, we get

$$L_k - L_k'' = \begin{cases} 
0 & \text{if } |L_k - L_k'| \neq d, \\
 d & \text{if } L_k - L_k' = d, \\
-d & \text{if } L_k' - L_k = d.
\end{cases}$$

By Step 1, the random walk $(R_{kr})_{k \geq 0}$ is a simple symmetric random walk on the lattice $dZ$. By Lemma 3.1, $(R_{kr})_{k \geq 0}$ will hit zero at an a.s. finite random time $K$, so

$$0 = R_{Kr} = S_{Kr} - S''_{Kr}.$$ 

Thus, $S$ and $S''$ meet at the a.s. finite random time $T := Kr$.

Step 4: Construction of a successful exact coupling.

Define the discrete-time stochastic process $S''' = (s' + X_1''' + \ldots + X_n'''_{n \geq 0})$, where

$$X_k''' = \begin{cases} 
X_k'' & \text{if } k \leq T, \\
X_k & \text{if } k > T.
\end{cases}$$

That is, $S'''$ follows $S''$ until it meets $S$ and then it switches to $S$.

The random elements $(S, S'''', T)$ give us a successful exact coupling of $S$ and $S'$ if we can establish that $S'''$ is indeed a $(s', \mu)$ random walk. It suffices to show that the step-lengths $X_1''', X_2''', \ldots$ are i.i.d. with distribution $\mu$, and to that end it suffices to prove that $Y_1''', Y_2''', \ldots$ are i.i.d. having the same distribution as $Y_1$ (where $Y_k''' := (X_{(k-1)r+1}''', \ldots, X_{kr}''')$): For $n \geq 1$ and $A_1, A_2, \ldots, A_n \in B(\mathbb{R}^r)$, we have (remember that $T = Kr$)

$$P(Y_1''' \in A_1, \ldots, Y_n''' \in A_n)$$

$$= \sum_{i=1}^{\infty} P(Y_1''' \in A_1, \ldots, Y_i''' \in A_i, K = i)$$

$$= \sum_{i=1}^{n-1} P(Y_1''' \in A_1, \ldots, Y_i''' \in A_i, K = i, Y_{i+1} \in A_{i+1}, \ldots, Y_n \in A_n)$$

$$+ \sum_{i=n}^{\infty} P(Y_1''' \in A_1, \ldots, Y_n''' \in A_n, K = i).$$
The event \( \{ K = i \} \) is determined by \( (Y_1, Y_2, \ldots, Y_i) \) and \( (Y'_1, Y'_2, \ldots, Y'_i) \) and is therefore independent of both \( (Y_{i+1}, Y_{i+2}, \ldots) \) and \( (Y'_{i+1}, Y'_{i+2}, \ldots) \). By Step 2, \( Y''_i, Y''_2, \ldots \) are i.i.d. with the same distribution as \( Y_1 \). Since \( Y''_{i+1}, Y''_{i+2}, \ldots \) are determined by \( (Y_1, Y_2, \ldots) \) and \( (Y'_1, Y'_2, \ldots) \), the random elements \( Y''_i, Y''_2, \ldots \) are also independent of \( \{ K = i \} \). Now, continuing the calculations above:

\[
P(Y'''_1 \in A_1, \ldots, Y'''_n \in A_n) \]
\[
= \sum_{i=1}^{n-1} P(Y''_1 \in A_1, \ldots, Y''_i \in A_i, K = i) P(Y_{i+1} \in A_{i+1}, \ldots, Y_n \in A_n) \\
+ \sum_{i=n}^{\infty} P(Y''_1 \in A_1, \ldots, Y''_n \in A_n, K = i) \\
= \sum_{i=1}^{n-1} P(Y''_1 \in A_1, \ldots, Y''_i \in A_i, K = i) P(Y''_{i+1} \in A_{i+1}, \ldots, Y''_n \in A_n) \\
+ \sum_{i=n}^{\infty} P(Y''_1 \in A_1, \ldots, Y''_n \in A_n, K = i) \\
= \sum_{i=1}^{n-1} P(Y''_1 \in A_1, \ldots, Y''_n \in A_n, K = i) \\
+ \sum_{i=n}^{\infty} P(Y''_1 \in A_1, \ldots, Y''_n \in A_n, K = i) \\
= \sum_{i=1}^{n-1} P(Y''_1 \in A_1, \ldots, Y''_n \in A_n, K = i) \\
= P(Y''_1 \in A_1, \ldots, Y''_n \in A_n) \\
= P(Y''_1 \in A_1) P(Y''_2 \in A_2) \cdots P(Y''_n \in A_n).
\]

Thus, \( Y''_1, Y''_2, \ldots \) are i.i.d. having the same distribution as \( Y_1 \) and therefore \( S'' \) is a \((s', \mu)\) random walk.

**Example 3.4.** Let \( \mu \) have a uniform distribution on the set \( \{ \sqrt{2}, \pi, e, 42 \} \). Let \( S \) and \( S' \) be \((\sqrt{2}, \mu)\) and \((\pi, \mu)\) random walks, respectively. By Theorem 3.3, there exist copies, \( \hat{S} \) and \( \hat{S}' \), of \( S \) and \( S' \), respectively, such that \( \hat{S} \) and \( \hat{S}' \) are the same from an a.s. finite random time onward.

### 4 Exact Coupling – Random Initial Positions

A random walk is a sequence of partial sums of random variables. Due to this additive nature it is natural to observe random walks in some additive sub-group of \( \mathbb{R} \). If a random walk has step-lengths in some lattice \( d\mathbb{Z}, d > 0 \), it is natural to restrict the
possible initial distributions to the same lattice to be sure that the walk is in \(d\mathbb{Z}\). If a random walk has step-lengths with continuous distributions, we observe it on the whole line \(\mathbb{R}\), allowing any initial distribution.

Therefore, if a random walk has some general discrete step-length distribution \(\mu\), it is natural to restrict the possible initial distribution to the additive sub-group generated by \(\mathbb{A}_\mu\). This is the aforementioned \(a(\mathbb{A}_\mu)\). In this natural setting we establish a theorem giving a necessary and sufficient condition on \(\mathbb{A}_\mu\) for there always to exist a successful exact coupling of two random walks.

**Theorem 4.1.** Let \(S\) and \(S'\) be discrete \((\lambda, \mu)\) and \((\lambda', \mu)\) random walks, respectively. The following are equivalent.

(i) There exists a successful exact coupling, \((\hat{S}, \hat{S}', T)\), of \(S\) and \(S'\), for all \(\lambda, \lambda' \in a(\mathbb{A}_\mu)\).

(ii) \(a(\mathbb{A}_\mu) = a(\mathbb{A}_\mu - \mathbb{A}_\mu)\).

Proof. \((i) \Rightarrow (ii)\): Let \(d \in a(\mathbb{A}_\mu)\) and choose \(\lambda\) and \(\lambda'\) to be concentrated at \(d\) and 0, respectively. Let \((\hat{S}, \hat{S}', T)\) be a successful exact coupling of \(S\) and \(S'\), defined on some probability space \((\Omega, \mathcal{F}, P)\), and let \(\hat{X}_1, \hat{X}_2, \ldots\) be the step-lengths of \(\hat{S}\) and \(\hat{X}'_1, \hat{X}'_2, \ldots\) the step-lengths of \(\hat{S}'\). Then there is an \(\omega \in \Omega\) such that \(T(\omega) < \infty\), and since

\[
\hat{S}_{T(\omega)}(\omega) = \hat{S}'_{T(\omega)}(\omega),
\]

we get

\[
d + \hat{X}_1(\omega) + \ldots + \hat{X}_{T(\omega)}(\omega) = 0 + \hat{X}'_1(\omega) + \ldots + \hat{X}'_{T(\omega)}(\omega),
\]

and isolating \(d\) gives

\[
d = (\hat{X}_1(\omega) - \hat{X}'_1(\omega)) + \ldots + (\hat{X}_{T(\omega)}(\omega) - \hat{X}'_{T(\omega)}(\omega)).
\]

The right hand side is in \(a(\mathbb{A}_\mu - \mathbb{A}_\mu)\), thus \(a(\mathbb{A}_\mu) \subseteq a(\mathbb{A}_\mu - \mathbb{A}_\mu)\). Since the reverse inclusion always holds, we have \(a(\mathbb{A}_\mu) = a(\mathbb{A}_\mu - \mathbb{A}_\mu)\).

\((ii) \Rightarrow (i)\): Now assume \(a(\mathbb{A}_\mu) = a(\mathbb{A}_\mu - \mathbb{A}_\mu)\). Let \(S\) and \(S'\) be \((\lambda, \mu)\) and \((\lambda', \mu)\) random walks, respectively, where \(\lambda\) and \(\lambda'\) are probability distributions on \(a(\mathbb{A}_\mu)\).

Let \(\hat{S}_0\) and \(\hat{S}'_0\) be independent \(a(\mathbb{A}_\mu)\)-valued random variables with distributions \(\lambda\) and \(\lambda'\), respectively, defined on some probability space \((\Omega, \mathcal{F}, P)\). On the same probability space let there be defined, independent of \((\hat{S}_0, \hat{S}'_0)\), a countable collection of independent random elements

\[
(\hat{S}_{(s,s')}, \hat{S}'_{(s,s')}, T_{(s,s')}, s, s' \in a(\mathbb{A}_\mu) = a(\mathbb{A}_\mu - \mathbb{A}_\mu),
\]
where for each pair \((s, s')\), with \(s, s' \in a(A_\mu)\),

\[
(\hat{S}_{(s, s')}, \hat{S}'_{(s, s')}, T_{(s, s')})
\]

is a successful exact coupling of a \((s, \mu)\) and a \((s', \mu)\) random walk. Because \(s - s' \in a(A_\mu) = a(A_\mu - A_\mu)\), we know these random elements exist by Theorem 3.3.

We define our candidate for a successful exact coupling of \(S\) and \(S'\) as

\[
(\hat{S}, \hat{S}', T) := (\hat{S}_{(\hat{s}_0, \hat{s}'_0)}, \hat{S}'_{(\hat{s}_0, \hat{s}'_0)}, T_{(\hat{s}_0, \hat{s}'_0)}).
\]

To see that \(P(\hat{S} \in C) = P(\hat{S}_{(\hat{s}_0, \hat{s}'_0)} \in C) = P(S \in C)\), for \(C \in \mathcal{B}(\mathbb{R}^\infty)\), note that

\[
P(\hat{S}_{(\hat{s}_0, \hat{s}'_0)} \in C) = \sum_{s' \in a(A_\mu)} \sum_{s \in a(A_\mu)} P(\hat{S}_{(\hat{s}_0, \hat{s}'_0)} \in C, \hat{S}_0 = s, \hat{S}'_0 = s')
\]

\[
= \sum_{s' \in a(A_\mu)} \sum_{s \in a(A_\mu)} P(\hat{S}_{(s, s')} \in C, \hat{S}_0 = s, \hat{S}'_0 = s')
\]

\[
= \sum_{s' \in a(A_\mu)} \sum_{s \in a(A_\mu)} P(\hat{S}_{(s, s')} \in C) P(\hat{S}_0 = s) P(\hat{S}'_0 = s')
\]

\[
= \sum_{s' \in a(A_\mu)} \left( \sum_{s \in a(A_\mu)} P(\hat{S}_{(s, s')} \in C) \lambda(s) \right) \lambda'(s')
\]

\[
= \sum_{s' \in a(A_\mu)} P(S \in C) \lambda'(s')
\]

\[
= P(S \in C).
\]

Therefore \(\hat{S}\) is indeed a \((\lambda, \mu)\) random walk and similarly we get that \(\hat{S}'\) is a \((\lambda', \mu)\) random walk. Finally, observe that for \(P\text{-a.e. } \omega\) in \(\{\hat{S}_0 = s, \hat{S}'_0 = s'\}\) we have

\[
(\theta_T \hat{S})(\omega) = \theta_T(\omega) \hat{S}_{(\omega, \hat{s}'_0(\omega))}(\omega) = \theta_T_{(s, s')}(\omega) \hat{S}_{(s, s')}(\omega)
\]

\[
= (\theta_T_{(s, s')}(\omega) \hat{S}_{(s, s')}(\omega)) = (\theta_T(\omega) \hat{S}'_{s,s'})(\omega)
\]

\[
= (\theta_T \hat{S}')(\omega).
\]

Since \(\bigcup_{s, s' \in a(A_\mu)} \{\hat{S}_0 = s, \hat{S}'_0 = s'\}\) is a countable partition of the probability space, \(T\) is a.s. finite and

\[
\theta_T \hat{S} = \theta_T \hat{S}'.
\]

\(\square\)

**Example 4.2.** Let \(\mu\) be a distribution on the rational numbers, \(\mathbb{Q}\), such that \(\mu(q) > 0\) for all \(q \in \mathbb{Q}\). Let \(\lambda\) and \(\lambda'\) be some distributions on \(\mathbb{Q}\) and let \(S\) and \(S'\) be \((\lambda, \mu)\) and \((\lambda', \mu)\) random walks, respectively. Since obviously \(a(\mathbb{Q}) = a(\mathbb{Q} - \mathbb{Q})\), by Theorem 4.1, there exist copies, \(\hat{S}\) and \(\hat{S}'\), of \(S\) and \(S'\), respectively, such that \(\hat{S}\) and \(\hat{S}'\) are the same from an a.s. finite random time onward.
According to the following Proposition, Theorem 4.1 is a direct extension of Ornstein’s result.

**Proposition 4.3.** Let \( A \subseteq \mathbb{R} \) be contained in some lattice \( d \mathbb{Z} \), \( d > 0 \), such that \( A \) is not contained in a strict sub-lattice of \( d \mathbb{Z} \) (i.e. \( A \) is aperiodic in \( d \mathbb{Z} \)). Then the following are equivalent.

(i) \( a(A) = a(A - A) \).

(ii) \( A \) is strongly aperiodic.

**Proof.** \((i) \Rightarrow (ii)\): If \( A \) is not strongly aperiodic, then \( A - a \) is in some strict sub-lattice of \( d \mathbb{Z} \) for all \( a \in A \). This means that \( a(A - A) \) is in some strict sub-lattice of \( d \mathbb{Z} \). But since \( A \) itself is in no such sub-lattice, we get \( A \not\subseteq a(A - A) \). Thus \( a(A) \neq a(A - A) \), by Lemma 3.2.

\((ii) \Rightarrow (i)\): If \( A \) is strongly aperiodic, there is some \( a \in A \) such that \( A - a \) is aperiodic in \( d \mathbb{Z} \). This means that \( d \mathbb{Z} = a(A - a) \subseteq a(A - A) \), since aperiodic sets in \( d \mathbb{Z} \) span the whole lattice. Thus \( A \subseteq a(A - A) \) which, by Lemma 3.2, is equivalent to \( a(A) = a(A - A) \).

Although the condition \( a(A) = a(A - A) \) is equivalent to strong aperiodicity when \( A \) is in a lattice, it is not clear if the condition has any simpler or “deeper” meaning when \( A \) is non-lattice, that is \( A \not\subseteq d \mathbb{Z} \) for all \( d > 0 \). One thing we can say though is that if \( \mathbb{A}_\mu \) is the set \( \{a_1, a_2, \ldots\} \) and \( a_1, a_2, \ldots \) are linearly independent over the integers, then \( a(\mathbb{A}_\mu) \neq a(\mathbb{A}_\mu - \mathbb{A}_\mu) \). To see this, define the mapping \( \varphi : a(\mathbb{A}_\mu) \to \mathbb{Z}/2\mathbb{Z} \) such that \( \varphi(a_i) = 1 \), for all \( a_i \in \mathbb{A}_\mu \). Then, since \( \varphi(a') = 0 \) for all \( a' \in a(\mathbb{A}_\mu - \mathbb{A}_\mu) \), \( a(\mathbb{A}_\mu - \mathbb{A}_\mu) \) can not be the same set as \( a(\mathbb{A}_\mu) \).

Theorems 3.3 and 4.1 yield asymptotic results for discrete random walks.

**Corollary 4.4.** Let \( S \) and \( S' \) be random walks with a discrete step-length distribution \( \mu \). Let \( s, s' \in \mathbb{R} \) be such that \( s - s' \in a(\mathbb{A}_\mu - \mathbb{A}_\mu) \) and let \( \lambda \) and \( \lambda' \) be probability distributions on \( a(\mathbb{A}_\mu) \). Suppose that either one of the following conditions is satisfied:

(a) \( S_0 = s \) a.s. and \( S'_0 = s' \) a.s.

(b) \( S_0 \sim \lambda, S'_0 \sim \lambda' \) and \( a(\mathbb{A}_\mu) = a(\mathbb{A}_\mu - \mathbb{A}_\mu) \).

Then we get asymptotic loss of memory of \( S \) and \( S' \), that is

\[ \|P(S_n \in \cdot) - P(S'_n \in \cdot)\| \to 0, \quad \text{as} \quad n \to \infty. \]
Proof. If \( S_0 = s \) a.s. and \( S_0' = s' \) a.s., by Theorem 3.3, there exists a successful exact coupling, \((\hat{S}, \hat{S}', T)\), of \( S \) and \( S' \). Then \( \{T \leq n\} \) is a coupling event of the coupling \((\hat{S}_n, \hat{S}_n')\) of \( S_n \) and \( S_n' \). By the coupling event inequality, we have

\[
\|P(S_n \in \cdot) - P(S_n' \in \cdot)\| \leq 2P(T > n),
\]

and since \( T \) is a.s. finite, we get

\[
\|P(S_n \in \cdot) - P(S_n' \in \cdot)\| \to 0, \quad \text{as} \quad n \to \infty.
\]

Theorem 4.1 gives the same result when \( S_0 \sim \lambda, S_0' \sim \lambda' \) and \( a(\Delta \mu) = a(\Delta \mu - \Delta \mu') \).

The method of proof in Theorem 3.3 works for a larger class of random walks than discrete with i.i.d. step-lengths: Let \( H \) and \( H' \) be two differently started, not necessarily discrete random walks. Assume \( H \) and \( H' \) are independent and let \( H_0 = s \) a.s. and \( H_0' = s' \) a.s. Let \( d = s - s' \) and let \( X_1, X_2, \ldots \) and \( X'_1, X'_2, \ldots \) be the step-lengths of \( H \) and \( H' \), respectively, where

\[
X_1 \sim \mu_1, X_2 \sim \mu_2, \ldots, X'_1 \sim \mu_1, X'_2 \sim \mu_2, \ldots,
\]

for some sequence \( \mu_1, \mu_2, \ldots \) of distributions. If there is an \( r \) such that \( \inf_{k \geq 1} P(L_k - L'_k = d) > 0 \), where \( L_k := X_{(k-1)r+1} + \ldots + X_{kr} \) and \( L'_k := X'_{(k-1)r+1} + \ldots + X'_{kr} \), then the same proof as in Theorem 3.3 will work to find a successful exact coupling of \( H \) and \( H' \).

The theorems of this section can be used to establish some similar results on the existence of a successful shift-coupling of two differently started, discrete random walks. This will be our next topic.

## 5 Shift-Coupling

We begin by proving an analog of Theorem 3.3 for shift-coupling.

**Theorem 5.1.** Let \( S \) and \( S' \) be discrete \((s, \mu)\) and \((s', \mu)\) random walks, respectively. There exists a successful shift-coupling, \((\hat{S}, \hat{S}', T, T')\), of \( S \) and \( S' \) if and only if \( s - s' \in a(\Delta \mu) \).

**Proof.** Let \((\hat{S}, \hat{S}', T, T')\) be a successful shift-coupling of \( S \) and \( S' \), on some probability space \((\Omega, \mathcal{F}, P)\). Then there is an \( \omega \in \Omega \) such that \( T(\omega) < \infty \), and since

\[
\hat{S}_{T(\omega)}(\omega) = \hat{S}_{T(\omega)}'(\omega),
\]
we get
\[ s + \hat{X}_1(\omega) + \ldots + \hat{X}_{T(\omega)}(\omega) = s' + \hat{X}'_1(\omega) + \ldots + \hat{X}'_{T(\omega)}(\omega), \]
which is equivalent to
\[ s - s' = \hat{X}_1(\omega) + \ldots + \hat{X}_{T(\omega)}(\omega) - (\hat{X}'_1(\omega) + \ldots + \hat{X}'_{T(\omega)}(\omega)). \]
The right hand side is in \( a(A_\mu) \), proving the only if part.

For the if part, let \( d := s - s' \in a(A_\mu) \). If \( 0 \in A_\mu \) then \( a(A_\mu) = a(A_\mu - A_\mu) \) and the result follows by Theorem 3.3. Suppose that \( 0 \notin A_\mu \) and define \( A_\mu^0 := A_\mu \cup \{ 0 \} \). Define a distribution \( \nu \) on \( A_\mu^0 \) by requiring that \( \nu(0) = 1/2 \) and \( \nu(a | A_\mu) = \mu(a) \) for \( a \in A_\mu \).

Let \( R \) and \( R' \) be \((s, \nu)\) and \((s', \nu)\) random walks, respectively. Since \( 0 \notin A_\nu = A_\mu^0 \), we get \( a(A_\nu) = a(A_\mu - A_\nu) \) and according to Theorem 4.1, there exists a successful exact coupling, \((\hat{R}, \hat{R}', K)\), of \( R \) and \( R' \). Call the step-lengths of \( R \) and \( \hat{R}' \), \( Y_1, Y_2, \ldots \) and \( \hat{Y}_1, \hat{Y}_2, \ldots \), respectively. Observe the random walks that emerge if we only look at \( \hat{R} \) and \( \hat{R}' \) when their step-lengths are not zero; define
\[ \tau_0 := 0 \quad \text{and, recursively,} \quad \tau_{i+1} := \min\{n > \tau_i : \hat{Y}_n \neq 0\}, \]
and define \( \tau_i' \) in a similar way from \( \hat{Y}'_1, \hat{Y}'_2, \ldots \). Now, define the stochastic processes
\[ \hat{S} := (s + \hat{Y}_{\tau_1} + \ldots + \hat{Y}_{\tau_n})_{n \geq 0} \quad \text{and} \quad \hat{S}' := (s' + \hat{Y}'_{\tau'_1} + \ldots + \hat{Y}'_{\tau'_n})_{n \geq 0}. \]
Since \( \hat{Y}_{\tau_1}, \hat{Y}_{\tau_2}, \ldots \) are independent, having distribution \( \mu \), \( \hat{S} \) is a \((s, \mu)\) random walk. Similarly, \( \hat{S}' \) is a \((s', \mu)\) random walk. Define
\[ O := \#\{n \geq 1 : \hat{Y}_n = 0, n \leq K\} \quad \text{and} \quad O' := \#\{n \geq 1 : \hat{Y}'_n = 0, n \leq K\}. \]
That is, \( O \) is the number of zero jumps of \( \hat{R} \) before the coupling time \( K \) and \( O' \) is the number of zero jumps of \( \hat{R}' \) before \( K \). Since \((\hat{R}, \hat{R}', K)\) is a successful exact coupling of \( R \) and \( R' \), \( \hat{R} \) and \( \hat{R}' \) are exactly the same from the time \( K \) onwards. Thus \( \hat{S} \) and \( \hat{S}' \) are exactly the same from the times \( T := K - O \) and \( T' := K - O' \), respectively, and \((\hat{S}, \hat{S}', T, T')\) is a successful shift-coupling of \( S \) and \( S' \).

Next, we establish the existence of a successful shift-coupling of discrete random walks in the natural setting.

**Theorem 5.2.** Let \( S \) and \( S' \) be discrete \((\lambda, \mu)\) and \((\lambda', \mu)\) random walks where \( \lambda, \lambda' \in a(A_\mu) \). Then there exists a successful shift-coupling, \((\hat{S}, \hat{S}', T, T')\), of \( S \) and \( S' \).

**Proof.** Let \( \hat{S}_0 \) and \( \hat{S}'_0 \) be independent with distributions \( \lambda \) and \( \lambda' \), respectively. Define on the same probability space, independent of \( \hat{S}_0 \) and \( \hat{S}'_0 \), the independent quadruples
\[ (\hat{S}_{(s,s')}, \hat{S}'_{(s,s')}, T_{(s,s')}, T'_{(s,s')}), \quad s, s' \in a(A_\mu), \]
and define
\[ \hat{T} := \min\{n > 0 : \hat{S}_n = \hat{S}'_n\}. \]
where \((\hat{S}_{(s,s')}, \hat{S}'_{(s,s')}, T_{(s,s')}, T'_{(s,s')})\) is a successful shift-coupling of a \((s, \mu)\) and a \((s', \mu)\) random walk as in Theorem 5.1. Define

\[
(\hat{S}, \hat{S}', T, T') := (\hat{S}_{(s_0,s'_0)}, \hat{S}'_{(s_0,s'_0)}, T_{(s_0,s'_0)}, T'_{(s_0,s'_0)}).
\]

By similar calculations as in the proof of Theorem 4.1, we get that \((\hat{S}, \hat{S}', T, T')\) is a successful shift-coupling of \(S\) and \(S'\).

We obtain asymptotic results on discrete random walks as a corollary to the above theorems.

**Corollary 5.3.** Let \(S\) and \(S'\) be random walks with a discrete step-length distribution \(\mu\). Let \(s, s' \in \mathbb{R}\) be such that \(s - s' \in \mathcal{A}(\mu)\) and let \(\lambda\) and \(\lambda'\) be probability distributions on \(\mathcal{A}(\mu)\). Suppose that either one of the following conditions is satisfied:

(a) \(S_0 = s\) a.s. and \(S'_0 = s'\) a.s.

(b) \(S_0 \sim \lambda\) and \(S'_0 \sim \lambda'\).

Then we get Cesaro (time-average) total variation convergence of \(S\) and \(S'\), that is,

\[
\left\| \frac{1}{n} \sum_{k=0}^{n-1} P(S_k \in \cdot) - \frac{1}{n} \sum_{k=0}^{n-1} P(S'_k \in \cdot) \right\| \to 0, \quad \text{as} \quad n \to \infty.
\]

**Proof.** By theorems 5.1 and 5.2, if either one of the conditions (a) or (b) are satisfied, there exists a successful shift-coupling, \((\hat{S}, \hat{S}', T, T')\), of \(S\) and \(S'\). By Proposition 2.4, we get

\[
\left\| \frac{1}{n} \sum_{k=0}^{n-1} P(S_k \in \cdot) - \frac{1}{n} \sum_{k=0}^{n-1} P(S'_k \in \cdot) \right\| = 2 \sup_{A \in \mathcal{B}(\mathbb{R})} \left\| \frac{1}{n} \sum_{k=0}^{n-1} P(S_k \in A) - \frac{1}{n} \sum_{k=0}^{n-1} P(S'_k \in A) \right\|
\]

\[
= 2 \sup_{A \in \mathcal{B}(\mathbb{R})} \left| \frac{1}{n} \sum_{k=0}^{n-1} P(\hat{S}_k \in A) - \frac{1}{n} \sum_{k=0}^{n-1} P(\hat{S}'_k \in A) \right|
\]

\[
= 2 \sup_{A \in \mathcal{B}(\mathbb{R})} |E|\left[ \frac{1}{n} \sum_{k=0}^{n-1} 1_{\{S_k \in A\}} - \frac{1}{n} \sum_{k=0}^{n-1} 1_{\{S'_k \in A\}} \right].
\]

Now, we can write

\[
|E|\left[ \sum_{k=0}^{n} 1_{\{S_k \in A\}} - \sum_{k=0}^{n} 1_{\{S'_k \in A\}} \right]
\]

\[
= |E|\left[ 1_{\{T < T'\}} \sum_{k=0}^{n} 1_{\{\hat{S}_k \in A\}} - \sum_{k=0}^{n} 1_{\{\hat{S}'_k \in A\}} + 1_{\{T \geq T'\}} \sum_{k=0}^{n} 1_{\{\hat{S}_k \in A\}} - \sum_{k=0}^{n} 1_{\{\hat{S}'_k \in A\}} \right],
\]

15
and since $\hat{S}_{T+k} = \hat{S}_{T'+k}$, for $k \geq 0$, we get, with the convention that $\sum_{k=0}^{n} x_k = 0$,

$$|E[1_{\{T<T'\}}(\sum_{k=0}^{n} 1_{\{S_k \in A\}} - \sum_{k=0}^{n} 1_{\{\hat{S}_k \in A\}})]|$$

$= |E[1_{\{T<T'\}}(\sum_{k=0}^{T \land n-1} 1_{\{S_k \in A\}} + \sum_{k=T \land n}^{n} 1_{\{\hat{S}_k \in A\}} - \sum_{k=0}^{T' \land n-1} 1_{\{S_k \in A\}} - \sum_{k=T' \land n}^{n} 1_{\{\hat{S}_k \in A\}})]|$

$= |E[1_{\{T<T'\}}(\sum_{k=0}^{T \land n-1} 1_{\{S_k \in A\}} + \sum_{k=T \land n+(n-T' \land n)}^{T'} 1_{\{S_k \in A\}} - \sum_{k=0}^{T' \land n-1} 1_{\{\hat{S}_k \in A\}})]|$

$\leq E[1_{\{T<T'\}}(T \land n + (n - (T \land n + (n - T' \land n))) + T' \land n)]$

$\leq 4E[1_{\{T<T'\}}(T' \land n)]$.

Similarly we obtain

$$|E[1_{\{T \geq T'\}}(\sum_{k=0}^{n} 1_{\{S_k \in A\}} - \sum_{k=0}^{n} 1_{\{\hat{S}_k \in A\}})]| \leq 4E[1_{\{T \geq T'\}}(T \land n)],$$

and thus

$$|E[\sum_{k=0}^{n} 1_{\{S_k \in A\}} - \sum_{k=0}^{n} 1_{\{\hat{S}_k \in A\}}]|$$

$= |E[1_{\{T<T'\}}(\sum_{k=0}^{n} 1_{\{S_k \in A\}} - \sum_{k=0}^{n} 1_{\{\hat{S}_k \in A\}}) + 1_{\{T \geq T'\}}(\sum_{k=0}^{n} 1_{\{\hat{S}_k \in A\}} - \sum_{k=0}^{n} 1_{\{\hat{S}_k \in A\}})]|$

$\leq 4E[1_{\{T<T'\}}(T' \land n)] + 4E[1_{\{T \geq T'\}}(T \land n)]$

$= 4E[(T \lor T') \land n].$

Finally, we get

$$\|\frac{1}{n} \sum_{k=0}^{n-1} P(S_k \in \cdot) - \frac{1}{n} \sum_{k=0}^{n-1} P(S'_k \in \cdot)\| = 2 \sup_{A \in B(\mathbb{R})} |E[\frac{1}{n} \sum_{k=0}^{n-1} 1_{\{S_k \in A\}} - \frac{1}{n} \sum_{k=0}^{n-1} 1_{\{\hat{S}_k \in A\}}]|$$

$\leq \frac{8}{n} E[(T \lor T') \land (n-1)] \to 0 \text{ as } n \to \infty.$

Where the convergence in the last step follows by a.s. finiteness of $T$ and $T'$ and dominated convergence.

\[\square\]

### 6 Classical Regeneration

The following introduction to classical regeneration is based on Chapter 10, Sections 2 and 3, in [3].
Let \((\Omega, \mathcal{F}, P)\) be a probability space, supporting the random elements

\[ Z = (Z_t)_{t \in [0, \infty)} \quad \text{and} \quad S = (S_n)_{n \geq 0}, \]

where \(Z\) is a one-sided continuous-time stochastic process taking values in some space \((E, \mathcal{E})\) and \(S\) is a sequence of random times, satisfying

\[ 0 \leq S_0 < S_1 < S_2 < \ldots \to \infty. \]

The process \(Z\) is a random element in \((G, \mathcal{G}) := (E^{[0, \infty)}, \mathcal{E}^{(0, \infty)})\), where \(\mathcal{E}^{(0, \infty)}\) is the \(\sigma\)-algebra spanned by the projection mappings \(\pi_t : E^{(0, \infty)} \to E, \pi(z) = z_t\). We regard \(S\) as an element in the sequence space \((L, \mathcal{L})\), where

\[ L := \{(s_n)_{n \geq 0} \in [0, \infty)^N : s_0 < s_1 < \ldots \to \infty\} \]

and \(\mathcal{L}\) are the Borel subsets of \(L\): \(\mathcal{L} = L \cap (\mathcal{B}(0, \infty))^N\). Thus, \((Z, S)\) is a random element in \((G \times L, \mathcal{G} \otimes \mathcal{L})\).

The random times \(S\) split \(Z\) into a delay

\[ D := (Z_t)_{t \in [0, S_0)} \]

and cycles

\[ C_n := (Z_{S_n-1+t})_{t \in [0, X_n)}, \]

where \(X_n\) are the cycle-lengths

\[ X_n = S_n - S_{n-1}. \]

After time \(S_0\), the delay, \(D\), is in some external state (cemetery-state) \(\Delta \notin E\) and \(C_n\) is in \(\Delta\) after time \(X_n\). Thus, the delay and cycles are one-sided continuous-time stochastic processes taking values in the measurable space \((E_{\Delta}, \mathcal{E}_{\Delta}) := (E \cup \Delta, \sigma(E \cup \Delta))\) and are random elements in \((E_{\Delta}^{(0, \infty)}, \mathcal{E}_{\Delta}^{(0, \infty)})\), where as before, \(\mathcal{E}_{\Delta}^{(0, \infty)}\) is spanned by the finite dimensional projections. Let \(\theta_t\), where \(t \geq 0\), be the shift map from \(G\) to \(G\),

\[ \theta_t(z) = (z_{t+s})_{s \in [0, \infty)}. \]

Let \(\theta_t\) also denote the joint shift map from \(G \times L\) to \(G \times L\):

\[ \theta_t(z, s) = (\theta_t(z), (s_{nt+k} - t)_{k \geq 0}), \]

where \(n_t := \min\{n \geq 1 : s_n \geq t\}\). If \(A\) is any countable set in \([0, \infty)\), then the mapping \((t, (z, s)) \mapsto \theta_t(z, s)\) is \(2^A \otimes \mathcal{G} \otimes \mathcal{L} / \mathcal{G} \otimes \mathcal{L}\)-measurable. Thus, if \(T\) is a random time on \((\Omega, \mathcal{F}, P)\), such that for each \(\omega \in \Omega\) we have \(T(\omega) \in \mathcal{A}_T\), where \(\mathcal{A}_T\) is countable
(we shall assume this is the case for all discrete random times in this section), then \( \theta_T(Z, S) \) is a random element in \( G \times L \). All the random times we will encounter in this section will be discrete and therefore we shall have no problems when shifting by a random time.

A *successful exact coupling* of two one-sided continuous-time processes \( Z \) and \( Z' \) are random elements \( (\hat{Z}, \hat{Z}', T) \) such that \( T \) is a.s. finite, \( \hat{Z} \overset{D}{=} Z, \hat{Z}' \overset{D}{=} Z' \) and \( \theta_T \hat{Z} = \theta_T \hat{Z}' \).

After these preliminaries, we come to the definition of this section’s main concept.

**Definition 6.1.** A one-sided, continuous-time stochastic process \( Z \) is said to be *classically regenerative* with regeneration times \( S \) if, for all \( n \geq 0 \), we have

\[
\theta_{S_n}(Z, S) \overset{D}{=} \theta_{S_0}(Z, S).
\]

and

\[
\theta_{S_n}(Z, S) \text{ is independent of } ((Z_s)_{s \in [0, S_n]}, S_0, S_1, \ldots, S_n).
\]

In other words, \( (Z, S) \) regenerates at the times \( S \), that is it starts anew, independently of the past. The above definition can be reformulated as follows (see [3], page 346 for a proof).

**Proposition 6.2.** \( Z \) is classically regenerative with regeneration times \( S \) if and only if the cycles \( C_1, C_2, C_3, \ldots \) are i.i.d. and independent of the delay \( D \).

Notice that \( S \) is a random walk with initial position \( S_0 \) and step-lengths \( X_1, X_2, \ldots \) when \( Z \) is classically regenerative with regeneration times \( S \). Another pair \((Z', S')\) is called a *version* of a classically regenerative \((Z, S)\) if \((Z', S')\) is itself classically regenerative and

\[
\theta_{S_0}(Z', S') \overset{D}{=} \theta_{S_0}(Z, S).
\]

Thus, a version, \((Z', S')\), of a classically regenerative process \((Z, S)\) has the same cycle-distribution but the delay-distributions may differ; \((Z', S')\) is a differently started version of \((Z, S)\). By Proposition 6.2, if \((Z, S)\) is classically regenerative then clearly \( S \) is a random walk with positive i.i.d. step-lengths.

An example of a classically regenerative process is the GI/GI/1 queue that empties infinitely often, with the successive entrance times to an empty system as regeneration times.

A *renewal process* is a \((\lambda, \mu)\) random walk, \( S \), with a positive initial position \( S_0 \) and strictly positive step-lengths \( X_1, X_2, \ldots \). We call \( S_0 \) the delay of \( S \) while the \( X_1, X_2, \ldots \) are called *life times* in the renewal context (the analogy being that the \( X_1, X_2, \ldots \) are the life times of light bulbs that are replaced by a new one the moment they burn out,
then \((S_n)_{n \geq 0}\) are the \textit{renewal times} of the light bulbs. A renewal process \(S\) determines the following one-sided continuous-time stochastic processes (let \(N_t = \inf\{k \geq 0 : S_k \geq t\}\)):

\[
A_t := t - S_{N_t - 1}, \quad \text{age (of light bulb) at time } t.
\]

\[
B_t := S_{N_t} - t, \quad \text{residual life time at time } t.
\]

\[
D_t := A_t + B_t = X_{N_t}, \quad \text{total life at time } t.
\]

\[
U_t := A_t / D_t, \quad \text{relative age at time } t.
\]

It is easy to see that all of these processes are classically regenerative with the same regeneration times \(S\). We now apply our results from the last section to obtain an exact coupling of these processes.

\textbf{Theorem 6.3.} Let \(S\) and \(S'\) be discrete \((\lambda, \mu)\) and \((\lambda', \mu)\) renewal processes, respectively, with \(\lambda, \lambda' \in \mathfrak{a}(A, \mu)\). Let \(Z\) and \(Z'\) be defined by \(Z_t = (A_t, B_t, D_t, U_t)\) and \(Z'_t = (A'_t, B'_t, D'_t, U'_t)\). Then there exists a successful exact coupling, \((\hat{Z}, \hat{Z}', T)\), of \(Z\) and \(Z'\).

\textit{Proof.} By Theorem 5.2, there exists a successful shift-coupling, \((\hat{S}, \hat{S}', K, K')\), of \(S\) and \(S'\). Let \(\hat{Z}\) and \(\hat{Z}'\) be the processes associated with \(\hat{S}\) and \(\hat{S}'\), respectively. Since \(Z\) and \(Z'\) are determined by \(S\) and \(S'\), and \(\hat{S} \overset{D}{=} S\) and \(\hat{S}' \overset{D}{=} S'\), we get that \(\hat{Z} \overset{D}{=} Z\) and \(\hat{Z}' \overset{D}{=} Z'\). Since \(\theta_K \hat{S} = \theta_{K'} \hat{S}'\), \(\hat{Z}\) and \(\hat{Z}'\) are exactly the same from the a.s. finite random time \(T := \hat{S}_K = \hat{S}'_{K'}\) onward (notice that \(T\) a.s. takes countably many values). Thus, \((\hat{Z}, \hat{Z}', T)\) is a successful exact coupling of \(Z\) and \(Z'\). \(\square\)

This gives us asymptotic results on \(Z_t = (A_t, B_t, D_t, U_t)\).

\textbf{Corollary 6.4.} Let \(Z\) and \(Z'\) be as in the above theorem. We have

\[
\|P(Z_t \in \cdot) - P(Z'_t \in \cdot)\| \to 0 \quad \text{as} \quad t \to \infty.
\]

\textit{Proof.} The event \(\{T \leq t\}\) is a coupling event of the coupling \((\hat{Z}_t, \hat{Z}'_t)\) of \(Z_t\) and \(Z'_t\) and the coupling event inequality gives us

\[
\|P(Z_t \in \cdot) - P(Z'_t \in \cdot)\| \leq 2P(T \geq t).
\]

The result follows since \(T\) is finite. \(\square\)
Thus, $A_t, B_t, D_t$ and $U_t$, in the end, forget how they started.

More generally, we can apply the results from the last two sections to find a successful exact coupling of two regenerative processes when the delay and cycle-lengths are discrete.

**Theorem 6.5.** Let $(Z, S)$ be classically regenerative with discrete delay-length distribution $\lambda$ and discrete cycle-length distribution $\mu$. Let $(Z', S')$ be a version of $(Z, S)$ with discrete delay-length distribution $\lambda'$. If $\lambda, \lambda' \in a(A_\mu)$, then there exists a successful exact coupling, $(\hat{Z}, \hat{Z}', T)$, of $Z$ and $Z'$.

**Proof.** Let $D, C_1, C_2, \ldots$ be the delay and cycles of $(Z, S)$ and let $D'$ be the delay of $(Z', S')$. By Proposition 6.2, $C_1, C_2, \ldots$ are i.i.d. and independent of $D$. By Theorem 5.2, there exists a successful shift-coupling, $(\hat{S}, \hat{S}', K, K')$, of $S$ and $S'$. Let $\hat{S}_n = \hat{S}_0 + \hat{X}_1 + \ldots + \hat{X}_n$ and $\hat{S}'_n = \hat{S}'_0 + \hat{X}'_1 + \ldots + \hat{X}'_n$. Define, independent of $(\hat{S}, \hat{S}', K, K')$, the countable independent collection

$$(C^x_1, C^x_2, C^x_3, \ldots), \quad x \in A_\mu,$$

where, for each $x \in A_\mu$, $C^x_1, C^x_2, C^x_3, \ldots$ are i.i.d. and for all $i \geq 1$,

$$P(C^x_i \in \cdot) = P(C'^x_i \in \cdot) = P(C_1 \in \cdot | X_1 = x).$$

Independent of all of these random elements, let $D^y, y \in a(A_\mu)$, be a countable collection of independent random elements with

$$P(D^y \in \cdot) = P(D \in \cdot | S_0 = y),$$

and independent of all the random elements above, let $D'^y, y \in a(A_\mu)$, be a countable collection of independent random elements with

$$P(D'^y \in \cdot) = P(D' \in \cdot | S'_0 = y).$$

Now define the stochastic processes $\hat{Z}$, with delay $D_{\hat{S}_0}$ and cycles $C^{\hat{X}_1}, C^{\hat{X}_2}, \ldots$ and $\hat{Z}'$ with delay $D'^{\hat{S}_0}$ and cycles $C'^{\hat{X}_1}, C'^{\hat{X}_2}, \ldots$.

The cycles $C^{\hat{X}_1}, C^{\hat{X}_2}, \ldots$ are obviously i.i.d. and independent of $D_{\hat{S}_0}$. For $i \geq 1$, we
get

\[
P(C_i^{\hat{X}_i} \in \cdot) = \sum_{x \in a(\mu)} P(C^{\hat{X}_i} \in \cdot, \hat{X}_i = x)
\]

\[
= \sum_{x \in a(\mu)} P(C_i^{\hat{x}} \in \cdot) P(\hat{X}_i = x)
\]

\[
= \sum_{x \in a(\mu)} P(C_i^{\hat{x}} \in \cdot) P(X_1 = x)
\]

\[
= \sum_{x \in a(\mu)} P(C_i^{\hat{x}} \in \cdot, X_1 = x)
\]

\[
= P(C_1 \in \cdot),
\]

where the fourth equality follows directly from the definition of \(C_i^{\hat{x}}\). Similarly, we get:

\[
D_{\hat{S}_0} \overset{D}{=} D, D'_{\hat{S}_0} \overset{D}{=} D'
\]

and for \(i \geq 1\), \(C_i^{\hat{X}_i} \overset{D}{=} C_1\). Thus \(\hat{Z}\) and \(\hat{Z}'\) are copies of \(Z\) and \(Z'\), respectively.

By definition, for \(i \geq 1\), the cycle \(C_i^{\hat{X}_i}\) has length \(\hat{X}_i\), while the cycle \(C_i^{\hat{X}_i'}\) has length \(\hat{X}_i'\). Now, since \((\hat{S}, \hat{S}', K, K')\) is a successful shift-coupling of \(S\) and \(S'\), we have that \(T := \hat{S}_K = \hat{S}'_{K'}\) is an a.s. finite random time at which \(\hat{Z}\) and \(\hat{Z}'\) share a regeneration time. Define a third process \(\hat{Z}''\), following \(\hat{Z}'\) until the time \(T\), and then switching to \(\hat{Z}\). The cycles of \(\hat{Z}''\) are

\[
C_i^{\hat{Z}''} = \begin{cases} 
C_i^{\hat{X}_i'} & \text{if } i \leq K' \\
C_{K+1+(i-K')} & \text{if } i > K'. 
\end{cases}
\]

By a similar argument as in step 4 of the proof of Theorem 3.3, \(C_1^{\hat{Z}''}, C_2^{\hat{Z}''}, \ldots\) are i.i.d. and independent of \(D_{\hat{S}_0}\), having the same distribution as \(C_1\). Thus, \((\hat{Z}, \hat{Z}''', T)\) is a successful exact coupling of \(Z\) and \(Z'\).

Again, we obtain a corollary on asymptotic loss of memory, the proof is the same as in Corollary 6.4.

**Corollary 6.6.** Let \((Z, S)\) and \((Z', S')\) be as in the above theorem. We have

\[
\|P(Z_t \in \cdot) - P(Z'_t \in \cdot)\| \to 0 \quad \text{as} \quad t \to \infty.
\]

If we could choose \(Z'\) to be stationary, that is \(Z' \overset{D}{=} \theta_t Z'\) for all \(t \geq 0\), the above corollary would yield that \(Z_t\) converges to stationarity in total variation. However, this can not be done since a stationary version of \(Z\) has a continuous delay-length distribution (while \(S'_0\) is discrete) with density \(\frac{P(X_1 > x)}{E[X_1]}\) (see [3], page 65 for proof).
7 Transfer and Splitting

Next, we will look at random walks whose step-lengths have singular components. Since we are dealing with components of the step-lengths, we shall need some new concepts and techniques before we can move on. This section is based on Chapter 3, Sections 4 and 5 of [3].

We say that a probability space, \((\bar{\Omega}, \bar{\mathcal{F}}, \bar{P})\), is an extension of the probability space \((\Omega, \mathcal{F}, P)\) if there is a \(\bar{\mathcal{F}}/\mathcal{F}\)-measurable mapping \(\xi\) such that \(\bar{P}^{-1}\xi = P\), that is \(\bar{P}(\xi \in A) = P(A), A \in \mathcal{F}\). The term extension stems from the fact that every random element on \((\Omega, \mathcal{F}, P)\) also exists on \((\bar{\Omega}, \bar{\mathcal{F}}, \bar{P})\), that is, every random element \(X\) on \((\Omega, \mathcal{F}, P)\) has the induced copy \(\bar{X}(\omega) := X(\xi(\omega))\), on \((\bar{\Omega}, \bar{\mathcal{F}}, \bar{P})\). \(\bar{X}\) is indeed a copy of \(X\) since \(\bar{P}(\bar{X} \in A) = \bar{P}(\xi \in X^{-1}(A)) = P(X^{-1}(A)) = P(X \in A)\).

After extending a probability space, we shall treat the induced copies \(\bar{X}\) as if they were the original \(X\).

Let \(X_0\) be a random element on some probability space \((\Omega, \mathcal{F}, P)\) taking values in some space \((E_0, \mathcal{E}_0)\) and let \(Q(\cdot, \cdot)\) be a \(((E_0, \mathcal{E}_0), (E_1, \mathcal{E}_1))\) probability kernel (this means that \(Q(x_0, \cdot)\) is a probability measure on \((E_1, \mathcal{E}_1)\) for all \(x_0 \in E_0\) and \(Q(\cdot, B)\) is \(\mathcal{E}_0/\mathcal{B}(\mathbb{R})\)-measurable for all \(B \in \mathcal{E}_1\)). Then we can extend \((\Omega, \mathcal{F}, P)\) to support a random element \(X_1\) in \((E_1, \mathcal{E}_1)\) such that \(Q(\cdot, \cdot)\) is a version of \(P(X_1 \in \cdot \mid X_0 = \cdot)\), that is, for all \(B \in \mathcal{B}(\mathbb{R})\), we have

\[
Q(\cdot, B) = P(X_1 \in B \mid X_0 = \cdot), \quad P\text{-a.s.}
\]

We say that \(Q(\cdot, \cdot)\) is a regular version of \(P(X_1 \in \cdot \mid X_0 = \cdot)\).

The product measure theorem (see [4], page 97) indicates how we should proceed. Define

\[
\bar{\Omega} = \Omega \times E_1, \quad \bar{\mathcal{F}} = \mathcal{F} \otimes \mathcal{E}_1,
\]

and for each \(C \in \bar{\mathcal{F}}\), let

\[
\bar{P}(C) = \int_{\Omega} Q(X_0(\omega), C(\omega))dP,
\]

where \(C(\omega) = \{x_1 \in E_1 : (\omega, x_1) \in C\}\).

Before proving that \((\bar{\Omega}, \bar{\mathcal{F}}, \bar{P})\) extends \((\Omega, \mathcal{F}, P)\), supporting an \(X_1\) such as above, we require a definition.

**Definition 7.1.** Let \(X, Y\) and \(Z\) be random elements on some probability space. We say that \(X\) is conditionally independent of \(Z\), given \(Y\) if

\[
P(X \in \cdot, Z \in \cdot \mid Y) = P(X \in \cdot \mid Y)P(Z \in \cdot \mid Y) \quad a.s.
\]
This is equivalent to (see [5], page 87)

\[ P(X \in \cdot \mid Y, Z) = P(X \in \cdot \mid Y) \quad \text{a.s.} \]

**Theorem 7.2** (Conditioning-in a new random element). Let \((\Omega, \mathcal{F}, P), (\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{P}), X_0\) and \(Q\) be as above. Then \((\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{P})\) extends \((\Omega, \mathcal{F}, P)\) and there is a random element \(\tilde{X}_1\) on \((\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{P})\) such that \(Q(\cdot, \cdot)\) is a regular version of \(P(\tilde{X}_1 \in \cdot \mid \tilde{X}_0 = \cdot)\). Moreover, if \(X\) is any random element on \((\Omega, \mathcal{F}, P)\) and \(\tilde{X}\) its induced copy, then \(\tilde{X}_1\) is conditionally independent of \(\tilde{X}\), given \(\tilde{X}_0\).

**Proof.** To see that \((\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{P})\) extends \((\Omega, \mathcal{F}, P)\), define \(\xi\) on \((\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{P})\) as

\[ \xi(\omega, x_1) = \omega. \]

Then

\[ \tilde{P}(\xi^{-1}(A)) = \tilde{P}(\xi \in A) = \tilde{P}(A \times E_1) = \int_A dP = P(A), \]

and \((\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{P})\) extends \((\Omega, \mathcal{F}, P)\).

Now define \(\tilde{X}_1(\omega, x_1) = x_1\). For \(A \in \mathcal{E}_0\) and \(B \in \mathcal{E}_1\), we get

\[
\tilde{P}(\tilde{X}_0 \in A, \tilde{X}_1 \in B) = \tilde{P}(\{X_0 \in A\} \times B) = \int_{X_0^{-1}(A)} Q(X_0(\cdot), B) dP
\]

\[ = \int_A Q(\cdot, B) dP_{X_0} = \int_A Q(\cdot, B) d\tilde{P}_{X_0}. \]

Thus, \(Q(\cdot, \cdot)\) is a version of \(\tilde{P}(\tilde{X}_1 \in \cdot \mid \tilde{X}_0 = \cdot)\).

To see that \(\tilde{X}_1\) is conditionally independent of \(\tilde{X}\), given \(\tilde{X}_0\), for any original random element \(X\) in some measurable space \((E, \mathcal{E})\), observe that, for \(B \in \mathcal{E}_1\) and \(C \in \mathcal{E} \otimes \mathcal{E}_0\), we have

\[
\tilde{P}(\tilde{X}_1 \in B, (\tilde{X}, \tilde{X}_0) \in C) = \tilde{P}(\{(X, X_0) \in C\} \times B)
\]

\[ = E[\mathbf{1}_{\{(X, X_0) \in C\}} Q(X_0, B)]. \]

Since \((\tilde{X}, \tilde{X}_0) \overset{D}{=} (X, X_0)\), we obtain

\[
\tilde{P}(\tilde{X}_1 \in B, (\tilde{X}, \tilde{X}_0) \in C) = \tilde{E}[\mathbf{1}_{\{(X, X_0) \in C\}} Q(\tilde{X}_0, B)].
\]

Thus, \(Q(\tilde{X}_0, \cdot) = \tilde{P}(\tilde{X}_1 \in \cdot \mid \tilde{X}_0)\) is a version of \(\tilde{P}(\tilde{X}_1 \in \cdot \mid X, \tilde{X}_0)\) and the result follows.

This “Markov-extension” of a probability space can be repeated countably many times due to the Ionescu Tulcea theorem (see [4], page 109). We are now ready for the transfer theorem.
Theorem 7.3. (Transfer) Let $X'_0$ and $X'_1$ be random elements in some spaces $(E_0, \mathcal{E}_0)$ and $(E_1, \mathcal{E}_1)$, respectively, defined on some probability space $(\Omega', \mathcal{F}', P')$. Suppose that $P'(X'_1 \in \cdot \mid X'_0 = \cdot)$ has a regular version $Q(\cdot, \cdot)$. Further, suppose we have a random element $X_0$ defined on some probability space, $(\Omega, \mathcal{F}, P)$, such that $X_0 \overset{D}{=} X'_0$. Then we can extend $(\Omega, \mathcal{F}, P)$ to support a random element $X_1$ in $(E_1, \mathcal{E}_1)$ such that $(X_0, X_1) \overset{D}{=} (X'_0, X'_1)$.

Proof. By Theorem 7.2, we can extend $(\Omega, \mathcal{F}, P)$ to support a random element $X_1$, such that $Q(\cdot, \cdot)$ is a version of $P(X_1 \in \cdot \mid X_0 = \cdot)$. Thus, since $P_{X'_0} = P_{X_0}$, we have, for $A \in \mathcal{E}_0, B \in \mathcal{E}_1$,

$$
P(X_0 \in A, X_1 \in B) = \int_A P(X_1 \in B \mid X_0 = \cdot) dP_{X_0}
= \int_A Q(\cdot, B) dP_{X_0}
= \int_A P(X'_1 \in B \mid X'_0 = \cdot) dP_{X'_0}
= P(X'_0 \in A, X'_1 \in B).$$

A complete (every Cauchy-sequence converges) and separable (has a countable, dense subset) metric space, is called Polish. Polish spaces are important in probability theory due to the fact that if $X$ and $Y$ are random elements in the measurable spaces $(E, \mathcal{E})$ and $(G, \mathcal{G})$, respectively, with $E$ Polish and $\mathcal{E}$ its Borel subsets, then $P(X \in \cdot \mid Y = \cdot)$ has a regular version (see [4], page 265).

Corollary 7.4. Let $E_1$ and $E_2$ be Polish spaces and $\mathcal{E}_1$ and $\mathcal{E}_2$ be their Borel subsets. Let $f$ be an $\mathcal{E}_1/\mathcal{E}_2$-measurable mapping and let $X$ be a random element in $(E_2, \mathcal{E}_2)$, defined on some probability space $(\Omega, \mathcal{F}, P)$. Let $V$ be a random element in $(E_1, \mathcal{E}_1)$, defined on some probability space $(\Omega', \mathcal{F}', P')$, such that $f(V) \overset{D}{=} X$. Then we can extend $(\Omega, \mathcal{F}, P)$ to support a random element $V'$ in $(E_1, \mathcal{E}_1)$ such that $X = f(V')$ a.s.

Proof. Since $(E_1, \mathcal{E}_1)$ is Polish, there exist a regular version of

$$
P(V \in \cdot \mid f(V) = \cdot).
$$

By the transfer theorem, we can extend $(\Omega, \mathcal{F}, P)$ to support a random element $V'$ in $(E_1, \mathcal{E}_1)$, such that $(X, V') \overset{D}{=} (f(V), V)$. Since the mapping $(x_1, x_2) \mapsto (x_1, f(x_2))$ is $\mathcal{E}_2 \otimes \mathcal{E}_1/\mathcal{E}_2 \otimes \mathcal{E}_1$-measurable, we get

$$(X, f(V')) \overset{D}{=} (f(V), f(V)).$$
Because the diagonal $D := \{(x_2, x_2) : x_2 \in E_2\}$ is measurable in $E_2 \otimes E_2$, and since $E_2$ is Polish (see [3], page 152), we get

$$P(X = f(V')) = P((X, f(V)) \in D) = P'(f(V), f(V)) \in D) = 1.$$ \hfill \qed

In the transfer theorem, the new random element $X_1$ is conditionally independent of all the original random elements given $X_0$. If $X_0$ itself is independent of some random element $X$, we obtain the following result.

**Lemma 7.5.** Let $X_0$, $X_1$ and $X$ be random elements in some general spaces $(E_0, \mathcal{E}_0)$, $(E_1, \mathcal{E}_1)$ and $(E, \mathcal{E})$, respectively, such that $X_1$ is independent of $X$ given $X_0$ and such that $X_0$ and $X$ are independent. Then $(X_0, X_1)$ is independent of $X$.

**Proof.** For $A \in \mathcal{E}_0$, $B \in \mathcal{E}_1$ and $C \in \mathcal{E}$ we have

$$P(X_0 \in A, X_1 \in B, X \in C) = \int_{X_0^{-1}(A) \cap X^{-1}(C)} P(X_1 \in B \mid X_0, X) dP$$

$$= \int_{X_0 \cap X^{-1}(C)} P(X_1 \in B \mid X_0 = \cdot, X = \cdot) dP_{(X_0, X)}$$

$$= \int_{X_0 \cap X^{-1}(C)} P(X_1 \in B \mid X_0 = \cdot) dP_{(X_0, X)}$$

$$= \int_{X_0 \cap X^{-1}(C)} \int_{X} P(X_1 \in B \mid X_0 = \cdot) dP_X dP_X$$

$$= P(X_0 \in A, X_1 \in B)P(X \in C),$$

where in the second equality we used the change of measure theorem, in the third equality the conditional independence of $X_1$ and $X_0$ given $X$, and in the fourth equality we used the independence of $X_0$ and $X$ and Fubini’s theorem. The result now follows from a monotone class argument. \hfill \qed

Let $X$ be a random element on $(\Omega, \mathcal{F}, P)$, taking values in some space $(E, \mathcal{E})$. Suppose $P(X \in \cdot) \geq \nu(\cdot) + \nu'(\cdot)$, where $\nu$ and $\nu'$ are some measures on $(E, \mathcal{E})$, that is, $\nu(\cdot) + \nu'(\cdot)$ is a *component* of $X$. The transfer theorem gives us a 0-1-2 random variable that determines when $X$ is governed by $\nu$ and when it is governed by $\nu'$. This is called a *splitting* of $X$. We only prove a splitting result for the case when $X$ has a sum, $\nu(\cdot) + \nu'(\cdot)$, of two measures as a component because that is precisely what we shall need in the next section. For a more detailed discussion of splitting, see [3], Chapter 3, Section 5.

**Theorem 7.6.** *(Splitting)* If $X, \nu$ and $\nu'$ are as above, then there exists a 0-1-2
random variable, $K$, such that

$$P(K = 1) = \|\nu\|, \quad P(K = 2) = \|\nu'\|, \quad \text{and} \quad P(X \in \cdot, K = 1) = \nu(\cdot), \quad P(X \in \cdot, K = 2) = \nu'(\cdot).$$

**Proof.** Let $U', V', W'$ and $K'$ be independent random elements, defined on some probability space such that

$$P(K' = 1) = \|\nu\|, \quad P(K' = 2) = \|\nu'\| \quad \text{and} \quad P(K' = 0) = 1 - (\|\nu\| + \|\nu'\|),$$

$$P(U' \in \cdot) = \frac{\nu(\cdot)}{\|\nu\|},$$

$$P(V' \in \cdot) = \frac{\nu'(\cdot)}{\|\nu'\|},$$

$$P(W' \in \cdot) = \frac{P(X \in \cdot) - (\nu(\cdot) + \nu'(\cdot))}{1 - (\|\nu\| + \|\nu'\|)}.$$

Now define

$$X' = \begin{cases} U' & \text{if } K' = 1, \\ V' & \text{if } K' = 2, \\ W' & \text{if } K' = 0. \end{cases}$$

Then $X'$ is a copy of $X$, since

$$P(X' \in \cdot) = P(X' \in \cdot, K' = 1) + P(X' \in \cdot, K' = 2) + P(X' \in \cdot, K' = 0)$$

$$= P(U' \in \cdot, K' = 1) + P(V' \in \cdot, K' = 2) + P(W' \in \cdot, K' = 0)$$

$$= P(U' \in \cdot)P(K' = 1) + P(V' \in \cdot)P(K' = 2) + P(W' \in \cdot)P(K' = 0)$$

$$= \frac{\nu(\cdot)}{\|\nu\|} \|\nu\| + \frac{\nu'(\cdot)}{\|\nu'\|} \|\nu'\| + \frac{P(X \in \cdot) - (\nu(\cdot) + \nu'(\cdot))}{1 - (\|\nu\| + \|\nu'\|)}(1 - (\|\nu\| + \|\nu'\|))$$

$$= P(X \in \cdot).$$

Since $K'$ takes values in the Polish space $\{0, 1, 2\}$, there exists a regular version of $P(K' \in \cdot \mid X' = \cdot)$. By the transfer theorem we can extend $(\Omega, \mathcal{F}, P)$ to support the random variable $K$, such that $(X, K) \overset{D}{=} (X', K')$ and thus

$$P(X \in \cdot, K = 1) = P(X' \in \cdot, K' = 1) = P(U' \in \cdot, K' = 1)$$

$$= P(U' \in \cdot)P(K' = 1) = \frac{\nu(\cdot)}{\|\nu\|} \|\nu\| = \nu(\cdot).$$

By the same argument, we get $P(X \in \cdot, K = 2) = \nu'(\cdot)$.

We are now ready for our final endeavour.
8 Non-Discrete Step-Lengths

We have been dealing with discrete random walks. On the other end of the spectrum
are those with step-lengths that are spread-out: Let $S$ be a $(\lambda, \mu)$ random walk and
suppose we have an independent sequence of step-lengths: $X_1, X_2, \ldots$, where $X_i \sim \mu$. We say that $S$ has spread-out step-lengths if there is an $r$ and a measurable function $f$, with $\int_{\mathbb{R}} f(x)dx > 0$, such that

$$P(X_1 + \ldots + X_r \in A) \geq \int_A f(x)dx, \quad A \in \mathcal{B}(\mathbb{R}).$$

If $S$ has spread out step-lengths and $S'$ is a differently started version of $S$, then there exists a successful exact coupling of $S$ and $S'$ (see [3], page 98).

In between discrete and spread-out random walks there lie random walks whose step-lengths are neither; random walks with singular-continuous step-lengths. 

**Definition 8.1.** We call a measure, $\nu$, on $\mathbb{R}$, singular-continuous, if for each $x \in \mathbb{R}$, $\nu(x) = 0$, and there is a set $A \in \mathcal{B}(\mathbb{R})$ with Lebesgue-measure zero, such that $\nu(\mathbb{R}) = \nu(A) > 0$.

The next theorem establishes the existence of a successful exact coupling of two differently started $(s, \mu)$ and $(s', \mu)$ random walks, where the sum of $r$ step-lengths has a component that satisfies a certain periodicity condition. The proof is an extension of the approach in the spread-out case given in [3].

**Theorem 8.2.** Let $S$ and $S'$ be $(s, \mu)$ and $(s', \mu)$ random walks, respectively. Let $X_1, X_2, \ldots$ be the step-lengths of $S$ and suppose there is an $r$ such that $X_1 + X_2 + \ldots + X_r$ has a component of the form $\nu(\cdot) + \nu(\cdot + a)$, $a > 0$, where $\nu$ is a non-trivial measure. Suppose that $s - s' \in a\mathbb{Z}$. Then there exists a successful exact coupling, $(\hat{S}, \hat{S}', T)$, of $S$ and $S'$.

**Proof.** It is no restriction to let $S$ and $S'$ be independent, defined on some probability space $(\Omega, \mathcal{F}, P)$. Let $L_k = X_{(k-1)r+1} + \ldots + X_{kr}$ for $k \geq 1$. Now, $P(L_1 \in \cdot) \geq \nu(\cdot) + \nu(\cdot + a)$, so we can use the splitting theorem to obtain a 0-1-2 random variable $K_1$ such that

$$P(K_1 = 1) = P(K_1 = 2) = \|\nu\|, \quad \text{and}$$

$$P(L_1 \in \cdot, K_1 = 1) = \nu(\cdot), \quad P(L_1 \in \cdot, K_1 = 2) = \nu(\cdot + a).$$

Since $L_1$ is independent of $L_2, L_3, \ldots$, by the transfer theorem and Lemma 7.5, $(L_1, K_1)$ is independent of $L_2, L_3, \ldots$ Since we can transfer countably many times we can obtain an i.i.d. sequence $(L_1, K_1), (L_2, K_2), \ldots$
For \( k \geq 1 \), define
\[
L''_k = \begin{cases} 
L_k & \text{if } K_k = 0, \\
L_k - a & \text{if } K_k = 1, \\
L_k + a & \text{if } K_k = 2.
\end{cases}
\]

Since \( L''_k \) is the same measurable function of \((L_k, K_k)\) for all \( k \geq 1 \), and since \((L_1, K_1), (L_2, K_2), \ldots\) are i.i.d., the \( L''_1, L''_2, \ldots \) are also i.i.d. To see that \( L''_k \overset{D}{=} L_k \), note that
\[
P(L''_k \in \cdot) = P(L''_k \in \cdot, K_k = 0) + P(L''_k \in \cdot, K_k = 1) + P(L''_k \in \cdot, K_k = 2)
= P(L_k \in \cdot, K_k = 0) + P(L_k - a \in \cdot, K_k = 1) + P(L_k + a \in \cdot, K_k = 2)
= P(L_k \in \cdot) - (\nu(\cdot) + \nu(\cdot + a)) + \nu(\cdot + a) + \nu(\cdot)
= P(L_k \in \cdot).
\]

Let \( R := (S_{kr})_{k \geq 0} \) and let \( R'' \) be the random walk beginning in \( s' \) and having step-lengths \( L''_1, L''_2, \ldots \). The difference walk \( R - R'' \) begins in \( s - s' \in a\mathbb{Z} \) and has step-lengths
\[
L_k - L''_k = \begin{cases} 
0 & \text{if } K_k = 0, \\
a & \text{if } K_k = 1, \\
-a & \text{if } K_k = 2.
\end{cases}
\]

Since \( P(K_k = 1) = P(K_k = 2) = ||\nu|| > 0 \), the walk \( R - R'' \) is a simple, symmetric random walk on the lattice \( a\mathbb{Z} \) and, by Lemma 3.1, will hit zero in an a.s. finite random time \( M \). Now, since \( L''_1 \overset{D}{=} L_1 \) and \( L_1 = X_1 + \ldots + X_r \), Corollary 7.4 allows us to extend \((\Omega, \mathcal{F}, P)\) to support new random variables \( X''_1, \ldots, X''_r \) such that \((X''_1, \ldots, X''_r) \overset{D}{=} (X_1, \ldots, X_r)\), making \( X''_1, \ldots, X''_r \) i.i.d., and such that \( X''_1 + \ldots + X''_r = L''_1 \). We can do this countably many times to obtain an i.i.d. sequence \( X''_1, X''_2, \ldots \) with \( X''_1 \sim \mu \) and \( L''_k = X''_{(k-1)r+1} + \ldots + X''_{kr}, k \geq 1 \). Now, let \( T := Mr \) and define \( S''' \) with initial position \( s' \) and step-lengths
\[
X'''_k = \begin{cases} 
X''_k & \text{if } k \leq T, \\
X_k & \text{if } k > T.
\end{cases}
\]

By the same argument as in Step 4 of Theorem 3.3, \( S''' \) is a \((s', \mu)\) random walk and thus, \((S, S''' , T)\) is a successful exact coupling of \( S \) and \( S' \).

This gives us asymptotic results on \( S \) and \( S' \).

**Corollary 8.3.** Let \( S \) and \( S' \) be as in the above theorem. We have
\[
\|P(S_n \in \cdot) - P(S'_n \in \cdot)\| \to 0 \quad \text{as} \quad t \to \infty.
\]
Proof. The event \( \{ T \leq n \} \) is a coupling event of the coupling \((\hat{S}_n, \hat{S}'_n)\) of \(S_n\) and \(S'_n\) and the coupling event inequality gives us

\[
\| P(S_n \in \cdot) - P(S'_n \in \cdot) \| \leq 2P(T \geq t).
\]

The result follows since \( T \) is finite.

A specific example of a singular-continuous random walk that fits nicely into this theorem is a random walk with step-lengths in the “double” Cantor set.

**Example 8.4.** Let \( D_1, D_2, \ldots \) be an i.i.d. sequence of random variables with \( P(D_i = 0) = P(D_i = 2) = 1/2 \). Define

\[
Y = \sum_{n=1}^{\infty} D_n 3^{-n}.
\]

So, \( Y \) is the random variable whose base-3 expansion is \(0.D_1D_2D_3\ldots\) and since \( D_i \) is either 0 or 2, for each \( i \geq 1 \), \( Y \) is concentrated on the Cantor set, \( C \). Now, \( C \) has Lebesgue-measure zero and obviously \( P(Y = y) = 0 \) for all \( y \in C \) so \( Y \) has a singular-continuous distribution, \( \nu \). Now, define the distribution \( \mu(\cdot) = \frac{1}{2} \nu(\cdot) + \frac{1}{2} \nu(\cdot + 1) \), concentrated on the union of \( C \) and \( C - 1 \). If \( S \) and \( S' \) are \((0, \mu)\) and \((1, \mu)\) random walks, respectively, by Theorem 8.2, there exists a successful coupling, \((\hat{S}, \hat{S}', T)\), of \( S \) and \( S' \).
References


